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Extragradient methods for solving bilevel split equilibrium problems with constraints

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ABSTRACT
This paper introduces and analyzes two novel iterative algorithms for addressing the monotone bilevel split equilibrium problem in real Hilbert spaces. The problem encompasses a general system of variational inequalities and a common fixed point problem involving a countable family of uniformly Lipschitzian pseudocontractive mappings alongside an asymptotically nonexpansive mapping. Our algorithms are predicated on a novel subgradient extragradient implicit method that utilizes the strong monotonicity of one bifunction at the upper-level equilibrium and the monotonicity of another bifunction at the lower level. We establish strong convergence results for the proposed algorithms under mild conditions. A detailed example demonstrates the practicality and effectiveness of our methods.

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1. Introduction

The theory of variational inequalities has been extensively applied in recent decades to address a wide range of problems encountered in various fields such as engineering, economics, mathematical programming, optimization, and finance. In particular, variational inequality problems (VIP) have become indispensable tools for modelling and solving problems involving equilibrium, optimization, and game theory, among others. The concept of variational inequalities is rooted in the study of fixed-point theory and monotone operator theory, providing a natural and unified framework for tackling these diverse problems.

Let $\mathcal{H}$ denote a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. A self-mapping A is defined on $\mathcal{H}$, and we consider a nonempty closed convex subset C of $\mathcal{H}$. The classical variational inequality problem (VIP) seeks to identify a point $b \in C$ that satisfies the following inequality:

$$\langle Ab, d - b \rangle \geq 0, \quad \forall d \in C.$$

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The solution set of the VIP is denoted by $\mathrm{VI}(C, A)$. This problem is of great importance in optimization, where the goal is often to find equilibrium points or to solve constrained optimization problems. The mapping A can often represent a gradient or an operator describing the relationship between the decision variable and the system's constraints. Therefore, solving the VIP can lead to the identification of optimal solutions in various applied settings, such as resource allocation problems, traffic equilibrium, and market equilibrium models.

In addition to the VIP, fixed point theory plays a crucial role in modelling a variety of real-world problems. Fixed-point problems (FPP) are fundamental in the study of iterative processes, where the goal is to find a point $x \in \mathcal{H}$ such that $T(x) = x$ for a given mapping T. This can be used to represent problems where the system's behaviour at each step is influenced by its state at the previous step, which is common in economics, game theory, and engineering. Let $\mathrm{Fix}(T)$ denote the set of fixed points of the mapping T. These problems often arise in optimization and equilibrium theory, where the search for a fixed point of a mapping corresponds to finding a steady state or equilibrium condition of the system under study.

The theory of equilibrium problems (EP) provides a powerful and cohesive framework for addressing a diverse array of real-world challenges that encompass VIPs, FPPs, complementarity problems, and Nash equilibrium problems. In an equilibrium problem, we seek to find a point $d \in C$ that satisfies the following condition involving a bifunction Θ:

$$\Theta(d, b) \geq 0, \quad \forall b \in C.$$

Here, $\Theta : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a bifunction that typically models the interactions between different agents or components of a system. It is often used to capture the constraints and interactions within an equilibrium system, such as market forces, optimization constraints, or the behaviour of agents in a game. The condition $\Theta(d, d) = 0$ for all $d \in C$ ensures that the equilibrium condition holds when the system is in balance. EPs generalize many classical problems in optimization and equilibrium theory, and their solutions correspond to points where the system's components are in a stable state or equilibrium.

One of the most effective methods for solving VIPs is the extragradient method introduced by Korpelevich [1]. This method, for any initial point $f_0 \in C$, generates a sequence $\{f_n\}$ according to the following iterative scheme:

$$b_n = \mathbf{Pj}_C(f_n - \ell A f_n), \quad f_{n+1} = \mathbf{Pj}_C(f_n - \ell A b_n), \quad \forall n \geq 1,$$

where $\mathbf{Pj}_C$ denotes the metric projection of $\mathcal{H}$ onto C, A is an L-Lipschitz continuous operator, and $\ell \in (0, \frac{1}{L})$. The sequence $\{f_n\}$ weakly converges to an element of $\mathrm{VI}(C, A)$. Extensive research on the extragradient method has highlighted its effectiveness and led to numerous extensions and improvements in various settings [2–14]. The convergence rate and convergence behaviour are crucial characteristics of any iterative algorithm. While weak convergence is often satisfactory in finite-dimensional spaces, strong convergence is often more desirable in infinite-dimensional settings. Furthermore, the extragradient method has been successfully applied to a variety of other optimization problems, including saddle-point problems, variational inequalities in machine learning, and network equilibrium models. In these contexts, the extragradient method's robustness and ability to handle large-scale problems with complex constraints have made it a preferred choice.

The algorithm's flexibility in adapting to different types of mappings, such as monotone or non-monotone operators, has contributed to its widespread adoption across multiple domains, including economics, optimization, and game theory.

Let $B_1, B_2 : \mathcal{H} \to \mathcal{H}$ be two nonlinear mappings. The general system of variational inequality problem (GSVI) aims to find $(d^*, b^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 b^* + d^* - b^*, d - d^* \rangle \geq 0, & \forall d \in C \\ \langle \mu_2 B_2 d^* + b^* - d^*, b - b^* \rangle \geq 0, & \forall b \in C, \end{cases} \tag{1}$$

where $\mu_1, \mu_2 \in (0, \infty)$ are constants. Notably, when $B_1 = B_2 = A$ and $d^* = b^*$, the GSVI (1) reduces to the aforementioned VIP. Importantly, problem (1) can be reformulated as a FPP, enabling the application of fixed point techniques for its solution.

Let Ξ denote the set of common solutions to the GSVI (1) for two inverse-strongly monotone operators B_1 and B_2, as well as the common fixed point problem (CFPP) for a countable family of ℓ-uniformly Lipschitzian pseudocontractive mappings $\{\mathcal{Z}_n\}_{n=1}^{\infty}$ and an asymptotically nonexpansive mapping $\mathcal{Z}_0$. In 2019, Ceng and Wen [4] proposed a novel hybrid extragradient-like implicit method to identify an element in Ξ. Starting from an arbitrary initial point $f_1 \in C$, this method generates a sequence $\{f_n\}$ through the following iterative process:

$$\begin{cases} u_n = \chi_n f_n + (1 - \chi_n)\mathcal{Z}_n u_n, \\ d_n = \mathbf{Pj}_C(u_n - \mu_2 B_2 u_n), \\ r_n = \mathbf{Pj}_C(d_n - \mu_1 B_1 d_n), \\ f_{n+1} = \mathbf{Pj}_C[\alpha_n g(f_n) + (\mathcal{I} - \alpha_n \rho F)\mathcal{Z}_0^n r_n], \quad \forall n \geq 1, \end{cases}$$

where $\mathcal{I}$ is the identity operator, and $g : C \to C$ is a ξ-contraction with $\xi \in [0, 1)$. Ceng and Wen [4] proved the strong convergence of $\{f_n\}$ to an element $g^* \in \Xi$. Recently, He et al. [15] investigated the monotone bilevel equilibrium problem (MBEP), which is constrained by the GSVI and CFPP. Specifically, they considered a strongly monotone equilibrium problem $\mathrm{EP}(\boldsymbol{\Omega}, \Gamma)$, where $\boldsymbol{\Omega}$ is the common solution set of another monotone equilibrium problem $\mathrm{EP}(C, \Phi)$, the GSVI, and the CFPP. By leveraging the subgradient extragradient implicit scheme, He et al. [15] developed two iterative algorithms to solve the MBEP. They established strong convergence theorems for these algorithms under suitable assumptions.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real Hilbert spaces, with $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$. Let $\mathcal{K} : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator, and $A, F : \mathcal{H}_1 \to \mathcal{H}_1$ and $B : \mathcal{H}_2 \to \mathcal{H}_2$ be nonlinear mappings. The bilevel split variational inequality problem (BSVIP), as introduced in [16], seeks a point $g^* \in \boldsymbol{\Omega}$ such that

$$\langle F g^*, z - g^* \rangle \geq 0, \quad \forall z \in \boldsymbol{\Omega},$$

where $\boldsymbol{\Omega}$ denotes the solution set of the split variational inequality problem (SVIP) defined as

$$\boldsymbol{\Omega} := \{z \in \mathrm{VI}(C, A) : \mathcal{K}z \in \mathrm{VI}(Q, B)\}.$$

Censor et al. [17] proposed a method for solving the SVIP with the following iterative procedure:

$$f_{n+1} = \mathbf{Pj}_C(\mathcal{I} - \lambda A)(f_n + \rho \mathcal{K}^*(\mathbf{Pj}_Q(\mathcal{I} - \lambda B) - \mathcal{I})\mathcal{K} f_n), \quad \forall n \geq 1,$$

where A and B are inverse-strongly monotone mappings. Under suitable conditions, They demonstrated that f_n converges weakly to $g^* \in \Omega$. It is worth noting that the VIP can be reformulated as a FPP. Consequently, we can now rephrase the BSVIP as follows. Let $A : \mathcal{H}_1 \to \mathcal{H}_1$ be L-Lipschitzian and quasimonotone, $F : \mathcal{H}_1 \to \mathcal{H}_1$ be κ-Lipschitzian and η-strongly monotone, and $\mathcal{Z} : \mathcal{H}_2 \to \mathcal{H}_2$ be a δ-demimetric mapping with $\delta \in (-\infty, 1)$. The problem seeks $g^* \in \Omega$ satisfying:

$$\langle Fg^*, z - g^* \rangle \geq 0, \quad \forall z \in \Omega,$$

where Ω is now defined as: $\Omega := \{z \in \mathrm{VI}(C, A) : \mathcal{K}z \in \mathrm{Fix}(\mathcal{Z})\}$. This particular BSVIP is often referred to as the bilevel split quasimonotone variational inequality problem (BSQVIP).

This paper explores the use of a novel subgradient extragradient implicit method to solve the monotone bilevel split equilibrium problem (MBSEP), which is subject to the GSVI and CFPP constraints. In this context, the CFPP involves finding a common fixed point for a countable family of uniformly Lipschitzian pseudocontractive mappings along with an asymptotically nonexpansive mapping. Our proposed method leverages the strong monotonicity of one bifunction at the upper-level equilibrium, while also incorporating the monotonicity of another bifunction at the lower level. Under relatively mild conditions, we establish strong convergence results for the algorithms we introduce. An illustrative example is provided to demonstrate the applicability and feasibility of the methods.

The paper is structured as follows: In Section 2, we introduce the basic concepts and tools that will be used throughout the study. Section 3 is devoted to proving the strong convergence of the proposed algorithms. Section 4 discusses the application of our main theorems to approximate a common solution to the GSVI, VIP, and split feasibility problem (SFP). In Section 5, we present an illustrative example that highlights the applicability and practical implementation of the proposed methods. Finally, we conclude with a summary in the last section.

2. Preliminaries

We denote strong convergence by the symbol $\to$ and weak convergence by the symbol $\rightharpoonup$. Given a sequence $\{f_n\} \subset \mathcal{H}$, the weak ω-limit set of $\{f_n\}$, denoted by $\omega_w(f_n)$, is defined as follows:

$$\omega_w(f_n) = \{x \in \mathcal{H} : f_{n_l} \rightharpoonup x \quad \text{for some subsequence } \{f_{n_l}\} \subset \{f_n\}\}.$$

A normal cone to C at a point $b \in C$ is defined as the set:

$$N_C(b) = \{w \in \mathcal{H} : \langle w, d - b \rangle \leq 0, \quad \forall d \in C\}.$$

Similarly, the subdifferential of a convex function $g : C \to \mathbb{R} \cup \{+\infty\}$ at a point $b \in C$ is defined by:

$$\partial g(b) = \{w \in \mathcal{H} : g(d) - g(b) \geq \langle w, d - b \rangle, \quad \forall d \in C\}.$$

Definition 2.1: A bifunction $\Theta : C \times C \to \mathbb{R}$ is said to be:

(i) χ-strongly monotone if there exists $\chi > 0$ such that $\Theta(b,d) + \Theta(d,b) \leq -\chi\|b - d\|^2$, $\forall b,d \in C$;
(ii) monotone if $\Theta(b,d) + \Theta(d,b) \leq 0, \quad \forall b,d \in C$;
(iii) Lipschitz-type continuous with constants $c_1, c_2 > 0$ if

$$\Theta(b,d) + \Theta(d,w) \geq \Theta(b,w) - c_1\|b-d\|^2 - c_2\|d-w\|^2, \quad \forall b,d,w \in C.$$

Definition 2.2: An operator $\mathcal{Q} : C \to \mathcal{H}$ is said to be:

(i) L-Lipschitz continuous if there exists $L > 0$ such that $\|\mathcal{Q}d - \mathcal{Q}b\| \leq L\|d - b\|$, $\forall d,b \in C$;
(ii) χ-strongly monotone if there exists $\chi > 0$ such that $\langle \mathcal{Q}d - \mathcal{Q}b, d - b\rangle \geq \chi\|d - b\|^2$, $\forall d,b \in C$;
(iii) α-inverse-strongly monotone if there exists $\alpha > 0$ such that $\langle \mathcal{Q}d - \mathcal{Q}b, d - b\rangle \geq \alpha\|\mathcal{Q}d - \mathcal{Q}b\|^2$, $\forall d,b \in C$;
(iv) monotone if $\langle \mathcal{Q}d - \mathcal{Q}b, d - b\rangle \geq 0$, $\forall d,b \in C$.
(v) pseudomonotone if $\langle \mathcal{Q}d, b - d\rangle \geq 0 \Rightarrow \langle \mathcal{Q}b, b - d\rangle \geq 0$, $\forall b,d \in C$;
(vi) quasimonotone if $\langle \mathcal{Q}d, b - d\rangle > 0 \Rightarrow \langle \mathcal{Q}b, b - d\rangle \geq 0$, $\forall b,d \in C$;
(vii) δ-demicontractive if there exists $\delta \in [0,1)$ such that $\|\mathcal{Q}b - d\|^2 \leq \|b - d\|^2 + \delta\|b - \mathcal{Q}b\|^2, \quad \forall b \in C,\ d \in \mathrm{Fix}(\mathcal{Q}) \neq \emptyset$;
(viii) δ-demimetric if there exists $\delta \in (-\infty, 1)$ such that $\langle b - \mathcal{Q}b, b - d\rangle \geq \frac{1-\delta}{2}\|b - \mathcal{Q}b\|^2$, $\forall b \in C,\ d \in \mathrm{Fix}(\mathcal{Q}) \neq \emptyset$.

Definition 2.3: A mapping $T : C \to C$ is classified as asymptotically nonexpansive if there exists a sequence $\{\theta_n\}_{n=1}^{\infty} \subset [0,\infty)$ such that $\lim_{n\to\infty} \theta_n = 0$ and the following inequality holds for all $d,b \in C$ and $n \geq 1$:

$$\theta_n\|d - b\| + \|d - b\| \geq \|T^n d - T^n b\|.$$

If $\theta_n = 0$ for all $n \geq 1$, the mapping T is referred to as nonexpansive.

Definition 2.4 ([18]): Let $\mathcal{Z} : C \to C$ be a self-mapping. The operator $(\mathcal{I} - \mathcal{Z})$ is said to be demiclosed at zero if, for any sequence $\{b_n\}$ in C such that $b_n \rightharpoonup b \in C$ and $(\mathcal{I} - \mathcal{Z})b_n \to 0$, then $(\mathcal{I} - \mathcal{Z})b = 0$.

Definition 2.5 ([4]): Let $\{\mathcal{Z}_n\}_{n=1}^{\infty}$ be a sequence of continuous pseudocontractive self-mappings on a nonempty, closed, and convex set C in a real Hilbert space $\mathcal{H}$. The sequence $\{\mathcal{Z}_n\}_{n=1}^{\infty}$ is called a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C if there exists a constant $\ell > 0$ such that each $\mathcal{Z}_n$ is ℓ-Lipschitz continuous.

For every point $b \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $\mathbf{Pj}_C b$, such that $\|b - \mathbf{Pj}_C b\| \leq \|b - d\|$ for all $d \in C$. Recall the following properties of the metric projection for all $b,d \in \mathcal{H}$ (see [18]):

(i) $\langle b - d, \mathbf{Pj}_C b - \mathbf{Pj}_C d\rangle \geq \|\mathbf{Pj}_C b - \mathbf{Pj}_C d\|^2$;
(ii) $d = \mathbf{Pj}_C b \iff \langle b - d, w - d\rangle \leq 0 \mathrm{forall} w \in C$;
(iii) $\|b - d\|^2 \geq \|b - \mathbf{Pj}_C b\|^2 + \|d - \mathbf{Pj}_C b\|^2$;

(iv) $\|b-d\|^2 = \|b\|^2 - \|d\|^2 - 2\langle b-d, d\rangle$;
(v) $\|sb+(1-s)d\|^2 = s\|b\|^2 + (1-s)\|d\|^2 - s(1-s)\|b-d\|^2$.

Lemma 2.6: *Let $\mathcal{Q}:\mathcal{H}\to\mathcal{H}$ be an α-inverse-strongly monotone mapping. Then, for any $\mu \geq 0$, we have*

$$\|(\mathcal{I}-\mu\mathcal{Q})d-(\mathcal{I}-\mu\mathcal{Q})b\|^2 \leq \|d-b\|^2 - \mu(2\alpha-\mu)\|\mathcal{Q}d-\mathcal{Q}b\|^2, \quad \forall d,b\in\mathcal{H}.$$

In particular, if $0\leq\mu\leq 2\alpha$, then $(\mathcal{I}-\mu\mathcal{Q})$ is nonexpansive.

Using Lemma 2.6, we immediately acquire the following lemma.

Lemma 2.7 ([6]): *Let $B_1:\mathcal{H}\to\mathcal{H}$ and $B_2:\mathcal{H}\to\mathcal{H}$ be α-inverse-strongly monotone and χ-inverse-strongly monotone mappings, respectively. Let the mapping $G:\mathcal{H}\to C$ be defined as $G := \mathbf{Pj}_C(\mathcal{I}-\mu_1 B_1)\mathbf{Pj}_C(\mathcal{I}-\mu_2 B_2)$. If $0<\mu_1\leq 2\alpha$ and $0<\mu_2\leq 2\chi$, then $G:\mathcal{H}\to C$ is nonexpansive.*

Lemma 2.8 ([19]): *Let C be a nonempty, closed, and convex subset of a Banach space X. If $\mathcal{Z}:C\to C$ is a continuous and strong pseudocontraction mapping, then there exists a unique fixed point of $\mathcal{Z}$ in C.*

Lemma 2.9 ([3, Theorem 2.1.3]): *Let $g: C\to\mathbb{R}\cup\{+\infty\}$ be a subdifferentiable function. Then, $\hat{u}$ is a solbtion to the convex minimization problem:* $\min\{g(b): b\in C\}$ *if and only if* $0\in\partial g(\hat{b}) + N_C(\hat{b})$.

Lemma 2.10 ([6]): *For given $d^*, b^*\in C$, the pair (d^*, b^*) is a solution of the problem GSVI (1) if and only if $d^*\in \mathrm{Fix}(G)$, where $G := \mathbf{Pj}_C(\mathcal{I}-\mu_1 B_1)\mathbf{Pj}_C(\mathcal{I}-\mu_2 B_2)$ and $b^* = \mathbf{Pj}_C(d^*-\mu_2 B_2 d^*)$.*

Lemma 2.11 ([20]): *Let C be a nonempty, closed, and convex subset of a Banach space X. Let $\{\mathcal{Z}_n\}_{n=1}^{\infty}$ be a sequence of self-mappings on C such that $\sum_{n=1}^{\infty}\sup\{\|\mathcal{Z}_{n+1}b-\mathcal{Z}_n u\| : b\in C\}<\infty$. Then, for each $d\in C$, the sequence $\{\mathcal{Z}_n d\}$ converges strongly to a point in C. Define $\mathcal{Z}:C\to C$ by $\mathcal{Z}d=\lim_{n\to\infty}\mathcal{Z}_n d$ for all $d\in C$, then $\lim_{n\to\infty}\sup\{\|\mathcal{Z}b-\mathcal{Z}_n u\| : b\in C\} = 0$.*

Lemma 2.12 ([21]): *Let C be a nonempty, closed, and convex subset of a Banach space X admitting a weakly continuous duality mapping. If $\mathcal{Z}:C\to C$ is an asymptotically nonexpansive mapping with nonempty fixed-point set, denoted by $\mathrm{Fix}(\mathcal{Z})$, then the operator $(\mathcal{I}-\mathcal{Z})$ is demiclosed at zero. In other words, if $\{d_n\}$ is a sequence in C such that d_n converges weakly to $d\in C$ and $(\mathcal{I}-\mathcal{Z})d_n$ converges strongly to 0, then $(\mathcal{I}-\mathcal{Z})d=0$.*

Lemma 2.13 ([22]): *Suppose $\{\Theta_k\}$ is a real sequence that does not decrease at infinity, meaning that there exists a subsequence $\{\Theta_{k_m}\}\subset\{\Theta_k\}$ satisfying $\Theta_{k_m}<\Theta_{k_m+1}$ for all $m\geq 1$. Define the integer sequence $\{\psi(k)\}_{k\geq k_0}$ as follows: $\psi(k)=\max\{m\leq k:\Theta_m<\Theta_{m+1}\}$, where $k_0\geq 1$ is an integer such that the set $\{m\leq k_0:\Theta_m<\Theta_{m+1}\}$ is nonempty. Then, the following properties hold:*

(i) $\psi(k_0)\leq\psi(k_0+1)\leq\cdots$ *and* $\psi(k)\to\infty$;
(ii) $\Theta_{\psi(k)}\leq\Theta_{\psi(k)+1}$ *and* $\Theta_k\leq\Theta_{\psi(k)+1}, \quad \forall k\geq k_0$.

3. Main results

In the rest of this paper, let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote two real Hilbert spaces. Consider a nonempty, closed, and convex feasible set $C \subset \mathcal{H}_1$. To address the MBSEP with GSVI and CFPP constraints, we make the following assumptions:

(A1) $\mathcal{Z}_0 : \mathcal{H}_1 \to C$ is an asymptotically nonexpansive mapping, with associated sequence $\{\theta_n\}$. $\{\mathcal{Z}_k\}_{k=1}^{\infty}$ is a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C. $\mathcal{Z}$ is a δ-demimetric self-mapping on $\mathcal{H}_2$ such that $\mathcal{I} - \mathcal{Z}$ is demiclosed at zero, where $\delta \in (-\infty, 1)$. $\mathcal{K} : \mathcal{H}_1 \to \mathcal{H}_2$ is a non-zero bounded linear operator with adjoint $\mathcal{K}^*$. Let $B_1, B_2 : \mathcal{H}_1 \to \mathcal{H}_1$ be α-inverse-strongly monotone and χ-inverse-strongly monotone mappings, respectively. Define $G : \mathcal{H}_1 \to C$ as $G = \mathbf{Pj}_C(\mathcal{I} - \mu_1 B_1)\mathbf{Pj}_C(\mathcal{I} - \mu_2 B_2)$, where $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\chi)$.

(A2) $\sum_{k=1}^{\infty} \sup_{p \in D} \|\mathcal{Z}_{k+1}p - \mathcal{Z}_k p\| < \infty$ for any bounded subset $D \subset C$. Define $\widetilde{S}$ by $\widetilde{S}u = \lim_{k\to\infty} \mathcal{Z}_k u$ for all $u \in C$, such that $\mathrm{Fix}(\widetilde{S}) = \bigcap_{k=1}^{\infty} \mathrm{Fix}(\mathcal{Z}_k)$.

(A3) $\Gamma : C \times C \to \mathbb{R} \cup \{+\infty\}$ and $\Phi : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{R} \cup \{+\infty\}$ are two bifunctions. We impose the following assumptions on Φ and Γ:

Ass$_\Phi$:

(Φ_1) The set $\Xi = \mathrm{Fix}(G) \cap \Omega \cap (\bigcap_{k=0}^{\infty} \mathrm{Fix}(\mathcal{Z}_k))$ is nonempty, where $\Omega := \{z \in \mathrm{Sol}(C, \Phi) : \mathcal{K}z \in \mathrm{Fix}(\mathcal{Z})\}$.

(Φ_2) The bifunction Φ is monotone and Lipschitz continuous with constants $c_1, c_2 > 0$, and Φ is weakly continuous in the sense that if $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$, then $\lim_{n\to\infty} \Phi(u_n, v_n) = \Phi(u, v)$.

Ass$_\Gamma$:

(Γ_1) The bifunction Γ is ν-strongly monotone and weakly continuous.

(Γ_2) For every $k \in \{1, \dots, m\}$, there exist mappings $\hat{\Gamma}_k, \widetilde{\gamma}_k : C \times C \to \mathcal{H}_1$ such that:

(i) $\hat{\Gamma}_k(b, d) + \hat{\Gamma}_k(d, b) = 0$ and $\|\hat{\Gamma}_k(b, d)\| \le \hat{\ell}_k \|b - d\|$ for all $b, d \in C$;

(ii) $\widetilde{\gamma}_k(b, b) = 0$ and $\|\widetilde{\gamma}_k(b, d) - \widetilde{\gamma}_k(m, p)\| \le \widetilde{\ell}_k \|(b - d) - (m - p)\|$ for all $b, d, m, p \in C$;

(iii) $\Gamma(b, d) + \Gamma(d, w) \ge \Gamma(b, w) + \sum_{k=1}^{m} \langle \hat{\Gamma}_k(b, d), \widetilde{\gamma}_k(d, w) \rangle$ for all $b, d, w \in C$.

(Γ_3) For any sequence $\{v_n\} \subset C$ such that $v_n \to v$, we have $\limsup_{n\to\infty} \dfrac{|\Gamma(v, v_n)|}{\|v_n - v\|} < +\infty$.

(A4) We select sequences $\{\zeta_n\}, \{\chi_n\}, \{\rho_n\}, \{\xi_n\} \subset (0, 1)$ and $\{\alpha_n\}, \{\sigma_n\} \subset (0, \infty)$ that satisfy the following conditions:

(H1) $\chi_n + \rho_n + \xi_n = 1$ for all $n \ge 1$, and $0 < \liminf_{n\to\infty} \chi_n$, $0 < \liminf_{n\to\infty} \xi_n$.

(H2) $0 < \liminf_{n\to\infty} \rho_n \le \limsup_{n\to\infty} \rho_n < 1$, and $0 < \liminf_{n\to\infty} \zeta_n \le \limsup_{n\to\infty} \zeta_n < 1$.

(H3) $\sum_{n=1}^{\infty} \sigma_n = \infty$, $\lim_{n\to\infty} \sigma_n = 0$, $\lim_{n\to\infty} \theta_n / \sigma_n = 0$, and $\sum_{n=1}^{\infty} \theta_n < \infty$.

(H4) $\{\alpha_n\} \subset (\underline{\alpha}, \overline{\alpha}) \subset (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$ and $\lim_{n\to\infty} \alpha_n = \widetilde{\alpha}$, where c_1 and c_2 are the Lipschitz constants of Φ.

(H5) The inequality $2\sigma_n\nu - \sigma_n^2\Upsilon^2 < 1$ holds, where $0 < \lambda < \min\{\nu, \Upsilon\}$, and $0 < \sigma_n < \min\{\frac{1}{\lambda}, \frac{2\nu-2\lambda}{\Upsilon^2-\lambda^2}, \frac{2\nu}{\Upsilon^2}\}$, with ν being the strongly monotone constant of Γ and $\Upsilon = \sum_{k=1}^{m} \hat{\ell}_k\widetilde{\ell}_k$.

The MBSEP with the GSVI and CFPP constraints is formulated as follows:

$$\text{Find} g^* \in \Xi = \text{Fix}(G) \cap \boldsymbol{\Omega} \cap \left(\bigcap_{k=0}^{\infty} \text{Fix}(\mathcal{Z}_k) \right), \quad \text{such that } x^* \in \text{Sol}(\Xi, \Gamma),$$

where $\boldsymbol{\Omega} := \{z \in \text{Sol}(C, \Phi) : \mathcal{K}z \in \text{Fix}(\mathcal{Z})\}$.

Now, we propose a novel subgradient extragradient implicit approach, as shown in Algorithm 1.

Algorithm 1

Initialization: Given $f_1 \in C$ and $\vartheta \geq 0$ arbitrarily. Let $\{\zeta_n\}, \{\chi_n\}, \{\rho_n\}, \{\xi_n\} \subset (0, 1)$, and $\{\alpha_n\}, \{\sigma_n\} \subset (0, \infty)$ be such that hypotheses (H1)–(H5) hold.

Iterative Steps: Calculate f_{n+1} as follows:

Step 1. Compute

$$\begin{cases} u_n = \zeta_n f_n + (1 - \zeta_n)\mathcal{Z}_n u_n, \\ r_n = \text{argmin} \left\{ \alpha_n \Phi(u_n, y) + \dfrac{1}{2}\|y - u_n\|^2 : y \in C \right\}. \end{cases} \tag{2}$$

Step 2. Choose $w_n \in \partial_2\Phi(u_n, r_n)$, and compute

$$\begin{cases} C_n = \{v \in \mathcal{H}_1 : \langle u_n - \alpha_n w_n - r_n, v - r_n\rangle \leq 0\}, \\ v_n = \text{argmin} \left\{ \alpha_n \Phi(r_n, z) + \dfrac{1}{2}\|z - u_n\|^2 : z \in C_n \right\}. \end{cases} \tag{3}$$

Step 3. Compute $t_n = v_n - \vartheta_n\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n$, where for any fixed $\epsilon > 0$, ϑ_n is chosen to be the bounded sequence satisfying

$$0 < \epsilon \leq \vartheta_n \leq \frac{(1 - \delta)\|(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2}{\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2} - \epsilon \quad \text{if } (\mathcal{I} - \mathcal{Z})\mathcal{K}v_n \neq 0. \tag{4}$$

Otherwise, set $\vartheta_n = \vartheta \geq 0$.

Step 4. Compute

$$\begin{cases} q_n = \mathbf{Pj}_C(d_n - \mu_2 B_2 d_n), \\ p_n = \mathbf{Pj}_C(q_n - \mu_1 B_1 q_n), \\ d_n = \chi_n f_n + \rho_n p_n + \xi_n \mathcal{Z}_0^n t_n. \end{cases}$$

Step 5. Compute $f_{n+1} = \text{argmin}\{\sigma_n\Gamma(d_n, t) + \frac{1}{2}\|t - d_n\|^2 : t \in C\}$.

Step 6. Set $n := n + 1$ and return to **Step 1**.

Remark 3.1: Suppose the bifunction Γ satisfies the condition $\text{Ass}_\Gamma(\Gamma_2)$. Then, for all $u, v, w \in C$:

$$\begin{aligned}\Gamma(u,v)+\Gamma(v,w) &\geq \Gamma(u,w)+\sum_{k=1}^{m}\langle \hat{\Gamma}_k(u,v),\widetilde{\gamma}_k(v,w)\rangle \\ &\geq \Gamma(u,w)-\sum_{k=1}^{m}\hat{\ell}_k\widetilde{\ell}_k\|u-v\|\|v-w\| \\ &\geq \Gamma(u,w)-\frac{1}{2}\Upsilon\|u-v\|^2-\frac{1}{2}\Upsilon\|v-w\|^2,\end{aligned}$$

where $\Upsilon=\sum_{k=1}^{m}\hat{\ell}_k\widetilde{\ell}_k$. Thus, Γ is Lipschitz continuous with constants $c_1=c_2=\frac{1}{2}\Upsilon$.

We are now in a position to state and prove the first main result of this paper.

Theorem 3.1: *Let the sequence* $\{f_n\}$ *be generated by Algorithm* 1*, and suppose that the conditions* (A1)-(A4) *hold. The sequence* $\{f_n\}$ *converges strongly to the unique solution* g^* *of the problem* $\text{EP}(\Xi,\Gamma)$*, provided that* $\mathcal{Z}_0^n f_n-\mathcal{Z}_0^{n+1}f_n\to 0$.

Proof: By Lemma 2.7, it follows that G is nonexpansive. Therefore, using Lemma 2.8 and Banach's contraction mapping principle, we deduce from the sequences $\{\zeta_n\},\{\rho_n\}\subset(0,1)$ that for each $n\geq 1$, the following hold:

(i) $\exists u_n\in C$ such that $u_n=\zeta_n f_n+(1-\zeta_n)\mathcal{Z}_n u_n$, and
(ii) $\exists d_n\in C$ such that $d_n=\chi_n f_n+\rho_n G d_n+\xi_n\mathcal{Z}_0^n t_n$.

We claim that the stepsize ϑ_n defined in (4) is well-defined. Indeed, it is sufficient to show that $\|\mathcal{K}^*(\mathcal{I}-\mathcal{Z})\mathcal{K}v_n\|^2\neq 0$. Take an arbitrary fixed point $p\in\Xi$. Since $\mathcal{Z}$ is a δ-demimetric mapping, we have

$$\begin{aligned}\|v_n-p\|\|\mathcal{K}^*(\mathcal{I}-\mathcal{Z})\mathcal{K}v_n\| &\geq \langle v_n-p,\mathcal{K}^*(\mathcal{I}-\mathcal{Z})\mathcal{K}v_n\rangle \\ &= \langle \mathcal{K}v_n-\mathcal{K}p,(\mathcal{I}-\mathcal{Z})\mathcal{K}v_n\rangle \geq \frac{1-\delta}{2}\|(\mathcal{I}-\mathcal{Z})\mathcal{K}v_n\|^2.\end{aligned}$$

When $(\mathcal{I}-\mathcal{Z})\mathcal{K}v_n\neq 0$, it follows that $\|(\mathcal{I}-\mathcal{Z})\mathcal{K}v_n\|^2>0$. As a result, we have $\|\mathcal{K}^*(\mathcal{I}-\mathcal{Z})\mathcal{K}v_n\|^2>0$. Since $\lim_{n\to\infty}\theta_n/\sigma_n=0$, we assume that $\theta_n\leq\frac{1}{2}\lambda\sigma_n$ for all $n\geq 1$. In what follows, we divide the remainder of the proof into few claims below.

Claim 1. We show that the following inequality holds:

$$\begin{aligned}\|t_n-p\|^2 &\leq \|u_n-p\|^2-\epsilon^2\|\mathcal{K}^*(\mathcal{I}-\mathcal{Z})\mathcal{K}v_n\|^2-(1-2\alpha_n c_1)\|r_n-u_n\|^2 \\ &\quad -(1-2\alpha_n c_2)\|v_n-r_n\|^2,\quad \forall n\geq 1.\end{aligned}$$

Indeed, by Lemma 2.9, we know that for r_n there exists $w_n \in \partial_2\Phi(u_n, r_n)$ such that $\alpha_n w_n + r_n - u_n \in -N_C(r_n)$. This leads to the inequality

$$\langle \alpha_n w_n + r_n - u_n, x - r_n \rangle \leq 0, \quad \forall x \in C.$$

From the definition of $w_n \in \partial_2\Phi(u_n, r_n)$, it follows that

$$\alpha_n[\Phi(u_n, x) - \Phi(u_n, r_n)] \geq \langle \alpha_n w_n, x - r_n \rangle, \quad \forall x \in \mathcal{H}_1. \tag{5}$$

Adding the last two inequalities, we get

$$\alpha_n[\Phi(u_n, x) - \Phi(u_n, r_n)] + \langle r_n - u_n, x - r_n \rangle \geq 0, \quad \forall x \in C. \tag{6}$$

It follows from $v_n \in C_n$ and the definition of C_n that $\langle u_n - \alpha_n w_n - r_n, v_n - r_n \rangle \leq 0$, and hence

$$\alpha_n \langle w_n, v_n - r_n \rangle \geq \langle u_n - r_n, v_n - r_n \rangle. \tag{7}$$

Substituting $x = v_n$ into (5), we obtain $\alpha_n[\Phi(u_n, v_n) - \Phi(u_n, r_n)] \geq \alpha_n \langle w_n, v_n - r_n \rangle$. Adding this inequality to (7), we conclude that

$$\alpha_n[\Phi(u_n, v_n) - \Phi(u_n, r_n)] \geq \langle u_n - r_n, v_n - r_n \rangle. \tag{8}$$

By Lemma 2.9, we know that for v_n there exist $h_n \in \partial_2\Phi(r_n, v_n)$ and $\varrho_n \in N_{C_n}(v_n)$ such that

$$\alpha_n h_n + v_n - u_n + \varrho_n = 0,$$

which leads to the inequality

$$\alpha_n \langle h_n, y - v_n \rangle \geq \langle u_n - v_n, y - v_n \rangle, \quad \forall y \in C_n,$$

and

$$\Phi(r_n, y) - \Phi(r_n, v_n) \geq \langle h_n, y - v_n \rangle, \quad \forall y \in \mathcal{H}_1.$$

Substituting $y = p \in C \subset C_n$ into the last two inequalities and adding them, we obtain

$$\alpha_n[\Phi(r_n, p) - \Phi(r_n, v_n)] \geq \langle u_n - v_n, p - v_n \rangle.$$

By the monotonicity of Φ, the fact that $p \in \mathrm{Sol}(C, \Phi)$, and that $r_n \in C$, we conclude that

$$\Phi(r_n, p) \leq -\Phi(p, r_n) \leq 0.$$

Thus,

$$-\alpha_n \Phi(r_n, v_n) \geq \langle u_n - v_n, p - v_n \rangle.$$

Note that the Lipschitz-type continuity of Φ implies

$$\Phi(u_n, r_n) + \Phi(r_n, v_n) \geq \Phi(u_n, v_n) - c_1\|u_n - r_n\|^2 - c_2\|r_n - v_n\|^2.$$

Therefore, it follows that

$$\langle u_n - v_n, v_n - p \rangle \geq \alpha_n \Phi(r_n, v_n)$$

$$\geq \alpha_n[\Phi(u_n, v_n) - \Phi(u_n, r_n)] - \alpha_n c_1\|u_n - r_n\|^2 - \alpha_n c_2\|r_n - v_n\|^2.$$

This, together with (8), yields

$$\langle u_n - v_n, v_n - p\rangle \geq \langle u_n - r_n, v_n - r_n\rangle - \alpha_n c_1\|u_n - r_n\|^2 - \alpha_n c_2\|r_n - v_n\|^2.$$

Accordingly, applying the equality

$$\langle \upsilon, u\rangle = \frac{1}{2}(\|\upsilon + u\|^2 - \|\upsilon\|^2 - \|u\|^2), \quad \forall \upsilon, u \in \mathcal{H}_1 \tag{9}$$

to the terms $\langle u_n - v_n, v_n - p\rangle$ and $\langle r_n - u_n, v_n - r_n\rangle$ in the last inequality, we obtain

$$\|v_n - p\|^2 \leq \|u_n - p\|^2 - (1 - \alpha_n c_1)\|r_n - u_n\|^2 - (1 - \alpha_n c_2)\|v_n - r_n\|^2, \quad \forall n \geq 1. \tag{10}$$

Furthermore, we have

$$\begin{aligned}\|t_n - p\|^2 &= \|v_n - \vartheta_n\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n - p\|^2\\ &= \|v_n - p\|^2 - 2\vartheta_n\langle v_n - p, \mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\rangle + \vartheta_n^2\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2\\ &= \|v_n - p\|^2 - 2\vartheta_n\langle \mathcal{K}(v_n - p), (\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\rangle + \vartheta_n^2\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2.\end{aligned}$$

Since the operator $\mathcal{Z}$ is δ-demimetric, it follows that

$$\begin{aligned}\|t_n - p\|^2 &\leq \|v_n - p\|^2 - \vartheta_n(1 - \delta)\|(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2 + \vartheta_n^2\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2\\ &= \|v_n - p\|^2 + \vartheta_n[\vartheta_n\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2 - (1 - \delta)\|(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2].\end{aligned} \tag{11}$$

From the stepsize ϑ_n in (4), we have $\vartheta_n + \epsilon \leq \frac{(1-\delta)\|(\mathcal{I}-\mathcal{Z})\mathcal{K}v_n\|^2}{\|\mathcal{K}^*(\mathcal{I}-\mathcal{Z})\mathcal{K}v_n\|^2}$ if and only if

$$\vartheta_n\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2 - (1 - \delta)\|(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2 \leq -\epsilon\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2.$$

This is equivalent to

$$\vartheta_n(\vartheta_n\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2 - (1 - \delta)\|(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2) \leq -\vartheta_n\epsilon\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2. \tag{12}$$

Using $0 < \epsilon \leq \vartheta_n$ from (4), we derive $-\epsilon^2 \geq -\vartheta_n\epsilon$, thus obtaining

$$-\vartheta_n\epsilon\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2 \leq -\epsilon^2\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2. \tag{13}$$

By combining (11), (12), and (13), we arrive at

$$\begin{aligned}\|t_n - p\|^2 &\leq \|v_n - p\|^2 - \vartheta_n\epsilon\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2\\ &\leq \|v_n - p\|^2 - \epsilon^2\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2.\end{aligned} \tag{14}$$

Therefore, substituting (10) into (14), we establish the desired claim.

Claim 2. We show that the following inequality holds for all $x \in C$:

$$\|f_{n+1} - x\|^2 \leq \|d_n - x\|^2 - \|f_{n+1} - d_n\|^2 + 2\sigma_n[\Gamma(d_n, x) - \Gamma(d_n, f_{n+1})].$$

To prove this, we note that f_{n+1} minimizes the function $\sigma_n \Gamma(d_n, x) + \frac{1}{2}\|x - d_n\|^2$ over the set C. Therefore, there exists $m_n \in \partial_2 \Gamma(d_n, f_{n+1})$, the subdifferential of Γ with respect to its second argument, such that

$$0 \in \sigma_n m_n + f_{n+1} - d_n + N_C(f_{n+1}),$$

where $N_C(f_{n+1})$ denotes the normal cone to C at f_{n+1}. By using the definitions of the normal cone and the subgradient, we derive the following inequalities for all $x \in C$:

$$\langle \sigma_n m_n + f_{n+1} - d_n, x - f_{n+1} \rangle \geq 0,$$

and

$$\sigma_n[\Gamma(d_n, x) - \Gamma(d_n, f_{n+1})] \geq \langle \sigma_n m_n, x - f_{n+1} \rangle.$$

Adding these two inequalities, we obtain

$$2\sigma_n[\Gamma(d_n, x) - \Gamma(d_n, f_{n+1})] + 2\langle f_{n+1} - d_n, x - f_{n+1} \rangle \geq 0, \quad \forall x \in C. \tag{15}$$

Next, we substitute $\upsilon = f_{n+1} - d_n$ and $u = x - f_{n+1}$ into (9), leading to:

$$2\sigma_n[\Gamma(d_n, x) - \Gamma(d_n, f_{n+1})] + \|d_n - x\|^2 - \|f_{n+1} - d_n\|^2 - \|f_{n+1} - x\|^2 \geq 0, \quad \forall x \in C.$$

This inequality directly establishes the desired claim.

Claim 3. We demonstrate that if g^* is a solution to the MBSEP with the GSVI and CFPP constraints, then

$$\|f_{n+1} - g_n^*\| \leq \eta_n \|d_n - g^*\|^2 \leq (1 - \lambda\sigma_n)\|d_n - g^*\|,$$

where $g_n^* = \operatorname{argmin}\{\sigma_n \Gamma(g^*, \nu) + \frac{1}{2}\|\nu - g^*\|^2 : \nu \in C\}$, $\eta_n = \sqrt{1 - 2\sigma_n \nu + \sigma_n^2 \Upsilon^2}$, $0 < \lambda < \min\{\nu, \Upsilon\}$, $0 < \sigma_n < \min\{\frac{1}{\lambda}, \frac{2\nu - 2\lambda}{\Upsilon^2 - \lambda^2}\}$, and $\Upsilon = \sum_{k=1}^{m} \hat{\ell}_k \widetilde{\ell}_k$. By applying similar reasoning to that utilized in (15), we obtain

$$\sigma_n[\Gamma(g^*, x) - \Gamma(g^*, g_n^*)] + \langle g_n^* - g^*, x - g_n^* \rangle \geq 0, \quad \forall x \in C. \tag{16}$$

By substituting $x = g_n^* \in C$ into (15) and $x = f_{n+1} \in C$ into (16), we derive the inequalities

$$\sigma_n[\Gamma(d_n, g_n^*) - \Gamma(d_n, f_{n+1})] + \langle f_{n+1} - d_n, g_n^* - f_{n+1} \rangle \geq 0,$$

and

$$\sigma_n[\Gamma(g^*, f_{n+1}) - \Gamma(g^*, g_n^*)] + \langle g_n^* - g^*, f_{n+1} - g_n^* \rangle \geq 0.$$

Adding these two inequalities yields

$$\begin{aligned} 0 &\leq 2\sigma_n[\Gamma(d_n, g_n^*) - \Gamma(d_n, f_{n+1}) + \Gamma(g^*, f_{n+1}) - \Gamma(g^*, g_n^*)] \\ &\quad + 2\langle f_{n+1} - d_n - g_n^* + g^*, g_n^* - f_{n+1} \rangle \\ &= 2\sigma_n[\Gamma(d_n, g_n^*) - \Gamma(d_n, f_{n+1}) + \Gamma(g^*, f_{n+1}) - \Gamma(g^*, g_n^*)] + \|d_n - g^*\|^2 \end{aligned}$$

$$- \|f_{n+1} - d_n - g_n^* + g^*\|^2 - \|f_{n+1} - g_n^*\|^2, \tag{17}$$

where the last equality follows directly from (9). Utilizing the assumption $\mathrm{Ass}_\Gamma(\Gamma_2)$, we derive the following inequalities:

$$\Gamma(d_n, g_n^*) - \Gamma(g^*, g_n^*) \leq \Gamma(d_n, g^*) - \sum_{k=1}^{m} \langle \hat{\Gamma}_k(d_n, g^*), \widetilde{\gamma}_k(g^*, g_n^*)\rangle,$$

and

$$\Gamma(g^*, f_{n+1}) - \Gamma(d_n, f_{n+1}) \leq \Gamma(g^*, d_n) - \sum_{k=1}^{m} \langle \hat{\Gamma}_k(g^*, d_n), \widetilde{\gamma}_k(d_n, f_{n+1})\rangle.$$

Thus, we can conclude that

$$\begin{aligned}
&\Gamma(d_n, g_n^*) - \Gamma(d_n, f_{n+1}) + \Gamma(g^*, f_{n+1}) - \Gamma(g^*, g_n^*) \\
&\leq \Gamma(d_n, g^*) + \Gamma(g^*, d_n) - \sum_{k=1}^{m} \langle \hat{\Gamma}_k(d_n, g^*), \widetilde{\gamma}_k(g^*, g_n^*)\rangle \\
&\quad - \sum_{k=1}^{m} \langle \hat{\Gamma}_k(g^*, d_n), \widetilde{\gamma}_k(d_n, f_{n+1})\rangle.
\end{aligned}$$

Using Assumptionz $\mathrm{Ass}_\Gamma(\Gamma_1)$ and $\mathrm{Ass}_\Gamma(\Gamma_2)$, which states that $\Gamma(v, u) + \Gamma(u, v) \leq -\nu\|v - u\|^2$ for all $v, u \in C$, we derive the following:

$$\begin{aligned}
&\Gamma(d_n, g_n^*) - \Gamma(d_n, f_{n+1}) + \Gamma(g^*, f_{n+1}) - \Gamma(g^*, g_n^*) \\
&\leq -\nu\|d_n - g^*\|^2 + \sum_{k=1}^{m} \langle \hat{\Gamma}_k(d_n, g^*), \widetilde{\gamma}_k(d_n, f_{n+1}) - \widetilde{\gamma}_k(g^*, g_n^*)\rangle \\
&\leq -\nu\|d_n - g^*\|^2 + \sum_{k=1}^{m} \hat{\ell}_k \widetilde{\ell}_k \|d_n - g^*\| \|(d_n - f_{n+1}) - (g^* - g_n^*)\| \\
&= -\nu\|d_n - g^*\|^2 + \Upsilon\|d_n - g^*\| \|d_n - f_{n+1} - g^* + g_n^*\|.
\end{aligned} \tag{18}$$

Combining (17) and (18), we obtain:

$$\begin{aligned}
0 &\leq (1 - 2\sigma_n\nu)\|d_n - g^*\|^2 + 2\sigma_n\Upsilon\|d_n - g^*\| \|d_n - f_{n+1} - g^* + g_n^*\| \\
&\quad - \|f_{n+1} - d_n - g_n^* + g^*\|^2 - \|f_{n+1} - g_n^*\|^2 \\
&= (1 - 2\sigma_n\nu)\|d_n - g^*\|^2 - \left(\|f_{n+1} - d_n - g_n^* + g^*\| - \sigma_n\Upsilon\|d_n - g^*\|\right)^2 \\
&\quad + \sigma_n^2\Upsilon^2\|d_n - g^*\|^2 - \|f_{n+1} - g_n^*\|^2 \\
&\leq (1 - 2\sigma_n\nu + \sigma_n^2\Upsilon^2)\|d_n - g^*\|^2 - \|f_{n+1} - g_n^*\|^2.
\end{aligned}$$

From the range of λ and σ_n, we have $0 \leq \eta_n < 1 - \lambda\sigma_n$. This establishes the desired claim.

Let us further review how we obtain $\eta_n = \sqrt{1 - 2\sigma_n\nu + \sigma_n^2\Upsilon^2}$. By combining inequalities (17) and (18), we get the following

$$0 \leq (1 - 2\sigma_n\nu)\|d_n - g^*\|^2 + 2\sigma_n\Upsilon\|d_n - g^*\| \|d_n - f_{n+1} - g^* + g_n^*\|$$

$$
\begin{aligned}
&- \|f_{n+1} - d_n - g_n^* + g^*\|^2 - \|f_{n+1} - g_n^*\|^2 \\
&\leq \cdots \\
&\leq \left(1 - 2\sigma_n\nu + \sigma_n^2\Upsilon^2\right) \|d_n - g^*\|^2 - \|f_{n+1} - g_n^*\|^2.
\end{aligned}
$$

This leads to the following estimate

$$
\|f_{n+1} - g_n^*\| \leq \eta_n \|d_n - g^*\|,
$$

where $\eta_n := \sqrt{1 - 2\sigma_n\nu + \sigma_n^2\Upsilon^2}$. Given that $0 < \lambda < \min\{\nu, \Upsilon\}$ and $0 < \sigma_n < \min\{\frac{1}{\lambda}, \frac{2\nu-2\lambda}{\Upsilon^2-\lambda^2}\}$, we obtain

$$
0 \leq \eta_n = \sqrt{1 - 2\sigma_n\nu + \sigma_n^2\Upsilon^2} < 1 - \lambda\sigma_n.
$$

Claim 4. We demonstrate that the sequence $\{f_n\}$ is bounded. Specifically, we define

$$
\begin{aligned}
&X := C, \quad Y := [0,1], \quad \mathcal{G} := C, \quad s := \sigma_n, \quad \forall x \in Y, \\
&W(z,s) := -s\Gamma(g^*, z) - \frac{1}{2}\|z - g^*\|^2, \quad \forall (z,s) \in X \times Y.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
M(\sigma_n) &= \operatorname{argmax}\{W(z,\sigma_n) : z \in C\} \\
&= \operatorname{argmin}\{\sigma_n\Gamma(g^*, z) + \frac{1}{2}\|z - g^*\|^2 : z \in C\} = \{g_n^*\}.
\end{aligned}
$$

It is important to note that M is continuous and that $\lim_{n\to\infty} g_n^* = g^*$. Given the continuity of Γ on C, we have $\lim_{n\to\infty} \Gamma(g^*, g_n^*) = \Gamma(g^*, g^*) = 0$. In accordance with $\mathrm{Ass}_\Gamma(\Gamma_3)$, there exists a constant $\widehat{M}(g^*) > 0$ such that

$$
|\Gamma(g^*, g_n^*)| \leq \widehat{M}(g^*)\|g_n^* - g^*\|, \quad \forall n \geq 1.
$$

Substituting $x = g^*$ into (16) and utilizing $\Gamma(g^*, g^*) = 0$, we derive

$$
-\sigma_n\Gamma(g^*, g_n^*) + \langle g_n^* - g^*, g^* - g_n^* \rangle \geq 0,
$$

which implies

$$
\|g_n^* - g^*\|^2 \leq -\sigma_n\Gamma(g^*, g_n^*) \leq \sigma_n\widehat{M}(g^*)\|g_n^* - g^*\|, \quad \forall n \geq 1.
$$

This result assures that

$$
\|g_n^* - g^*\| \leq \sigma_n\widehat{M}(g^*), \quad \forall n \geq 1.
$$

Furthermore, invoking Lemma 2.6, it can be established that $\mathcal{I} - \mu_1 B_1$ and $\mathcal{I} - \mu_2 B_2$ are nonexpansive mappings for $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\chi)$. We denote $y^* = \mathbf{Pj}_C(\mathcal{I} - \mu_2 B_2)g^*$. Consequently, by applying Lemma 2.10, we obtain $g^* = \mathbf{Pj}_C(\mathcal{I} - \mu_1 B_1)y^* = Gg^*$. Given that each mapping $\mathcal{Z}_n : C \to C$ is a pseudocontraction, we derive

$$
\|u_n - g^*\|^2 = \zeta_n\langle f_n - g^*, u_n - g^* \rangle + (1 - \zeta_n)\langle \mathcal{Z}_n u_n - g^*, u_n - g^* \rangle
$$

$$\leq \zeta_n \|f_n - g^*\| \|u_n - g^*\| + (1 - \zeta_n)\|u_n - g^*\|^2,$$

which leads to

$$\|u_n - g^*\| \leq \|f_n - g^*\|, \quad \forall n \geq 1. \tag{19}$$

It follows from (10), (14), and (19) that

$$\|t_n - g^*\| \leq \|v_n - g^*\| \leq \|u_n - g^*\| \leq \|f_n - g^*\|, \quad \forall n \geq 1. \tag{20}$$

Considering the nonexpansivity of G and the asymptotic nonexpansivity of $\mathcal{Z}_0$, we deduce from (20) that

$$\begin{aligned}
&\|d_n - g^*\|^2 \\
&\quad = \chi_n \langle f_n - g^*, d_n - g^* \rangle + \rho_n \langle G d_n - g^*, d_n - g^* \rangle + \xi_n \langle \mathcal{Z}_0^n t_n - g^*, d_n - g^* \rangle \\
&\quad \leq \chi_n \|f_n - g^*\| \|d_n - g^*\| + \rho_n \|d_n - g^*\|^2 + \xi_n (1 + \theta_n) \|t_n - g^*\| \|d_n - g^*\| \\
&\quad \leq \chi_n (1 + \theta_n) \|f_n - g^*\| \|d_n - g^*\| + \rho_n \|d_n - g^*\|^2 + \xi_n (1 + \theta_n) \|f_n - g^*\| \|d_n - g^*\| \\
&\quad = (1 - \rho_n)(1 + \theta_n) \|f_n - g^*\| \|d_n - g^*\| + \rho_n \|d_n - g^*\|^2,
\end{aligned}$$

which immediately leads to $\|d_n - g^*\| \leq (1 + \theta_n)\|f_n - g^*\|$, $\forall n \geq 1$. Therefore,

$$\begin{aligned}
&\|f_{n+1} - g^*\| \\
&\quad \leq \|f_{n+1} - g_n^*\| + \|g_n^* - g^*\| \leq (1 - \lambda\sigma_n)\|d_n - g^*\| + \|g_n^* - g^*\| \\
&\quad \leq (1 - \lambda\sigma_n)(1 + \theta_n)\|f_n - g^*\| + \sigma_n \widehat{M}(g^*) \\
&\quad \leq [1 - \lambda\sigma_n + \theta_n]\|f_n - g^*\| + \sigma_n \widehat{M}(g^*) \\
&\quad \leq [1 - \lambda\sigma_n + \frac{1}{2}\lambda\sigma_n]\|f_n - g^*\| + \sigma_n \widehat{M}(g^*) \\
&\quad \leq \max\left\{\|f_n - g^*\|, \frac{2\widehat{M}(g^*)}{\lambda}\right\}.
\end{aligned}$$

By induction, we conclude that $\|f_n - g^*\| \leq \max\{\|f_1 - g^*\|, \frac{2\widehat{M}(g^*)}{\lambda}\}, \forall n \geq 1$. Consequently, the sequence $\{f_n\}$ is bounded, and similarly, the sequences $\{p_n\}$, $\{q_n\}$, $\{r_n\}$, $\{d_n\}$, $\{u_n\}$, $\{v_n\}$, and $\{t_n\}$ are also bounded.

Claim 5. We demonstrate that if $f_n - u_n \to 0$ and $u_n - r_n \to 0$, then $\omega_w(f_n) \subset \mathrm{Sol}(C, \Phi)$. To illustrate this, let us consider an arbitrary fixed element $\widetilde{z} \in \omega_w(f_n)$. Then, there exists a subsequence $\{f_{n_k}\} \subset \{f_n\}$ such that $f_{n_k} \rightharpoonup \widetilde{z}$. Given that $f_n - u_n \to 0$ and $u_n - r_n \to 0$, we obtain

$$\|f_{n_k} - r_{n_k}\| \leq \|f_{n_k} - u_{n_k}\| + \|u_{n_k} - r_{n_k}\| \to 0, \quad (k \to \infty).$$

Thus, it follows from $f_{n_k} \rightharpoonup \widetilde{z}$ that $u_{n_k} \rightharpoonup \widetilde{z}$ and $r_{n_k} \rightharpoonup \widetilde{z}$. Since $\{r_n\} \subset C$, and $r_{n_k} \rightharpoonup \widetilde{z}$, with C being weakly closed, we conclude that $\widetilde{z} \in C$. Employing (6), we have

$$\alpha_{n_k} \Phi(u_{n_k}, x) \geq \alpha_{n_k} \Phi(u_{n_k}, r_{n_k}) + \langle r_{n_k} - u_{n_k}, r_{n_k} - x \rangle, \quad \forall x \in C.$$

Taking the limit as $k \to \infty$ and using the conditions that $\lim_{n\to\infty} \alpha_n = \widetilde{\alpha} > 0$, $\Phi(\widetilde{z}, \widetilde{z}) = 0$, the boundedness of $\{r_{n_k}\}$, and the weak continuity of Φ, we deduce that $\widetilde{\alpha}\Phi(\widetilde{z}, x) \geq 0, \quad \forall x \in C$. This demonstrates that $\widetilde{z} \in \mathrm{Sol}(C, \Phi)$.

Claim 6. We demonstrate that $f_n \to g^*$, a unique solution of the MBSEP with GSVI and CFPP constraints. To begin, we define $\Re_n = \|f_n - g^*\|^2$. By acknowledging the nonexpansivity of the operator G and the asymptotic nonexpansivity of $\mathcal{Z}_0$, we derive the following inequality:

$$\begin{aligned}
\|d_n - g^*\|^2 &= \chi_n\langle f_n - g^*, d_n - g^*\rangle + \rho_n\langle Gd_n - g^*, d_n - g^*\rangle + \xi_n\langle \mathcal{Z}_0^n t_n - g^*, d_n - g^*\rangle \\
&\le \frac{\chi_n}{2}\left[\|f_n - g^*\|^2 + \|d_n - g^*\|^2 - \|f_n - d_n\|^2\right] + \rho_n\|d_n - g^*\|^2 \\
&\quad + \frac{\xi_n}{2}\left[\|\mathcal{Z}_0^n t_n - g^*\|^2 + \|d_n - g^*\|^2 - \|\mathcal{Z}_0^n t_n - d_n\|^2\right] \\
&= \frac{\chi_n}{2}\|f_n - g^*\|^2 + \frac{1+\rho_n}{2}\|d_n - g^*\|^2 + \frac{\xi_n}{2}\|\mathcal{Z}_0^n t_n - g^*\|^2 \\
&\quad - \frac{\chi_n}{2}\|f_n - d_n\|^2 - \frac{\xi_n}{2}\|\mathcal{Z}_0^n t_n - d_n\|^2 \\
&\le \frac{\chi_n}{2}\|f_n - g^*\|^2 + \frac{1+\rho_n}{2}\|d_n - g^*\|^2 + \frac{\xi_n(1+\theta_n)^2}{2}\|t_n - g^*\|^2 \\
&\quad - \frac{\chi_n}{2}\|f_n - d_n\|^2 - \frac{\xi_n}{2}\|\mathcal{Z}_0^n t_n - d_n\|^2.
\end{aligned}$$

That is

$$\begin{aligned}
\|d_n - g^*\|^2 \le \frac{\chi_n}{2}\|f_n - g^*\|^2 + \frac{1+\rho_n}{2}\|d_n - g^*\|^2 + \frac{\xi_n}{2}\|t_n - g^*\|^2 + \frac{\theta_n\widetilde{M}}{2} \\
- \frac{\chi_n}{2}\|f_n - d_n\|^2 - \frac{\xi_n}{2}\|\mathcal{Z}_0^n t_n - d_n\|^2,
\end{aligned}$$

where $\sup_{n\ge1}(2+\theta_n)\|f_n - g^*\|^2 \le \widetilde{M}$ for some $\widetilde{M} > 0$. This leads to the conclusion that

$$\begin{aligned}
\|d_n - g^*\|^2 \le \frac{1}{1-\rho_n}\Big[\chi_n\|f_n - g^*\|^2 + \xi_n\|t_n - g^*\|^2 \\
+\theta_n\widetilde{M} - \chi_n\|f_n - d_n\|^2 - \xi_n\|\mathcal{Z}_0^n t_n - d_n\|^2\Big]. \qquad (21)
\end{aligned}$$

By the results presented in Claims 1 and 2, we can deduce from Equations (20) and (21) that

$$\begin{aligned}
&\|f_{n+1} - g^*\|^2 \le \|d_n - g^*\|^2 - \|f_{n+1} - d_n\|^2 + 2\sigma_n[\Gamma(d_n, g^*) - \Gamma(d_n, f_{n+1})] \\
&\quad \le \frac{1}{1-\rho_n}[\chi_n\|f_n - g^*\|^2 + \xi_n\|t_n - g^*\|^2 + \theta_n\widetilde{M} - \chi_n\|f_n - d_n\|^2 - \xi_n\|\mathcal{Z}_0^n t_n - d_n\|^2] \\
&\quad - \|f_{n+1} - d_n\|^2 + 2\sigma_n[\Gamma(d_n, g^*) - \Gamma(d_n, f_{n+1})] \\
&\le \frac{1}{1-\rho_n}\{\chi_n\|f_n - g^*\|^2 + \xi_n[\|u_n - g^*\|^2 - \epsilon^2\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2 - (1-2\alpha_n c_1) \\
&\quad \times \|r_n - u_n\|^2 - (1-2\alpha_n c_2)\|v_n - r_n\|^2] + \theta_n\widetilde{M} - \chi_n\|f_n - d_n\|^2 \\
&\quad - \xi_n\|\mathcal{Z}_0^n t_n - d_n\|^2\} - \|f_{n+1} - d_n\|^2 + 2\sigma_n[\Gamma(d_n, g^*) - \Gamma(d_n, f_{n+1})].
\end{aligned}$$

Hence,

$$\|f_{n+1} - g^*\|^2 \le \|f_n - g^*\|^2 - \frac{\xi_n}{1-\rho_n}[\epsilon^2\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2 + (1-2\alpha_n c_1)\|r_n - u_n\|^2$$

$$+ (1 - 2\alpha_n c_2)\|v_n - r_n\|^2] + \frac{\theta_n \widetilde{M}}{1-\rho_n} - \frac{1}{1-\rho_n}[\chi_n \|f_n - d_n\|^2 + \xi_n \|\mathcal{Z}_0^n t_n - d_n\|^2]$$
$$- \|f_{n+1} - d_n\|^2 + \sigma_n K, \tag{22}$$

where $\sup_{n\geq 1}\{2|\Gamma(d_n, g^*) - \Gamma(d_n, f_{n+1})|\} \leq K$ for some $K > 0$. In the following, we demonstrate the convergence of the sequence $\{\Re_n\}$ to zero from two distinct perspectives.

Aspect 1. Let us assume that there exists an integer $n_0 \geq 1$ such that the sequence $\{\Re_n\}$ is non-increasing. Consequently, we have $\lim_{n\to\infty} \Re_n = \hbar < +\infty$ and $\lim_{n\to\infty}(\Re_n - \Re_{n+1}) = 0$. From Equation (22), one can derive the following inequality:

$$\begin{aligned}
&\xi_n \left[\epsilon^2 \|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2 + (1 - 2\alpha_n c_1)\|r_n - u_n\|^2 + (1 - 2\alpha_n c_2)\|v_n - r_n\|^2\right] \\
&\quad + \chi_n \|f_n - d_n\|^2 + \xi_n \|\mathcal{Z}_0^n t_n - d_n\|^2 + \|f_{n+1} - d_n\|^2 \\
&\quad \leq \frac{\xi_n}{1-\rho_n}\left[\epsilon^2 \|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2 + (1 - 2\alpha_n c_1)\|r_n - u_n\|^2 + (1 - 2\alpha_n c_2)\|v_n - r_n\|^2\right] \\
&\quad + \frac{1}{1-\rho_n}\left[\chi_n \|f_n - d_n\|^2 + \xi_n \|\mathcal{Z}_0^n t_n - d_n\|^2\right] + \|f_{n+1} - d_n\|^2 \\
&\quad \leq \Re_n - \Re_{n+1} + \frac{\theta_n \widetilde{M}}{1-\rho_n} + \sigma_n K.
\end{aligned}$$

Since $\sigma_n \to 0$, $\theta_n \to 0$, $\Re_n - \Re_{n+1} \to 0$, $0 < \liminf_{n\to\infty} \chi_n$, $0 < \liminf_{n\to\infty} \xi_n$, $0 < \liminf_{n\to\infty}(1-\rho_n)$, $0 < \epsilon$, and $\{\alpha_n\} \subset (\underline{\alpha}, \overline{\alpha}) \subset (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$, we have

$$\lim_{n\to\infty} \|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\| = \lim_{n\to\infty} \|f_n - d_n\| = \lim_{n\to\infty} \|\mathcal{Z}_0^n t_n - d_n\| = 0, \tag{23}$$
$$\lim_{n\to\infty} \|r_n - u_n\| = \lim_{n\to\infty} \|v_n - r_n\| = \lim_{n\to\infty} \|f_{n+1} - d_n\| = 0. \tag{24}$$

Next, we demonstrate that $\|d_n - p_n\| \to 0$ as $n \to \infty$. To this end, we define $y^* = \mathbf{Pj}_C(g^* - \mu_2 B_2 g^*)$. It follows from the definitions of q_n and p_n that $p_n = Gd_n$. By applying Lemma 2.6, we have the following inequality:

$$\|q_n - y^*\|^2 \leq \|d_n - g^*\|^2 - \mu_2(2\chi - \mu_2)\|B_2 d_n - B_2 g^*\|^2, \tag{25}$$
$$\|p_n - g^*\|^2 \leq \|q_n - y^*\|^2 - \mu_1(2\alpha - \mu_1)\|B_1 q_n - B_1 y^*\|^2. \tag{26}$$

Substituting (25) into (26), and utilizing (20) and (21), we obtain

$$\begin{aligned}
\|p_n - g^*\|^2 &\leq \|d_n - g^*\|^2 - \mu_2(2\chi - \mu_2)\|B_2 d_n - B_2 g^*\|^2 \\
&\quad - \mu_1(2\alpha - \mu_1)\|B_1 q_n - B_1 y^*\|^2 \\
&\leq \|f_n - g^*\|^2 + \frac{\theta_n \widetilde{M}}{1-\rho_n} - \mu_2(2\chi - \mu_2)\|B_2 d_n - B_2 g^*\|^2 \\
&\quad - \mu_1(2\alpha - \mu_1)\|B_1 q_n - B_1 y^*\|^2.
\end{aligned} \tag{27}$$

Furthermore, by substituting (27) into (22) and referencing (20), we derive

$$\|f_{n+1} - g^*\|^2 \leq \|d_n - g^*\|^2 + \sigma_n K$$

$$
\begin{aligned}
&\leq \chi_n\|f_n - g^*\|^2 + \rho_n\|p_n - g^*\|^2 + \xi_n\|\mathcal{Z}_0^n t_n - g^*\|^2 + \sigma_n K \\
&\leq \chi_n(1+\theta_n)^2\|f_n - g^*\|^2 + \rho_n\|p_n - g^*\|^2 + \xi_n(1+\theta_n)^2\|t_n - g^*\|^2 + \sigma_n K \\
&\leq (1-\rho_n)[1+\theta_n(2+\theta_n)]\|f_n - g^*\|^2 + \rho_n[\|f_n - g^*\|^2 + \frac{\theta_n\widetilde{M}}{1-\rho_n} \\
&\quad - \mu_2(2\chi - \mu_2)\|B_2 d_n - B_2 g^*\|^2 - \mu_1(2\alpha - \mu_1)\|B_1 q_n - B_1 y^*\|^2] + \sigma_n K \\
&\leq \|f_n - g^*\|^2 + \theta_n\widetilde{M} + \frac{\rho_n\theta_n\widetilde{M}}{1-\rho_n} - \rho_n[\mu_2(2\chi - \mu_2)\|B_2 d_n - B_2 g^*\|^2 \\
&\quad + \mu_1(2\alpha - \mu_1)\|B_1 q_n - B_1 y^*\|^2] + \sigma_n K,
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|f_{n+1} - g^*\|^2 \leq \|f_n - g^*\|^2 + \frac{\theta_n\widetilde{M}}{1-\rho_n} - \rho_n[\mu_2(2\chi - \mu_2)\|B_2 d_n - B_2 g^*\|^2 \\
+ \mu_1(2\alpha - \mu_1)\|B_1 q_n - B_1 y^*\|^2] + \sigma_n K,
\end{aligned}
$$

which consequently yields

$$
\begin{aligned}
&\rho_n[\mu_2(2\chi - \mu_2)\|B_2 d_n - B_2 g^*\|^2 + \mu_1(2\alpha - \mu_1)\|B_1 q_n - B_1 y^*\|^2] \\
&\quad \leq \Re_n - \Re_{n+1} + \frac{\theta_n\widetilde{M}}{1-\rho_n} + \sigma_n K.
\end{aligned}
$$

Given that $\sigma_n \to 0, \theta_n \to 0, \Re_n - \Re_{n+1} \to 0,\ \liminf_{n\to\infty} \rho_n > 0$ and $\liminf_{n\to\infty}(1 - \rho_n) > 0$, we deduce from $\mu_2 \in (0, 2\chi)$ and $\mu_1 \in (0, 2\alpha)$ that

$$
\lim_{n\to\infty} \|B_2 d_n - B_2 g^*\| = \lim_{n\to\infty} \|B_1 q_n - B_1 y^*\| = 0. \tag{28}
$$

On the other hand, it can be observed that

$$
\begin{aligned}
\|p_n - g^*\|^2 &\leq \langle q_n - y^*, p_n - g^*\rangle + \mu_1\langle B_1 y^* - B_1 q_n, p_n - g^*\rangle \\
&\leq \frac{1}{2}[\|q_n - y^*\|^2 + \|p_n - g^*\|^2 - \|q_n - p_n + g^* - y^*\|^2] \\
&\quad + \mu_1\|B_1 y^* - B_1 q_n\|\|p_n - g^*\|.
\end{aligned}
$$

This leads to the conclusion that

$$
\|p_n - g^*\|^2 \leq \|q_n - y^*\|^2 - \|q_n - p_n + g^* - y^*\|^2 + 2\mu_1\|B_1 y^* - B_1 q_n\|\|p_n - g^*\|. \tag{29}
$$

In a similar manner, we derive that

$$
\|q_n - y^*\|^2 \leq \|d_n - g^*\|^2 - \|d_n - q_n + y^* - g^*\|^2 + 2\mu_2\|B_2 g^* - B_2 d_n\|\|q_n - y^*\|. \tag{30}
$$

By combining (29) and (30), we can infer from (20) and (21) that

$$
\|p_n - g^*\|^2 \leq \|d_n - g^*\|^2 - \|d_n - q_n + y^* - g^*\|^2 - \|q_n - p_n + g^* - y^*\|^2
$$

$$
\begin{aligned}
&+2\mu_1\|B_1y^*-B_1q_n\|\|p_n-g^*\|+2\mu_2\|B_2g^*-B_2d_n\|\|q_n-y^*\| \\
&\le \|f_n-g^*\|^2+\frac{\theta_n\widetilde{M}}{1-\rho_n}-\|d_n-q_n+y^*-g^*\|^2-\|q_n-p_n+g^*-y^*\|^2 \\
&\quad +2\mu_1\|B_1y^*-B_1q_n\|\|p_n-g^*\|+2\mu_2\|B_2g^*-B_2d_n\|\|q_n-y^*\|. \qquad (31)
\end{aligned}
$$

Substituting (31) into (22), we find from (20) that

$$
\begin{aligned}
\|f_{n+1}-g^*\|^2 &\le \|d_n-g^*\|^2+\sigma_nK \\
&\le \chi_n(1+\theta_n)^2\|f_n-g^*\|^2+\rho_n\|p_n-g^*\|^2+\xi_n(1+\theta_n)^2\|t_n-g^*\|^2+\sigma_nK \\
&\le (1-\rho_n)[1+\theta_n(2+\theta_n)]\|f_n-g^*\|^2+\rho_n[\|f_n-g^*\|^2+\frac{\theta_n\widetilde{M}}{1-\rho_n} \\
&\quad -\|d_n-q_n+y^*-g^*\|^2-\|q_n-p_n+g^*-y^*\|^2+2\mu_1\|B_1y^*-B_1q_n\| \\
&\quad \times\|p_n-g^*\|+2\mu_2\|B_2g^*-B_2d_n\|\|q_n-y^*\|]+\sigma_nK \\
&\le \|f_n-g^*\|^2+\theta_n\widetilde{M}+\frac{\rho_n\theta_n\widetilde{M}}{1-\rho_n}-\rho_n[\|d_n-q_n+y^*-g^*\|^2 \\
&\quad +\|q_n-p_n+g^*-y^*\|^2]+2\mu_1\|B_1y^*-B_1q_n\|\|p_n-g^*\| \\
&\quad +2\mu_2\|B_2g^*-B_2d_n\|\|q_n-y^*\|]+\sigma_nK \\
&= \|f_n-g^*\|^2+\frac{\theta_n\widetilde{M}}{1-\rho_n}-\rho_n[\|d_n-q_n+y^*-g^*\|^2+\|q_n-p_n+g^*-y^*\|^2] \\
&\quad +2\mu_1\|B_1y^*-B_1q_n\|\|p_n-g^*\|+2\mu_2\|B_2g^*-B_2d_n\|\|q_n-y^*\|+\sigma_nK.
\end{aligned}
$$

This thus results in

$$
\begin{aligned}
\rho_n[\|d_n-q_n+y^*-g^*\|^2+\|q_n-p_n+g^*-y^*\|^2] &\le \mathfrak{R}_n-\mathfrak{R}_{n+1}+\frac{\theta_n\widetilde{M}}{1-\rho_n} \\
&\quad +2\mu_1\|B_1y^*-B_1q_n\|\|p_n-g^*\|+2\mu_2\|B_2g^*-B_2d_n\|\|q_n-y^*\|+\sigma_nK.
\end{aligned}
$$

We can conclude from (28) that

$$
\lim_{n\to\infty}\|d_n-q_n+y^*-g^*\|=\lim_{n\to\infty}\|q_n-p_n+g^*-y^*\|=0.
$$

As a result, we obtain

$$
\begin{aligned}
\|d_n-Gd_n\|=\|d_n-p_n\| &\le \|d_n-q_n+y^*-g^*\|+\|q_n-p_n+g^*-y^*\| \\
&\to 0,\quad (n\to\infty). \qquad (32)
\end{aligned}
$$

Noting that $u_n=\zeta_nf_n+(1-\zeta_n)\mathcal{Z}_nu_n$, from (20) and the pseudocontractiveness of $\mathcal{Z}_n$, we arrive at

$$
\begin{aligned}
\|u_n-g^*\|^2 &= \zeta_n\langle f_n-g^*,u_n-g^*\rangle+(1-\zeta_n)\langle\mathcal{Z}_nu_n-g^*,u_n-g^*\rangle \\
&\le \zeta_n\langle f_n-g^*,u_n-g^*\rangle+(1-\zeta_n)\|u_n-g^*\|^2,
\end{aligned}
$$

which consequently leads to

$$\|u_n - g^*\|^2 \leq \langle f_n - g^*, u_n - g^* \rangle = \frac{1}{2}[\|f_n - g^*\|^2 + \|u_n - g^*\|^2 - \|f_n - u_n\|^2].$$

Hence, it follows that $\|u_n - g^*\|^2 \leq \|f_n - g^*\|^2 - \|f_n - u_n\|^2$. This together with (20), (21), and (22) implies that

$$\begin{aligned}\|f_{n+1} - g^*\|^2 &\leq \|d_n - g^*\|^2 + \sigma_n K \\ &\leq \chi_n \|f_n - g^*\|^2 + \rho_n \|d_n - g^*\|^2 + \xi_n (1+\theta_n)^2 \|t_n - g^*\|^2 + \sigma_n K \\ &\leq \chi_n \|f_n - g^*\|^2 + \rho_n [\|f_n - g^*\|^2 + \frac{\theta_n \widetilde{M}}{1-\rho_n}] \\ &\quad + \xi_n (1+\theta_n)^2 [\|f_n - g^*\|^2 - \|f_n - u_n\|^2] + \sigma_n K \\ &\leq (1+\theta_n)^2 \|f_n - g^*\|^2 + \frac{\rho_n \theta_n \widetilde{M}}{1-\rho_n} - \xi_n (1+\theta_n)^2 \|f_n - u_n\|^2 + \sigma_n K \\ &\leq \|f_n - g^*\|^2 + \theta_n \widetilde{M} + \frac{\rho_n \theta_n \widetilde{M}}{1-\rho_n} - \xi_n (1+\theta_n)^2 \|f_n - u_n\|^2 + \sigma_n K \\ &= \|f_n - g^*\|^2 + \frac{\theta_n \widetilde{M}}{1-\rho_n} - \xi_n (1+\theta_n)^2 \|f_n - u_n\|^2 + \sigma_n K.\end{aligned}$$

Thus, it follows that $\xi_n (1+\theta_n)^2 \|f_n - u_n\|^2 \leq \mathfrak{R}_n - \mathfrak{R}_{n+1} + \frac{\theta_n \widetilde{M}}{1-\rho_n} + \sigma_n K$. Hence $\lim_{n\to\infty} \|f_n - u_n\| = 0$. It is noteworthy that

$$(1-\zeta_n)\|\mathcal{Z}_n u_n - u_n\| = \zeta_n \|f_n - u_n\| \leq \|f_n - u_n\| \to 0, \quad (n \to \infty).$$

Utilizing $\liminf_{n\to\infty}(1-\zeta_n) > 0$, we establish

$$\lim_{n\to\infty} \|\mathcal{Z}_n u_n - u_n\| = \lim_{n\to\infty} \|f_n - u_n\| = 0. \tag{33}$$

Employing (23) and (24), we obtain

$$\begin{aligned}\|f_n - f_{n+1}\| &\leq \|f_n - d_n\| + \|d_n - f_{n+1}\| \to 0, \quad (n \to \infty), \\ \|v_n - u_n\| &\leq \|v_n - r_n\| + \|r_n - u_n\| \to 0, \quad (n \to \infty),\end{aligned} \tag{34}$$

and

$$\begin{aligned}\|t_n - f_n\| &\leq \|t_n - v_n\| + \|v_n - u_n\| + \|u_n - f_n\| \\ &= \vartheta_n \|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K} v_n\| + \|v_n - u_n\| + \|u_n - f_n\| \to 0, \quad (n \to \infty).\end{aligned} \tag{35}$$

By combining (23) and (32), we have

$$\begin{aligned}\|f_n - Gf_n\| &\leq \|f_n - d_n\| + \|d_n - Gd_n\| + \|Gd_n - Gf_n\| \\ &\leq 2\|f_n - d_n\| + \|d_n - Gd_n\| \to 0, \quad (n \to \infty).\end{aligned} \tag{36}$$

We assert that $\|f_n - \overline{\mathcal{Z}} f_n\| \to 0$ as $n \to \infty$, where $\overline{\mathcal{Z}} := (2\mathcal{I} - \widetilde{\mathcal{Z}})^{-1}$. It is evident that $\widetilde{\mathcal{Z}} : C \to C$ is pseudocontractive and ℓ-Lipschitzian, defined by $\widetilde{\mathcal{Z}} x = \lim_{n\to\infty} \mathcal{Z}_n x$ for all $x \in C$. We also claim that $\lim_{n\to\infty} \|\widetilde{\mathcal{Z}}f_n - f_n\| = 0$. By utilizing the boundedness of the sequence $\{f_n\}$ and denoting $D = \overline{\text{conv}}\{f_n : n \geq 1\}$ (the closed convex hull of the set $\{f_n : n \geq 1\}$), we infer from the assumptions that $\sum_{n=1}^{\infty} \sup_{x\in D} \|\mathcal{Z}_{n+1}x - \mathcal{Z}_n x\| < \infty$. Consequently, by invoking Lemma 2.11, we obtain $\lim_{n\to\infty} \sup_{x\in D} \|\mathcal{Z}_n x - \widetilde{\mathcal{Z}}x\| = 0$, which further implies that $\lim_{n\to\infty} \|\mathcal{Z}_n f_n - \widetilde{\mathcal{Z}}f_n\| = 0$. This, combining with (33) yields

$$\begin{aligned}\|f_n - \widetilde{\mathcal{Z}}f_n\| &\leq \|f_n - u_n\| + \|u_n - \mathcal{Z}_n u_n\| + \|\mathcal{Z}_n u_n - \mathcal{Z}_n f_n\| + \|\mathcal{Z}_n f_n - \widetilde{\mathcal{Z}}f_n\| \\ &\leq (1+\ell)\|f_n - u_n\| + \|u_n - \mathcal{Z}_n u_n\| + \|\mathcal{Z}_n f_n - \widetilde{\mathcal{Z}}f_n\| \to 0,\ (n\to\infty).\end{aligned} \tag{37}$$

Noting that $\overline{\mathcal{Z}} := (2\mathcal{I} - \widetilde{\mathcal{Z}})^{-1}$, we establish that $\overline{\mathcal{Z}}$ is nonexpansive and that $\text{Fix}(\overline{\mathcal{Z}}) = \text{Fix}(\widetilde{\mathcal{Z}}) = \bigcap_{n=1}^{\infty} \text{Fix}(\mathcal{Z}_n)$ as a consequence of Theorem 6 of [23]. From (37), it follows that

$$\begin{aligned}\|f_n - \overline{\mathcal{Z}}f_n\| &= \|\overline{\mathcal{Z}}\,\overline{\mathcal{Z}}^{-1}f_n - \overline{\mathcal{Z}}f_n\| \leq \|\overline{\mathcal{Z}}^{-1}f_n - f_n\| \\ &= \|(2\mathcal{I} - \widetilde{\mathcal{Z}})f_n - f_n\| = \|f_n - \widetilde{\mathcal{Z}}f_n\| \to 0 \quad (n\to\infty).\end{aligned} \tag{38}$$

By combining (23) and (35), we acquire

$$\begin{aligned}\|\mathcal{Z}_0^n f_n - f_n\| &\leq \|\mathcal{Z}_0^n f_n - \mathcal{Z}_0^n t_n\| + \|\mathcal{Z}_0^n t_n - d_n\| + \|d_n - f_n\| \\ &\leq (1+\theta_n)\|f_n - t_n\| + \|\mathcal{Z}_0^n t_n - d_n\| + \|d_n - f_n\| \to 0 \quad (n\to\infty).\end{aligned}$$

This, together with the condition $\|\mathcal{Z}_0^n f_n - \mathcal{Z}_0^{n+1} f_n\| \to 0$, implies that

$$\begin{aligned}\|f_n - \mathcal{Z}_0 f_n\| &\leq \|f_n - \mathcal{Z}_0^n f_n\| + \|\mathcal{Z}_0^n f_n - \mathcal{Z}_0^{n+1} f_n\| + \|\mathcal{Z}_0^{n+1} f_n - \mathcal{Z}_0 f_n\| \\ &\leq (2+\theta_1)\|f_n - \mathcal{Z}_0^n f_n\| + \|\mathcal{Z}_0^n f_n - \mathcal{Z}_0^{n+1} f_n\| \to 0 \quad (n\to\infty).\end{aligned} \tag{39}$$

Next, we demonstrate that $\lim_{n\to\infty} \|f_n - g^*\| = 0$. Indeed, since the sequences $\{d_n\}$ and $\{f_n\}$ are bounded, there exists a subsequence $\{d_{n_k}\} \subset \{d_n\}$ such that $d_{n_k} \rightharpoonup \widetilde{z} \in C$, and

$$\liminf_{n\to\infty} [\Gamma(g^*, d_n) + \Gamma(d_n, f_{n+1})] = \lim_{k\to\infty} [\Gamma(g^*, d_{n_k}) + \Gamma(d_{n_k}, f_{n_k+1})]. \tag{40}$$

From (23) and (24), it follows that $f_{n_k} \rightharpoonup \widetilde{z}$ and $f_{n_k+1} \rightharpoonup \widetilde{z}$. Consequently, by the result stated in Claim 5, we conclude that $\widetilde{z} \in \text{Sol}(C, \Phi)$. We note that G and $\overline{\mathcal{Z}}$ are nonexpansive, and that $\mathcal{Z}_0$ is asymptotically nonexpansive. Given that $(\mathcal{I} - G)f_n \to 0$, $(\mathcal{I} - \overline{\mathcal{Z}})f_n \to 0$, and $(\mathcal{I} - \mathcal{Z}_0)f_n \to 0$ (due to (36), (38), and (39)), we can invoke Lemma 2.12 to deduce that $\widetilde{z} \in \text{Fix}(G)$, $\widetilde{z} \in \text{Fix}(\overline{\mathcal{Z}}) = \text{Fix}(\widetilde{\mathcal{Z}}) = \bigcap_{k=1}^{\infty} \text{Fix}(\mathcal{Z}_k)$, and $\widetilde{z} \in \text{Fix}(\mathcal{Z}_0)$. As a result, we have $\widetilde{z} \in \text{Fix}(G) \cap \text{Sol}(C, \Phi) \cap \bigcap_{k=0}^{\infty} \text{Fix}(\mathcal{Z}_k)$. Additionally, we demonstrate that $\mathcal{K}\widetilde{z} \in \text{Fix}(\mathcal{Z})$. Specifically, utilizing the δ-demimetric nature of $\mathcal{Z}$, it from (23) follows that

$$\begin{aligned}\frac{1-\delta}{2}\|(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2 &\leq \langle (\mathcal{I} - \mathcal{Z})\mathcal{K}v_n, \mathcal{K}(v_n - g^*)\rangle \\ &\leq \|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|\|v_n - g^*\| \to 0, \quad (n\to\infty).\end{aligned} \tag{41}$$

Observing that $v_n - f_n \to 0$ and $f_{n_k} \rightharpoonup \widetilde{z}$, we conclude that $v_{n_k} \rightharpoonup \widetilde{z}$. Since $\mathcal{K}$ is a bounded linear operator, it is readily apparent that $\mathcal{K}$ is weakly continuous on $\mathcal{H}_1$. Thus, we obtain $\mathcal{K}v_{n_k} \rightharpoonup \mathcal{K}\widetilde{z}$. Based on the assumption regarding $\mathcal{Z}$, we know that $\mathcal{I} - \mathcal{Z}$ is demi-closed at zero. Consequently, from (41) we infer that $\mathcal{K}\widetilde{z} \in \mathrm{Fix}(\mathcal{Z})$. Hence, it follows that $\widetilde{z} \in \mathrm{Fix}(G) \cap \Omega \cap \bigcap_{k=0}^{\infty} \mathrm{Fix}(\mathcal{Z}_k) = \Xi$, where $\Omega = \{z \in \mathrm{Sol}(C, \Phi) : \mathcal{K}z \in \mathrm{Fix}(\mathcal{Z})\}$. In terms of (40), we have

$$\liminf_{n\to\infty}[\Gamma(g^*, d_n) + \Gamma(d_n, f_{n+1})] = \Gamma(g^*, \widetilde{z}) \geq 0. \tag{42}$$

Given that Γ is ν-strongly monotone, from $f_n - d_n \to 0$ (due to (23)), it follows that

$$\limsup_{n\to\infty}[\Gamma(g^*, d_n) + \Gamma(d_n, g^*)] \leq \limsup_{n\to\infty}(-\nu\|d_n - g^*\|^2) = -\nu\hbar. \tag{43}$$

By combining (42) and (43), we arrive at

$$\begin{aligned}
&\limsup_{n\to\infty}\left[\Gamma(d_n, g^*) - \Gamma(d_n, f_{n+1})\right] \\
&\quad \times \limsup_{n\to\infty}\left[\Gamma(d_n, g^*) + \Gamma(g^*, d_n) - \Gamma(g^*, d_n) - \Gamma(d_n, f_{n+1})\right] \\
&\leq \limsup_{n\to\infty}[\Gamma(d_n, g^*) + \Gamma(g^*, d_n)] + \limsup_{n\to\infty}[-\Gamma(g^*, d_n) - \Gamma(d_n, f_{n+1})] \\
&= \limsup_{n\to\infty}[\Gamma(d_n, g^*) + \Gamma(g^*, d_n)] - \liminf_{n\to\infty}[\Gamma(g^*, d_n) + \Gamma(d_n, f_{n+1})] \leq -\nu\hbar.
\end{aligned}$$

Next, to achieve the objective, it is sufficient to demonstrate that $\hbar = 0$. Conversely, we assume that $\hbar > 0$. Without loss of generality, we can assume the existence of $n_0 \geq 1$ such that:

$$\Gamma(d_n, g^*) - \Gamma(d_n, f_{n+1}) \leq -\frac{\nu\hbar}{2}, \quad \forall n \geq n_0.$$

This condition, in conjunction with (22), guarantees that for all $n \geq n_0$,

$$\begin{aligned}
\|f_{n+1} - g^*\|^2 &\leq \|f_n - g^*\|^2 - \frac{\xi_n}{1-\rho_n}\left[\epsilon^2\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2 + (1 - 2\alpha_n c_1)\|r_n - u_n\|^2\right. \\
&\quad \left. + (1 - 2\alpha_n c_2)\|v_n - r_n\|^2\right] + \frac{\theta_n\widetilde{M}}{1-\rho_n} - \frac{1}{1-\rho_n}\left[\chi_n\|f_n - d_n\|^2 + \xi_n\|\mathcal{Z}_0^n t_n - d_n\|^2\right] \\
&\quad - \|f_{n+1} - d_n\|^2 + 2\sigma_n[\Gamma(d_n, g^*) - \Gamma(d_n, f_{n+1})] \\
&\leq \|f_n - g^*\|^2 + \frac{\theta_n\widetilde{M}}{1-\rho_n} + 2\sigma_n[\Gamma(d_n, g^*) - \Gamma(d_n, f_{n+1})] \\
&\leq \|f_n - g^*\|^2 + \frac{\theta_n\widetilde{M}}{1-\rho_n} - \sigma_n\nu\hbar.
\end{aligned}$$

This thus establishes that for all $n \geq n_0$,

$$\Re_n - \Re_{n_0} \leq \widetilde{M}\sum_{k=n_0}^{n-1}\frac{\theta_k}{1-\rho_k} - \nu\hbar\sum_{k=n_0}^{n-1}\sigma_k. \tag{44}$$

Given that $\sum_{k=1}^{\infty}\sigma_k=\infty$, $\sum_{k=1}^{\infty}\theta_k<\infty$, and $0<\liminf_{n\to\infty}(1-\rho_n)$, as well as $\lim_{n\to\infty}\Re_n=\hbar$, we take the limit in (44) as $n\to\infty$ to obtain:

$$-\infty<\hbar-\Re_{n_0}=\lim_{n\to\infty}(\Re_n-\Re_{n_0})\leq\lim_{n\to\infty}\left[\widetilde{M}\sum_{k=n_0}^{n-1}\frac{\theta_k}{1-\rho_k}-\nu\hbar\sum_{k=n_0}^{n-1}\sigma_k\right]=-\infty.$$

This leads to a contradiction. Therefore, we conclude that $\lim_{n\to\infty}\Re_n=0$ and consequently $f_n\to g^*\in\mathrm{Sol}(\Xi,\Gamma)$, where g^* is the unique solution to the problem $\mathrm{EP}(\Xi,\Gamma)$.

Aspect 2. Suppose that there exists a sequence $\{\Re_{n_k}\}\subset\{\Re_n\}$ such that $\Re_{n_k}<\Re_{n_k+1}$ for all $k\in\mathbb{N}$. We define the mapping $\phi:\mathbb{N}\to\mathbb{N}$ by $\phi(n):=\max\{k\leq n:\Re_k<\Re_{k+1}\}$. By applying Lemma 2.13, we obtain:

$$\Re_{\phi(n)}\leq\Re_{\phi(n)+1},\quad\text{and}\quad\Re_n\leq\Re_{\phi(n)+1}.$$

Using similar reasoning as presented in (24) and (34), we can infer that:

$$\lim_{n\to\infty}\|r_{\phi(n)}-u_{\phi(n)}\|=\lim_{n\to\infty}\|v_{\phi(n)}-r_{\phi(n)}\|=\lim_{n\to\infty}\|f_{\phi(n)+1}-d_{\phi(n)}\|=0,\tag{45}$$

$$\lim_{n\to\infty}\|f_{\phi(n)}-f_{\phi(n)+1}\|=0.\tag{46}$$

Since the sequence $\{d_n\}$ is bounded, there exists a subsequence of $\{d_{\phi(n)}\}$, which we will still denote by $\{d_{\phi(n)}\}$, such that $d_{\phi(n)}\rightharpoonup\widetilde{z}$. Subsequently, utilizing the same reasoning as in Aspect 1, we deduce that $\widetilde{z}\in\Xi$. From $d_{\phi(n)}\rightharpoonup\widetilde{z}$ and (45), we conclude that $f_{\phi(n)+1}\rightharpoonup\widetilde{z}$. Given the assumption on $\{\alpha_n\}$, it follows that $1-2\alpha_{\phi(n)}c_1>0$ and $1-2\alpha_{\phi(n)}c_2>0$. Thus, from (22), we can infer that

$$\begin{aligned}&2\sigma_{\phi(n)}[\Gamma(d_{\phi(n)},f_{\phi(n)+1})-\Gamma(d_{\phi(n)},g^*)]\\&\leq\Re_{\phi(n)}-\Re_{\phi(n)+1}-\frac{\xi_{\phi(n)}}{1-\rho_{\phi(n)}}\left[\epsilon^2\|\mathcal{K}^*(\mathcal{I}-\mathcal{Z})\mathcal{K}v_{\phi(n)}\|^2\right.\\&\quad\left.+(1-2\alpha_{\phi(n)}c_1)\|r_{\phi(n)}-u_{\phi(n)}\|^2+(1-2\alpha_{\phi(n)}c_2)\|v_{\phi(n)}-r_{\phi(n)}\|^2\right]+\frac{\theta_{\phi(n)}\widetilde{M}}{1-\rho_{\phi(n)}}\\&\quad-\frac{1}{1-\rho_{\phi(n)}}\left[\chi_{\phi(n)}\|f_{\phi(n)}-d_{\phi(n)}\|^2+\xi_{\phi(n)}\|\mathcal{Z}_0^{\phi(n)}t_{\phi(n)}-d_{\phi(n)}\|^2\right]\\&\quad-\|f_{\phi(n)+1}-d_{\phi(n)}\|^2\\&\leq\frac{\theta_{\phi(n)}\widetilde{M}}{1-\rho_{\phi(n)}},\end{aligned}$$

which leads to the conclusion that

$$\Gamma(d_{\phi(n)},f_{\phi(n)+1})-\Gamma(d_{\phi(n)},g^*)\leq\frac{\theta_{\phi(n)}}{\sigma_{\phi(n)}}\cdot\frac{\widetilde{M}}{2(1-\rho_{\phi(n)})}.\tag{47}$$

Note that Γ is ν-strongly monotone on C. Hence, we obtain

$$\nu\|d_{\phi(n)}-g^*\|^2\leq-\Gamma(d_{\phi(n)},g^*)-\Gamma(g^*,d_{\phi(n)}).\tag{48}$$

By combining (47) and (48), we derive from $\mathrm{Ass}_\Gamma(\Gamma_1)$ and $\widetilde{z} \in \Xi$ that

$$\begin{aligned}\nu \limsup_{n\to\infty} \|d_{\phi(n)} - g^*\|^2 &= \limsup_{n\to\infty}\left[-\frac{\theta_{\phi(n)}}{\sigma_{\phi(n)}}\cdot\frac{\widetilde{M}}{2(1-\rho_{\phi(n)})} + \nu\|d_{\phi(n)} - g^*\|^2\right]\\ &\le \limsup_{n\to\infty}\left[-\Gamma(d_{\phi(n)}, f_{\phi(n)+1}) - \Gamma(g^*, d_{\phi(n)})\right]\\ &= -\Gamma(\widetilde{z},\widetilde{z}) - \Gamma(g^*,\widetilde{z}) \le 0.\end{aligned}$$

Thus, it follows that $\limsup_{n\to\infty} \|f_{\phi(n)} - g^*\|^2 \le 0$, leading to

$$\lim_{n\to\infty} \|f_{\phi(n)} - g^*\|^2 = 0.$$

From (46), we can express

$$\begin{aligned}&\|f_{\phi(n)+1} - g^*\|^2 - \|f_{\phi(n)} - g^*\|^2\\ &\quad = 2\langle f_{\phi(n)+1} - f_{\phi(n)}, f_{\phi(n)} - g^*\rangle + \|f_{\phi(n)+1} - f_{\phi(n)}\|^2\\ &\quad \le 2\|f_{\phi(n)+1} - f_{\phi(n)}\|\,\|f_{\phi(n)} - g^*\| + \|f_{\phi(n)+1} - f_{\phi(n)}\|^2 \to 0 \quad (n\to\infty).\end{aligned}$$

Given that $\Re_n \le \Re_{\phi(n)+1}$, it follows that

$$\begin{aligned}\|f_n - g^*\|^2 &\le \|f_{\phi(n)+1} - g^*\|^2\\ &\le \|f_{\phi(n)} - g^*\|^2 + 2\|f_{\phi(n)+1} - f_{\phi(n)}\|\,\|f_{\phi(n)} - g^*\| + \|f_{\phi(n)+1} - f_{\phi(n)}\|^2.\end{aligned}$$

Consequently, from (46), we conclude that $f_n \to g^*$ as $n \to \infty$. This completes the proof. ■

In the case when $\{\mathcal{Z}_k\}_{k=1}^\infty$ is a countable family of 1-uniformly Lipschitzian pseudo-contractive self-mappings on C, we propose another iterative algorithm through a new subgradient extragradient implicit approach.

Theorem 3.2: *Let the sequence $\{f_n\}$ be generated by Algorithm 2, and assume that the conditions Ass_Φ–Ass_Γ hold for bifunctions Γ and Φ. Then, under hypotheses (H1)-(H5), the sequence $\{f_n\}$ converges strongly to the unique solution g^* of the problem $\mathrm{EP}(\Xi,\Gamma)$ provided that $\mathcal{Z}_0^n f_n - \mathcal{Z}_0^{n+1} f_n \to 0$.*

Proof: According to Lemma 2.7, it follows that G is nonexpansive. Thus, employing Banach's contraction mapping principle, we derive from the sequence $\{\rho_n\} \subset (0,1)$ that for all $n \ge 1$, there exists $d_n \in C$ such that

$$d_n = \chi_n u_n + \rho_n G d_n + \xi_n \mathcal{Z}_0^n t_n.$$

Let us arbitrarily select a fixed point $p \in \Xi = \mathrm{Fix}(G) \cap \Omega \cap \bigcap_{k=0}^\infty \mathrm{Fix}(\mathcal{Z}_k)$. Given that $\lim_{n\to\infty} \theta_n/\sigma_n = 0$, we may assume that $\theta_n \le \frac{1}{2}\lambda\sigma_n, \forall n \ge 1$. We will divide the remainder of the proof into several claims presented below.

Claims 1–3. We demonstrate that the results in Claims 1–3 of the proof of Theorem 3.1 continue to hold. Indeed, by utilizing the same inferences as those in the proof of Theorem 3.1, we derive the required results.

Algorithm 2

Initialization: Given $f_1 \in C$ and $\vartheta \geq 0$ arbitrarily. Let $\{\zeta_n\}, \{\chi_n\}, \{\rho_n\}, \{\xi_n\} \subset (0, 1)$ and $\{\alpha_n\}, \{\sigma_n\} \subset (0, \infty)$ such that hypotheses (H1)-(H5) hold.
Iterative Steps: Calculate f_{n+1} as follows:
Step 1. Compute u_n and r_n by (2).
Step 2. Choose $w_n \in \partial_2\Phi(u_n, r_n)$, and compute C_n and v_n according to (3).
Step 3. Compute $t_n = v_n - \vartheta_n\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n$, where for any fixed $\epsilon > 0$, ϑ_n is chosen to be the bounded sequence satisfying (4); otherwise set $\vartheta_n = \vartheta \geq 0$.
Step 4. Compute

$$\begin{cases} q_n = \mathbf{Pj}_C(d_n - \mu_2 B_2 d_n), \\ p_n = \mathbf{Pj}_C(q_n - \mu_1 B_1 q_n), \\ d_n = \chi_n u_n + \rho_n p_n + \xi_n \mathcal{Z}_0^n t_n. \end{cases}$$

Step 5. Compute $f_{n+1} = \operatorname{argmin}\{\sigma_n\Gamma(d_n, t) + \frac{1}{2}\|t - d_n\|^2 : t \in C\}$.
Step 6. Again set $n := n + 1$ and return to **Step 1**.

Claim 4. We establish that the sequence $\{f_n\}$ is bounded. Specifically, by employing reasoning analogous to that in the proof of Theorem 3.1, we assert that the relationship (20) remains valid. Noting the nonexpansivity of G and the asymptotic nonexpansivity of $\mathcal{Z}_0$, we infer from (20) that

$$\begin{aligned} \|d_n - g^*\|^2 &= \chi_n\langle u_n - g^*, d_n - g^*\rangle + \rho_n\langle Gd_n - g^*, d_n - g^*\rangle + \xi_n\langle \mathcal{Z}_0^n t_n - g^*, d_n - g^*\rangle \\ &\leq \chi_n\|u_n - g^*\|\|d_n - g^*\| + \rho_n\|d_n - g^*\|^2 + \xi_n(1 + \theta_n)\|t_n - g^*\|\|d_n - g^*\| \\ &\leq \chi_n(1 + \theta_n)\|f_n - g^*\|\|d_n - g^*\| + \rho_n\|d_n - g^*\|^2 \\ &\quad + \xi_n(1 + \theta_n)\|f_n - g^*\|\|d_n - g^*\| \\ &= (1 - \rho_n)(1 + \theta_n)\|f_n - g^*\|\|d_n - g^*\| + \rho_n\|d_n - g^*\|^2, \end{aligned}$$

which immediately implies that $\|d_n - g^*\| \leq (1 + \theta_n)\|f_n - g^*\|$. Consequently, we have

$$\begin{aligned} \|f_{n+1} - g^*\| &\leq \|f_{n+1} - d_n^*\| + \|d_n^* - g^*\| \leq (1 - \lambda\sigma_n)\|d_n - g^*\| + \|d_n^* - g^*\| \\ &\leq (1 - \lambda\sigma_n)(1 + \theta_n)\|f_n - g^*\| + \sigma_n\widehat{M}(g^*) \leq \max\left\{\|f_n - g^*\|, \frac{2\widehat{M}(g^*)}{\lambda}\right\}. \end{aligned}$$

By induction, we conclude that $\|f_n - g^*\| \leq \max\{\|f_1 - g^*\|, \frac{2\widehat{M}(g^*)}{\lambda}\}, \forall n \geq 1$. Therefore, the sequence $\{f_n\}$ is bounded, as are the sequences $\{p_n\}$, $\{q_n\}$, $\{r_n\}$, $\{d_n\}$, $\{u_n\}$, $\{v_n\}$, and $\{t_n\}$.

Claim 5. We demonstrate that if $f_n - u_n \to 0$ and $u_n - r_n \to 0$, then $\omega_w(f_n) \subset \mathrm{Sol}(C, \Phi)$. In fact, by applying the same reasoning as in the proof of Theorem 3.1, we obtain the desired result.

Claim 6. We establish that $f_n \to g^*$, which is a unique solution of the MBSEP with GSVI and CFPP constraints.

To that end, we define $\Re_n = \|f_n - g^*\|^2$. By observing the nonexpansivity of G and the asymptotic nonexpansivity of $\mathcal{Z}_0$, we obtain

$$\begin{aligned}\|d_n - g^*\|^2 &= \chi_n\langle u_n - g^*, d_n - g^*\rangle + \rho_n\langle Gd_n - g^*, d_n - g^*\rangle + \xi_n\langle \mathcal{Z}_0^n t_n - g^*, d_n - g^*\rangle \\ &\le \frac{\chi_n}{2}\|u_n - g^*\|^2 + \frac{1+\rho_n}{2}\|d_n - g^*\|^2 + \frac{\xi_n}{2}\|t_n - g^*\|^2 + \frac{\theta_n\widetilde{M}}{2} \\ &\quad - \frac{\chi_n}{2}\|u_n - d_n\|^2 - \frac{\xi_n}{2}\|\mathcal{Z}_0^n t_n - d_n\|^2,\end{aligned}$$

where $\sup_{n\ge1}(2+\theta_n)\|f_n - g^*\|^2 \le \widetilde{M}$ for some $\widetilde{M} > 0$. This ensures that

$$\begin{aligned}\|d_n - g^*\|^2 \le \frac{1}{1-\rho_n}[\chi_n\|u_n - g^*\|^2 + \xi_n\|t_n - g^*\|^2 + \theta_n\widetilde{M} - \chi_n\|u_n - d_n\|^2 \\ - \xi_n\|\mathcal{Z}_0^n t_n - d_n\|^2]. \qquad (49)\end{aligned}$$

By the results presented in Claims 1 and 2, we derive from Equations (20) and (49) that

$$\begin{aligned}\|f_{n+1} - g^*\|^2 &\le \|d_n - g^*\|^2 - \|f_{n+1} - d_n\|^2 + 2\sigma_n[\Gamma(d_n, g^*) - \Gamma(d_n, f_{n+1})] \\ &\le \|f_n - g^*\|^2 - \frac{\xi_n}{1-\rho_n}[\epsilon^2\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2 + (1 - 2\alpha_n c_1)\|r_n - u_n\|^2 \\ &\quad + (1 - 2\alpha_n c_2)\|v_n - r_n\|^2] + \frac{\theta_n\widetilde{M}}{1-\rho_n} - \frac{1}{1-\rho_n} \\ &\quad \times [\chi_n\|u_n - d_n\|^2 + \xi_n\|\mathcal{Z}_0^n t_n - d_n\|^2] - \|f_{n+1} - d_n\|^2 + \sigma_n K, \qquad (50)\end{aligned}$$

where $\sup_{n\ge1}\{2|\Gamma(d_n, g^*) - \Gamma(d_n, f_{n+1})|\} \le K$ for some $K > 0$. Finally, we establish the convergence of the sequence $\{\Re_n\}$ to zero in the following two respects.

Aspect 1. Suppose there exists an integer $n_0 \ge 1$ such that the sequence $\{\Re_n\}$ is non-increasing. Under these conditions, it follows that $\lim_{n\to\infty}\Re_n = \hbar < +\infty$ and that $\lim_{n\to\infty}(\Re_n - \Re_{n+1}) = 0$. From Equation (50), one obtains:

$$\begin{aligned}&\frac{\xi_n}{1-\rho_n}[\epsilon^2\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\|^2 + (1 - 2\alpha_n c_1)\|r_n - u_n\|^2 + (1 - 2\alpha_n c_2)\|v_n - r_n\|^2] \\ &\quad + \frac{1}{1-\rho_n}[\chi_n\|u_n - d_n\|^2 + \xi_n\|\mathcal{Z}_0^n t_n - d_n\|^2] + \|f_{n+1} - d_n\|^2 \\ &\le \Re_n - \Re_{n+1} + \frac{\theta_n\widetilde{M}}{1-\rho_n} + \sigma_n K.\end{aligned}$$

Since $\sigma_n \to 0, \theta_n \to 0, \Re_n - \Re_{n+1} \to 0, 0 < \liminf_{n\to\infty}\chi_n, 0 < \liminf_{n\to\infty}\xi_n$ and $0 < \liminf_{n\to\infty}(1-\rho_n)$, we deduce from $0 < \epsilon$ and $\{\alpha_n\} \subset (\underline{\alpha}, \overline{\alpha}) \subset (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$ that

$$\lim_{n\to\infty}\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\| = \lim_{n\to\infty}\|u_n - d_n\| = \lim_{n\to\infty}\|\mathcal{Z}_0^n t_n - d_n\| = 0, \qquad (51)$$

$$\lim_{n\to\infty}\|r_n - u_n\| = \lim_{n\to\infty}\|v_n - r_n\| = \lim_{n\to\infty}\|f_{n+1} - d_n\| = 0. \qquad (52)$$

Next, we aim to demonstrate that $\lim_{n\to\infty}\|f_n - g^*\| = 0$. Indeed, by employing analogous reasoning to that presented in (27), we obtain

$$\|p_n - g^*\|^2 \le \|f_n - g^*\|^2 + \frac{\theta_n\widetilde{M}}{1-\rho_n} - \mu_2(2\chi - \mu_2)\|B_2 d_n - B_2 g^*\|^2$$

$$- \mu_1(2\alpha - \mu_1)\|B_1 q_n - B_1 y^*\|^2,$$

which together with (50) arrives at

$$\begin{aligned}
\|f_{n+1} - g^*\|^2 &\le \|d_n - g^*\|^2 + \sigma_n K \\
&\le \chi_n \|u_n - g^*\|^2 + \rho_n \|p_n - g^*\|^2 + \xi_n \|\mathcal{Z}_0^n t_n - g^*\|^2 + \sigma_n K \\
&\le \chi_n (1+\theta_n)^2 \|f_n - g^*\|^2 + \rho_n \|p_n - g^*\|^2 + \xi_n (1+\theta_n)^2 \|t_n - g^*\|^2 + \sigma_n K \\
&\le (1-\rho_n)[1+\theta_n(2+\theta_n)]\|f_n - g^*\|^2 + \rho_n \Big[\|f_n - g^*\|^2 + \frac{\theta_n \widetilde{M}}{1-\rho_n} \\
&\quad - \mu_2(2\chi - \mu_2)\|B_2 d_n - B_2 g^*\|^2 - \mu_1(2\alpha - \mu_1)\|B_1 q_n - B_1 y^*\|^2\Big] + \sigma_n K \\
&\le \|f_n - g^*\|^2 + \frac{\theta_n \widetilde{M}}{1-\rho_n} - \rho_n\Big[\mu_2(2\chi - \mu_2)\|B_2 d_n - B_2 g^*\|^2 \\
&\quad + \mu_1(2\alpha - \mu_1)\|B_1 q_n - B_1 y^*\|^2\Big] + \sigma_n K,
\end{aligned}$$

which hence leads to

$$\begin{aligned}
&\rho_n[\mu_2(2\chi - \mu_2)\|B_2 d_n - B_2 g^*\|^2 + \mu_1(2\alpha - \mu_1)\|B_1 q_n - B_1 y^*\|^2] \\
&\quad \le \Re_n - \Re_{n+1} + \frac{\theta_n \widetilde{M}}{1-\rho_n} + \sigma_n K.
\end{aligned}$$

We conclude from $\mu_2 \in (0, 2\chi)$ and $\mu_1 \in (0, 2\alpha)$ that

$$\lim_{n\to\infty} \|B_2 d_n - B_2 g^*\| = \lim_{n\to\infty} \|B_1 q_n - B_1 y^*\| = 0. \tag{53}$$

On the other hand, applying the same reasoning as in Equation (31), we obtain

$$\begin{aligned}
\|p_n - g^*\|^2 &\le \|f_n - g^*\|^2 + \frac{\theta_n \widetilde{M}}{1-\rho_n} - \|d_n - q_n + y^* - g^*\|^2 - \|q_n - p_n + g^* - y^*\|^2 \\
&\quad + 2\mu_1 \|B_1 y^* - B_1 q_n\| \|p_n - g^*\| + 2\mu_2 \|B_2 g^* - B_2 d_n\| \|q_n - y^*\|,
\end{aligned}$$

which combined with (50) yields

$$\begin{aligned}
\|f_{n+1} - g^*\|^2 &\le \|d_n - g^*\|^2 + \sigma_n K \\
&\le \chi_n (1+\theta_n)^2 \|u_n - g^*\|^2 + \rho_n \|p_n - g^*\|^2 + \xi_n (1+\theta_n)^2 \|t_n - g^*\|^2 + \sigma_n K \\
&\le (1-\rho_n)[1+\theta_n(2+\theta_n)]\|f_n - g^*\|^2 + \rho_n \Big[\|f_n - g^*\|^2 + \frac{\theta_n \widetilde{M}}{1-\rho_n} \\
&\quad - \|d_n - q_n + y^* - g^*\|^2 - \|q_n - p_n + g^* - y^*\|^2 + 2\mu_1 \|B_1 y^* - B_1 q_n\| \\
&\quad \times \|p_n - g^*\| + 2\mu_2 \|B_2 g^* - B_2 d_n\| \|q_n - y^*\|\Big] + \sigma_n K \\
&\le \|f_n - g^*\|^2 + \frac{\theta_n \widetilde{M}}{1-\rho_n} - \rho_n[\|d_n - q_n + y^* - g^*\|^2 + \|q_n - p_n + g^* - y^*\|^2] \\
&\quad + 2\mu_1 \|B_1 y^* - B_1 q_n\| \|p_n - g^*\| + 2\mu_2 \|B_2 g^* - B_2 d_n\| \|q_n - y^*\| + \sigma_n K.
\end{aligned}$$

This leads to

$$\rho_n[\|d_n - q_n + y^* - g^*\|^2 + \|q_n - p_n + g^* - y^*\|^2] \leq \Re_n - \Re_{n+1} + \frac{\theta_n \widetilde{M}}{1-\rho_n} + 2\mu_1\|B_1 y^* - B_1 q_n\|\|p_n - g^*\| + 2\mu_2\|B_2 g^* - B_2 d_n\|\|q_n - y^*\| + \sigma_n K.$$

We obtain from (53) that $\lim_{n\to\infty}\|d_n - q_n + y^* - g^*\| = \lim_{n\to\infty}\|q_n - p_n + g^* - y^*\| = 0$. Consequently,

$$\|d_n - Gd_n\| = \|d_n - p_n\| \leq \|d_n - q_n + y^* - g^*\| + \|q_n - p_n + g^* - y^*\| \to 0 \quad (n \to \infty). \tag{54}$$

Applying the analogous reasoning to that of (33), we obtain $\lim_{n\to\infty}\|\mathcal{Z}_n f_n - u_n\| = \lim_{n\to\infty}\|f_n - u_n\| = 0$, which hence leads to

$$\|f_n - \mathcal{Z}_n f_n\| \leq \|f_n - u_n\| + \|u_n - \mathcal{Z}_n f_n\| \to 0 \quad (n \to \infty).$$

Concurrently, (51) and (52) imply that

$$\|f_n - d_n\| \leq \|f_n - u_n\| + \|u_n - d_n\| \to 0 \quad (n \to \infty), \tag{55}$$

and hence

$$\|f_n - f_{n+1}\| \leq \|f_n - d_n\| + \|d_n - f_{n+1}\| \to 0 \quad (n \to \infty).$$

We assert that $\|\mathcal{Z}_0 f_n - f_n\| \to 0$ and $\|Gf_n - f_n\| \to 0$ as $n \to \infty$. Notably, by acknowledging the nonexpansivity of G, we can infer from Equations (54) and (55) that

$$\begin{aligned}\|Gf_n - f_n\| &\leq \|Gf_n - Gd_n\| + \|Gd_n - d_n\| + \|d_n - f_n\| \\ &\leq 2\|f_n - d_n\| + \|Gd_n - d_n\| \to 0 \quad (n \to \infty).\end{aligned}$$

We infer from (51), (52), and (55) that

$$\begin{aligned}&\|v_n - f_n\| \leq \|v_n - r_n\| + \|r_n - u_n\| + \|u_n - f_n\| \to 0 \quad (n \to \infty), \\ &\|t_n - f_n\| \leq \|t_n - v_n\| + \|v_n - f_n\| = \vartheta_n\|\mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n\| + \|v_n - f_n\| \\ &\qquad \to 0 \quad (n \to \infty),\end{aligned}$$

and hence

$$\begin{aligned}\|\mathcal{Z}_0^n f_n - f_n\| &\leq \|\mathcal{Z}_0^n f_n - \mathcal{Z}_0^n t_n\| + \|\mathcal{Z}_0^n t_n - d_n\| + \|d_n - f_n\| \\ &\leq (1+\theta_n)\|f_n - t_n\| + \|\mathcal{Z}_0^n t_n - d_n\| + \|d_n - f_n\| \to 0 \quad (n \to \infty).\end{aligned}$$

Using the same reasoning as in (39), we obtain

$$\lim_{n\to\infty}\|f_n - \mathcal{Z}_0 f_n\| = 0.$$

Furthermore, employing the same reasoning as presented in Aspect 1 of the proof of Theorem 3.1, we establish that $\lim_{n\to\infty}\Re_n = 0$. Consequently, it follows that $f_n \to g^* \in \mathrm{Sol}(\Xi, \Gamma)$, where g^* denotes the unique solution of the problem $\mathrm{EP}(\Xi, \Gamma)$.

Aspect 2. Suppose there exists a subsequence $\{\Re_{n_k}\} \subset \{\Re_n\}$ such that $\Re_{n_k} < \Re_{n_k+1}$ for all $k \in \mathbb{N}$. We define the mapping $\phi : \mathcal{N} \to \mathcal{N}$ by $\phi(n) := \max\{k \le n : \Re_k < \Re_{k+1}\}$. In the remainder of the proof, we will utilize the same reasoning as presented in Aspect 2 of the proof of Theorem 3.1 to derive the desired result. This concludes the proof. ■

Our algorithms are more general and refined than the existing ones, as they address the solution of the MBSEP under GSVI and CFPP constraints. The theoretical results obtained in this paper extend and improve upon those found in [4, 15]. In comparison to the findings presented in Ceng and Wen [4], and He et al. [15], our results offer improvements and extensions in several key aspects.

(i) The accelerated subgradient extragradient method is particularity useful for solving the more general problem that is MBSEP with GSVI and CFPP constraints. Specifically, the original iterative step $d_n = v_n - \vartheta_n \mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n$ is transformed into the composite Mann implicit iteration $d_n = \chi_n f_n + \rho_n G d_n + \xi_n \mathcal{Z}_0^n t_n$, where $t_n = v_n - \vartheta_n \mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n$. Additionally, the original hybrid deepest-descent step $f_{n+1} = \rho_n f_n + ((1-\rho_n)\mathcal{I} - \chi_n \rho F)d_n$ is developed into the projection iteration $f_{n+1} = \operatorname{argmin}\{\sigma_n \Gamma(d_n, t) + \frac{1}{2}\|t - d_n\|^2 : t \in C\}$.

(ii) The hybrid extragradient-like implicit approach for approximating an element of $\mathrm{Fix}(G) \cap (\bigcap_{k=0}^{\infty} \mathrm{Fix}(\mathcal{Z}_k))$ developed in [4] has been extended to create a new subgradient extragradient implicit approach for solving the MBSEP with GSVI and CFPP constraints. Furthermore, it was demonstrated in [4] that $f_n \to g^* \in \mathrm{Fix}(G) \cap (\bigcap_{k=0}^{\infty} \mathrm{Fix}(\mathcal{Z}_k))$. In this paper, we show that $f_n \to g^* \in \mathrm{Fix}(G) \cap \Omega \cap (\bigcap_{k=0}^{\infty} \mathrm{Fix}(\mathcal{Z}_k))$, where g^* is a solution of $\mathrm{EP}(\Xi, \Gamma)$.

(iii) The problem of finding an element of the MBEP with the GSVI and CFPP constraints in [15] is expanded to formulate the MBSEP with the same GSVI and CFPP constraints. The subgradient extragradient implicit rule from [15] is generalized to establish a novel subgradient extragradient implicit approach for addressing the MBSEP with the GSVI and CFPP constraints. For instance, the original Mann implicit iteration given by $u_n = \zeta_n f_n + (1-\zeta_n) W_n u_n$ (with W_n being nonexpansive) in [15] is refined into the new Mann implicit iteration $u_n = \zeta_n f_n + (1-\zeta_n)\mathcal{Z}_n u_n$ (where $\mathcal{Z}_n$ is an ℓ-uniformly Lipschitzian pseudocontraction). Similarly, the original Mann implicit iteration $d_n = \chi_n f_n + \rho_n G d_n + \xi_n \mathcal{Z}_0^n v_n$ with $v_n = \operatorname{argmin}\{\alpha_n \Phi(r_n, z) + \frac{1}{2}\|z - u_n\|^2 : z \in C_n\}$ is transformed into the composite Mann implicit iteration $d_n = \chi_n f_n + \rho_n G d_n + \xi_n \mathcal{Z}_0^n t_n$ with $t_n = v_n - \vartheta_n \mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K}v_n$.

4. Applicability and implementability

Define mappings $B_1, B_2 : \mathcal{H}_1 \to \mathcal{H}_1$ as α-inverse-strongly monotone and χ-inverse-strongly monotone, respectively. Furthermore, define $G : \mathcal{H}_1 \to C$ as $G := \mathbf{Pj}_C(\mathcal{I} - \mu_1 B_1)\mathbf{Pj}_C(\mathcal{I} - \mu_2 B_2)$ for $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\chi)$. Let $\mathcal{Z}_0 : \mathcal{H}_1 \to C$ be an asymptotically nonexpansive mapping with a sequence $\{\theta_n\}$, and let $\mathcal{Z}_n = \mathcal{Z}_1 : C \to C$ be a nonexpansive mapping for all $n \ge 1$. Assume that $\mathcal{K} : \mathcal{H}_1 \to \mathcal{H}_2$ is a non-zero bounded linear operator with adjoint $\mathcal{K}^*$, and that $\mathcal{Z}$ is a δ-demimetric self-mapping on $\mathcal{H}_2$ such that $\mathcal{I} - \mathcal{Z}$ is demiclosed at zero, where $\delta \in (-\infty, 1)$. Let $\Xi = \mathrm{Fix}(G) \cap \Omega \cap \mathrm{VI}(C, \mathcal{A}) \neq$

Algorithm 3

Initialization: Given $f_1 \in C$ and $\vartheta \geq 0$ arbitrarily. Let $\{\zeta_n\}, \{\chi_n\}, \{\rho_n\}, \{\xi_n\} \subset (0,1)$ and $\{\alpha_n\}, \{\sigma_n\} \subset (0,\infty)$ such that hypotheses (H1)-(H5) hold.
Iterative Steps: Calculate f_{n+1} as follows:
Step 1. Compute

$$\begin{cases} u_n = \zeta_n f_n + (1-\zeta_n)\mathcal{Z}_1 u_n, \\ r_n = \mathbf{Pj}_C(u_n - \alpha_n \mathcal{A} u_n). \end{cases}$$

Step 2. Choose $w_n = \mathcal{A}u_n$, and compute

$$\begin{cases} C_n = \{v \in \mathcal{H}_1 : \langle u_n - \alpha_n w_n - r_n, v - r_n\rangle \leq 0\}, \\ v_n = \mathbf{Pj}_{C_n}(u_n - \alpha_n \mathcal{A} r_n). \end{cases}$$

Step 3. Compute $t_n = v_n - \vartheta_n \mathcal{K}^*(\mathcal{I} - \mathcal{Z})\mathcal{K} v_n$, where for any fixed $\epsilon > 0$, ϑ_n is chosen to be the bounded sequence satisfying (3); otherwise set $\vartheta_n = \vartheta \geq 0$.
Step 4. Compute

$$\begin{cases} q_n = \mathbf{Pj}_C(d_n - \mu_2 B_2 d_n), \\ p_n = \mathbf{Pj}_C(q_n - \mu_1 B_1 q_n), \\ d_n = \chi_n f_n + \rho_n p_n + \xi_n \mathcal{Z}_0^n t_n. \end{cases}$$

Step 5. Compute $f_{n+1} = \operatorname{argmin}\{\sigma_n \Gamma(d_n, t) + \frac{1}{2}\|t - d_n\|^2 : t \in C\}$.
Step 6. Set $n := n+1$ and return to **Step 1**.

$\emptyset$ where $\Omega = \{z \in \bigcap_{i=0}^{1} \mathrm{Fix}(\mathcal{Z}_i) : \mathcal{K}z \in \mathrm{Fix}(\mathcal{Z})\}$. Assume that $\mathcal{A}$ satisfies the following conditions:

(B1) $\mathcal{A}$ is monotone.
(B2) $\mathcal{A}$ is weakly to strongly continuous, i.e. for all $\{u_n\} \subset \mathcal{H}_1$, it holds that $u_n \rightharpoonup u \Rightarrow \mathcal{A}u_n \to \mathcal{A}u$.
(B3) $\mathcal{A}$ is L-Lipschitz continuous for some constant $L > 0$.

Let bifunction Γ and the positive sequences $\{\alpha_n\}$, $\{\sigma_n\}$, $\{\zeta_n\}$, $\{\chi_n\}$, $\{\rho_n\}$, and $\{\xi_n\}$ as defined in Algorithm 1. We define the bifunction $\Phi : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{R}$ as follows: $\Phi(u,v) := \langle \mathcal{A}u, v - u\rangle, \quad \forall u, v \in \mathcal{H}_1$. It is readily verified that the bifunction Φ satisfies the conditions $\mathrm{Ass}_\Phi(\Phi_1)$-$\mathrm{Ass}_\Phi(\Phi_2)$ and is Lipschitz continuous with constants $c_1 = c_2 = \frac{L}{2}$. Thus, the subgradient extragradient implicit Algorithm 1 simplifies to the following Algorithm 3 for solving the GSVI, VIP, and SFP.

Using Theorem 3.1, we can immediately derive the following result.

Theorem 4.1: *Let the sequence $\{f_n\}$ be generated by Algorithm 3. Then $\{f_n\}$ converges strongly to the unique solution g^* of the problem* $\mathrm{EP}(\Xi, \Gamma)$ *provided that* $\mathcal{Z}_0^n f_n - \mathcal{Z}_0^{n+1} f_n \to 0$.

Algorithm 4

Initialization: Given $f_1 \in C$ and $\vartheta \geq 0$ arbitrarily. Let $\{\zeta_n\}, \{\chi_n\}, \{\rho_n\}, \{\xi_n\} \subset (0,1)$ and $\{\alpha_n\}, \{\sigma_n\} \subset (0,\infty)$ such that hypotheses (H1)-(H5) hold.

Iterative Steps: Calculate f_{n+1} as follows:

Step 1–**Step 3** are the same as Algorithm 3.

Step 4. Compute

$$\begin{cases} q_n = \mathbf{Pj}_C(d_n - \mu_2 B_2 d_n), \\ p_n = \mathbf{Pj}_C(q_n - \mu_1 B_1 q_n), \\ d_n = \chi_n u_n + \rho_n p_n + \xi_n \mathcal{Z}_0^n t_n. \end{cases}$$

Step 5. Compute $f_{n+1} = \operatorname{argmin}\{\sigma_n \Gamma(d_n, t) + \frac{1}{2}\|t - d_n\|^2 : t \in C\}$.

Step 6. Set $n := n + 1$ and return to **Step 1**.

Similarly, the subgradient extragradient implicit Algorithm 2 can be reduced to the following Algorithm 4 for solving the GSVI, VIP, and SFP.

Using Theorem 3.2, we derive the following result.

Theorem 4.2: *Let the sequence $\{f_n\}$ be generated by Algorithm 4. Then $\{f_n\}$ converges strongly to the unique solution g^* of the problem* EP(Ξ, Γ) *provided that* $\mathcal{Z}_0^n f_n - \mathcal{Z}_0^{n+1} f_n \to 0$.

5. Numerical illustration

This section showcases a set of numerical experiments aimed at evaluating the performance of the proposed methods through a representative example. The main goal is to offer practical insights into the parameter selection process for the algorithms under consideration. The proposed algorithms were implemented using MATLAB and executed on a machine equipped with an Intel(R) Core(TM) i5-6200 CPU @ 2.30 GHz and 8 GB of RAM.

Example 5.1: In this example, the proposed Algorithms 1 and 2 are applied to solve the GSVI, VIP, and SFP. Let $C = [-2, 2]$ and $\mathcal{H}_1 = \mathcal{H}_2 := \mathcal{H} = \mathbb{R}$. Define the mappings $\mathcal{Z}_1 : C \to C$, $\mathcal{Z}_0 : \mathcal{H} \to C$, $\mathcal{A} : \mathcal{H} \to \mathcal{H}$, $B_k : \mathcal{H} \to \mathcal{H}$ $(k = 1, 2)$, $\hat{\Gamma}_1, \tilde{\gamma}_1 : C \times C \to \mathcal{H}$, $\Gamma : C \times C \to \mathbb{R}$, as well as $\mathcal{Z}, \mathcal{K} : \mathcal{H} \to \mathcal{H}$, as follows:

$$\mathcal{Z}_1(u) = \sin u, \quad \mathcal{Z}_0(u) = \frac{5}{6}\sin u, \quad \mathcal{A}(u) = u + \sin u, \quad B_k(u) = u + \frac{1}{2}\sin u,$$

$$\Gamma(u, v) = \langle u + \frac{1}{2}\sin u, v - u\rangle, \quad \hat{\Gamma}_1(u, v) = u - v + \frac{1}{2}(\sin u - \sin v),$$

$$\tilde{\gamma}_1(u, v) = u - v, \quad \mathcal{Z}(u) = \frac{1}{5}u + \frac{3}{5}\sin(u),, \quad \mathcal{K}(v) = v.$$

According to the above definition, we have

(1) $\mathcal{Z}_1$ is nonexpansive with $\mathrm{Fix}(\mathcal{Z}_1) = \{0\}$, and $\mathcal{Z}_0$ is asymptotically nonexpansive with $\theta_n = (5/6)^n$ for all $n \geq 1$, such that $\|\mathcal{Z}_0^{n+1} f_n - \mathcal{Z}_0^n f_n\| \to 0$ as $n \to \infty$.

(2) The set $\mathrm{Fix}(\mathcal{Z}_0) = \{0\}$ is well known. Furthermore, the operator $\mathcal{A}$ is monotone and L-Lipschitz continuous with $L = 2$. It can be verified that $c_1 = c_2 = \frac{L}{2} = 1$ and $0 \in \mathrm{VI}(C, \mathcal{A})$.
(3) For $k = 1, 2$, the operator B_k is $\frac{2}{9}$-inverse-strongly monotone. Note that $G(0) = \mathbf{Pj}_C(\mathcal{I} - \frac{1}{3}B_1)\mathbf{Pj}_C(\mathcal{I} - \frac{1}{3}B_2)0 = 0$, and thus $0 \in \mathrm{Fix}(G)$.
(4) $\mathcal{K}$ is a bounded linear operator on $\mathcal{H}$. It is clear that $\mathcal{Z}$ is a δ-demicontractive mapping with $\delta = \frac{1}{5}$, and $\mathrm{Fix}(\mathcal{Z}) = \{0\}$. In fact, $\mathcal{Z}$ is δ-strictly pseudocontractive with $\delta = \frac{1}{5}$

Note that $\Omega = \{z \in \bigcap_{k=0}^{1} \mathrm{Fix}(\mathcal{Z}_k) : \mathcal{K}(z) \in \mathrm{Fix}(\mathcal{Z})\} = \{0\}$. Therefore, $\Xi = \mathrm{Fix}(G) \cap \Omega \cap \mathrm{VI}(C, A) = \{0\} \neq \emptyset$. Observe that $0 < \frac{1}{5} = \epsilon \leq \vartheta_n \leq \frac{(1-\delta)\|(\mathcal{I}-\mathcal{Z})\mathcal{K}(v_n)\|^2}{\|\mathcal{K}^*(\mathcal{I}-\mathcal{Z})\mathcal{K}(v_n)\|^2} - \epsilon = \frac{3}{5}$, if $(\mathcal{I} - \mathcal{Z})\mathcal{K}(v_n) \neq 0$, and $\vartheta_n = \vartheta = \frac{1}{5}$ otherwise. Hence, we set $\vartheta_n = \frac{1}{5}$ for all $n \geq 1$. Moreover, it is readily known that:

(a) Γ is ν-strongly monotone with $\nu = \frac{1}{2}$;
(b) For $\hat{\ell}_1 = \frac{3}{2}$ and $\widetilde{\ell}_1 = 1$, the mappings $\hat{\Gamma}_1$ and $\widetilde{\gamma}_1$ satisfy the following properties: $\hat{\Gamma}_1(u, v) + \hat{\Gamma}_1(v, u) = 0$, $\|\hat{\Gamma}_1(u, v)\| \leq \hat{\ell}_1\|u - v\|$, $\widetilde{\gamma}_1(u, u) = 0$, $\|\widetilde{\gamma}_1(u, v) - \widetilde{\gamma}_1(x, y)\| = \widetilde{\ell}_1\|(u - v) - (x - y)\|$, and

$$\begin{aligned}
\Gamma(u, v) + \Gamma(v, w) &= \left\langle u + \frac{1}{2}\sin u, v - u\right\rangle + \left\langle v + \frac{1}{2}\sin v, w - v\right\rangle \\
&= \left\langle u + \frac{1}{2}\sin u, w - u\right\rangle + \left\langle u - v + \frac{1}{2}(\sin u - \sin v), v - w\right\rangle \\
&= \Gamma(u, w) + \langle \hat{\Gamma}_1(u, v), \widetilde{\gamma}_1(v, w)\rangle \\
&\geq \Gamma(u, w) - \hat{\ell}_1\widetilde{\ell}_1\|u - v\|\|v - w\| \\
&\geq \Gamma(u, w) - \frac{1}{2}\Upsilon\|u - v\|^2 - \frac{1}{2}\Upsilon\|v - w\|^2,
\end{aligned}$$

where $\Upsilon = \hat{\ell}_1\widetilde{\ell}_1 = \frac{3}{2}$.
(c) For any sequence $\{v_n\} \subset C$ such that $v_n \to v$, we have: $\limsup_{n\to\infty} \frac{|\Gamma(v, v_n)|}{\|v_n - v\|} = \limsup_{n\to\infty} \frac{|\langle v + \frac{1}{2}\sin v, v_n - v\rangle|}{\|v_n - v\|} \leq \|v + \frac{1}{2}\sin v\| \leq \frac{5}{2} < +\infty$.

Let

$$\begin{aligned}
&\mu_1 = \mu_2 = \frac{1}{3}, \zeta_n = \lambda = \frac{2}{9}, \underline{\alpha} = \frac{1}{6}, \overline{\alpha} = \frac{3}{7}, \alpha_n = \frac{1}{3}, \chi_n = \frac{1}{3(n+1)} + \frac{1}{6}, \\
&\rho_n = \frac{3n + 2}{3(n+1)} - \frac{1}{3}, \xi_n = \frac{1}{6}, \quad \text{and} \quad \sigma_n = \frac{1}{3(n+1)}.
\end{aligned}$$

Note that $\lim_{n\to\infty}\theta_n/\sigma_n = 0$, $\sum_{n=1}^{\infty}\theta_n < \infty$. In addition, it is evident that the sequences $\{\zeta_n\},\{\chi_n\},\{\rho_n\},\{\xi_n\} \subset (0,1)$ and $\{\alpha_n\},\{\sigma_n\} \subset (0,\infty)$ satisfy the hypotheses (H1)-(H4). Next, we verify that (H5) holds as well. Indeed, observe that

$$2\sigma_n\nu - \sigma_n^2\Upsilon^2 = \frac{1}{3(n+1)}\left(1 - \frac{3}{4}\cdot\frac{1}{n+1}\right) < 1,$$
$$0 < \lambda = \frac{2}{9} < \frac{1}{2} = \min\left\{\frac{1}{2},\frac{3}{2}\right\} = \min\{\nu,\Upsilon\},$$

and

$$0 < \sigma_n = \frac{1}{3(n+1)} \le \frac{1}{6} < \frac{1620}{6417} = \frac{1-\frac{4}{9}}{\frac{9}{4}-\frac{4}{81}} = \min\left\{\frac{9}{2},\frac{1-\frac{4}{9}}{\frac{9}{4}-\frac{4}{81}},\frac{1}{\frac{9}{4}}\right\}$$
$$= \min\left\{\frac{1}{\lambda},\frac{2\upsilon-2\lambda}{\Upsilon^2-\lambda^2},\frac{2\nu}{\Upsilon^2}\right\}.$$

In this case, Algorithm 3 can be reformulated as follows:

$$\begin{cases} u_n = \frac{2}{9}f_n + \frac{7}{9}\mathcal{Z}_1 u_n, \\ r_n = \mathbf{Pj}_C(u_n - \frac{1}{3}\mathcal{A}u_n), \\ v_n = \mathbf{Pj}_{C_n}(u_n - \frac{1}{3}\mathcal{A}r_n), \\ t_n = v_n - \frac{1}{5}(\mathcal{I}-\mathcal{Z})v_n, \\ d_n = \left(\frac{1}{3(n+1)}+\frac{1}{6}\right)f_n + \left(\frac{3n+2}{3(n+1)}-\frac{1}{3}\right)p_n + \frac{1}{6}\mathcal{Z}_0^n t_n, \\ q_n = \mathbf{Pj}_C\left(d_n - \frac{1}{3}B_2 d_n\right), \\ p_n = \mathbf{Pj}_C\left(q_n - \frac{1}{3}B_1 q_n\right), \\ f_{n+1} = \operatorname{argmin}\left\{\frac{1}{3(n+1)}\Gamma(d_n,t) + \frac{1}{2}\|t-d_n\|^2 : t \in C\right\}, \end{cases}$$

where C_n is chosen as in Algorithm 3 for each $n \ge 1$.

On the other hand, Algorithm 4 can be reformulated as follows:

$$\begin{cases} u_n = \frac{2}{9} f_n + \frac{7}{9} \mathcal{Z}_1 f_n, \\ r_n = \mathbf{Pj}_C(u_n - \frac{1}{3}\mathcal{A}u_n), \\ v_n = \mathbf{Pj}_{C_n}(u_n - \frac{1}{3}\mathcal{A}r_n), \\ t_n = v_n - \frac{1}{5}(\mathcal{I} - \mathcal{Z})v_n, \\ d_n = \left(\frac{1}{3(n+1)} + \frac{1}{6}\right) u_n + \left(\frac{3n+2}{3(n+1)} - \frac{1}{3}\right) p_n + \frac{1}{6}\mathcal{Z}_0^n t_n, \\ q_n = \mathbf{Pj}_C\left(d_n - \frac{1}{3}B_2 d_n\right), \\ p_n = \mathbf{Pj}_C\left(q_n - \frac{1}{3}B_1 q_n\right), \\ f_{n+1} = \operatorname{argmin}\left\{\frac{1}{3(n+1)}\Gamma(d_n, t) + \frac{1}{2}\|t - d_n\|^2 : t \in C\right\}, \end{cases}$$

where C_n is chosen as in Algorithm 4 for each $n \geq 1$.

Next, we present a series of numerical experiments based on Example 5.1 to evaluate the performance of Algorithm 3 and Algorithm 4. The performance is assessed in terms of *execution time* (t, in seconds) and the *number of iterations* (n) required for convergence. Specifically, we aim to examine how the performance of our algorithms is influenced by variations in the following parameters:

(i) Different initial values of f_1;
(ii) Different values of the sequence $\{\zeta_n\} \subset (0, 1)$.
(iii) Different values of $\{\alpha_n\}$ such that $0 < \alpha_n < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\} = \frac{1}{2}$;
(iv) Different values of $\{\vartheta_n\}$ such that $\frac{1}{5} \leq \vartheta_n \leq \frac{3}{5}$;
(v) Different values of $\mu_1 \in (0, 2\alpha)$ such that $0 < \mu_1 < \frac{4}{9}$;
(vi) Different combinations of the sequences $\{\chi_n\}$, $\{\rho_n\}$, and $\{\xi_n\}$, where each sequence belongs to $(0, 1)$ and satisfies Conditions (H1)–(H2).

In all numerical experiments, the stopping criterion is set to $E_n = \|f_{n+1} - f_n\| \leq 10^{-6}$. Unless otherwise specified in the individual experiments, the default values for the parameters and sequences are as follows:

$$f_1 = 2, \quad \zeta_n = \frac{2}{9}, \quad \alpha_n = \frac{1}{3}, \quad \vartheta_n = \frac{1}{5}, \quad \chi_n = \frac{1}{3(n+1)} + \frac{1}{6},$$
$$\rho_n = \frac{3n+2}{3(n+1)} - \frac{1}{3}, \quad \xi_n = \frac{1}{6}, \quad \mu_1 = \mu_2 = \frac{1}{3}, \quad \sigma_n = \frac{1}{3(n+1)}.$$

Experiment 1. (Impact of the chosen initial point f_1). Table 1 summarizes the numerical results obtained for six distinct values of f_1.

Table 1. Numerical data for Experiment 1.

	Algorithm 3		Algorithm 4	
f_1	n	t	n	t
−1.87	46	0.653	50	0.666
−1.23	45	0.561	49	0.660
−0.54	42	0.510	47	0.566
0.79	43	0.533	48	0.576
1.46	45	0.550	49	0.588
1.92	46	0.550	50	0.609

Table 2. Numerical data for Experiment 2.

	Algorithm 3		Algorithm 4	
ζ_n	n	t	n	t
0.05	45	0.544	42	0.525
0.25	46	0.548	45	0.571
0.45	47	0.585	46	0.585
0.65	47	0.551	47	0.596
0.85	47	0.588	47	0.570
0.95	48	0.598	48	0.633

Table 3. Numerical data for Experiment 3.

	Algorithm 3		Algorithm 4	
α_n	n	t	n	t
0.05	84	0.669488	82	0.655022
0.15	49	0.382301	47	0.412077
0.25	42	0.333577	41	0.331343
0.35	48	0.394391	47	0.390338
0.45	82	0.653176	80	0.622670
0.48	121	0.943851	116	0.913177

Based on the data presented in Table 1, we have: (i) Both the number of iterations and CPU time demonstrate minimal sensitivity to changes in f_1. This indicates that Algorithms 3 and 4 exhibit relative stability with respect to this parameter. (ii) For a majority of f_1 values, the iteration count (n) for Algorithm 4 is slightly higher than that of Algorithm 3. The CPU time (t) follows a similar trend, generally decreasing as f_1 increases.

Experiment 2. (Impact of the chosen parameter ζ_n). Table 2 presents the numerical results obtained for six distinct values of ζ_n.

Based on the numerical data in Table 2, we obatin: (i) Both algorithms exhibit stable iteration counts as ζ_n increases. Algorithm 3 generally requires more iterations than Algorithm 4 for smaller values of ζ_n. (ii) Generally, lower values of ζ_n appear more advantageous for both algorithms in terms of achieving reduced CPU time and iteration counts.

Experiment 3. (Impact of the chosen parameter α_n). Table 3 gives numerical results for six distinct values of α_n.

The data in Table 3 reveal the following trends regarding the impact of α_n on the iteration count n and CPU time t for both algorithms: (i) For both Algorithms 3 and 4, the iteration count n decreases as α_n increases from 0.05 to 0.25, reaching a minimum of 42 iterations

Table 4. Numerical data for Experiment 4.

	Algorithm 3		Algorithm 4	
ϑ_n	n	t	n	t
0.20	46	0.373672	45	0.358463
0.35	46	0.369676	44	0.351850
0.45	45	0.376370	44	0.348053
0.50	45	0.376668	43	0.328172
0.55	45	0.371132	43	0.335943
0.60	44	0.346172	43	0.326724

Table 5. Numerical data for Experiment 5.

	Algorithm 3		Algorithm 4	
μ_1	n	t	n	t
0.05	46	0.578	45	0.513
0.12	46	0.592	45	0.583
0.23	46	0.553	45	0.567
0.30	46	0.586	45	0.561
0.37	46	0.584	45	0.568
0.42	46	0.581	45	0.566

for Algorithm 3 and 41 iterations for Algorithm 4 at $\alpha_n = 0.25$. The CPU time t exhibits a similar trend. (ii) In both algorithms, $\alpha_n = 0.25$ consistently results in the lowest iteration count and CPU time, making it the most efficient choice for convergence.

Experiment 4. (Impact of the chosen parameter ϑ_n). Numerical results for six distinct values of ϑ_n are presented in Table 4.

Based on the numerical data in Table 4, we have: (i) For Algorithm 3, the performance remains largely unaffected by variations in ϑ_n, both in terms of the number of iterations and CPU time. (ii) For Algorithm 4, higher values of ϑ_n lead to a reduction in the number of iterations and faster convergence in terms of CPU time. This suggests that the range $\vartheta_n \geq 0.50$ is optimal for Algorithm 4, as it results in faster execution and a lower number of iterations.

Experiment 5. (Impact of the chosen parameter μ_1). Numerical results for six different values of μ_1 are displayed in Table 5.

Based on Table 5, we obtain: (i) Both Algorithms 3 and 4 maintain a stable iteration count across all tested values of μ_1. (ii) In terms of CPU time, Algorithm 4 generally performs slightly faster than Algorithm 3, with the lowest CPU time observed at $\mu_1 = 0.05$.

Experiment 6. (Impact of the chosen sequences χ_n, ρ_n, and ξ_n). Table 6 presents the numerical results obtained for ten distinct sets of values for χ_n, ρ_n, and ξ_n.

Based on the numerical results presented in Table 6, several key observations can be made regarding the behaviour and performance of the proposed algorithms under varying values of the parameters $\{\chi_n\}$, $\{\rho_n\}$, and $\{\xi_n\}$: (i) The numerical experiments indicate that a balanced approach to selecting the sequences $\{\chi_n\}$, $\{\rho_n\}$, and $\{\xi_n\}$ is key to optimizing the convergence rate of the proposed algorithms. The optimal sequences typically involve values that are moderate or gradually decreasing, which allows the algorithms to achieve both stability and efficiency. (ii) The most efficient sequences for achieving faster convergence are as follows: $\{\chi_n\}$ should have moderate, slowly decreasing values around $\frac{1}{4}$ or $\frac{1}{3}$;

Table 6. Numerical data corresponding to Experiment 6.

Case	χ_n	ρ_n	ξ_n	Algorithm 3		Algorithm 4	
				n	t	n	t
c1	$\frac{1}{2}+\frac{1}{4k}$	$\frac{1}{3}$	$\frac{1}{6}-\frac{1}{4k}$	77	0.991	73	0.991
c2	$\frac{1}{4}+\frac{1}{4(k+1)}$	$\frac{1}{2}-\frac{1}{4(k+1)}$	$\frac{1}{4}$	50	0.607	48	0.688
c3	$\frac{1}{3}+\frac{1}{6k}$	$\frac{1}{3}-\frac{1}{6k}$	$\frac{1}{3}$	51	0.618	49	0.631
c4	$\frac{1}{3}+\frac{1}{3k}$	$\frac{1}{3}-\frac{1}{6k}$	$\frac{1}{3}-\frac{1}{6k}$	53	0.655	51	0.624
c5	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	41	0.502	41	0.518
c6	$\frac{1}{4}-\frac{1}{12(k+1)}$	$\frac{1}{2}$	$\frac{1}{4}+\frac{1}{12(k+1)}$	48	0.549	47	0.638
c7	$\frac{1}{2}+\frac{1}{4k}$	$\frac{1}{4}$	$\frac{1}{4}-\frac{1}{4k}$	70	0.826	67	0.753
c8	$\frac{3}{10}+\frac{1}{10k}$	$\frac{3}{10}-\frac{1}{10k}$	$\frac{4}{10}$	47	0.537	45	0.518
c9	$\frac{1}{4}+\frac{1}{8k}$	$\frac{1}{2}-\frac{1}{8k}$	$\frac{1}{4}$	49	0.575	48	0.572
c10	$\frac{1}{3}+\frac{1}{6(k+1)}$	$\frac{1}{2}-\frac{1}{6(k+1)}$	$\frac{1}{6}$	58	0.707	57	0.715

$\{\rho_n\}$ should maintain stable values between $\frac{1}{3}$ and $\frac{1}{2}$; and $\{\xi_n\}$ should hold fixed or stable values around $\frac{1}{4}$ or $\frac{1}{3}$. This conclusion is supported by the best-performing cases, such as case c5.

6. Conclusions

In this article, we have introduced two novel iterative algorithms based on the subgradient extragradient implicit approach to solve the monotone bilevel split equilibrium problem (MBSEP), which incorporates the generalized split variational inequality (GSVI) and common fixed point problem (CFPP) constraints. Our approach leverages the subgradient projection onto a constructible half-space, circumventing the need for a second minimization over a closed convex set typically required in traditional methods. Additionally, by employing Mann's implicit iteration scheme, we have developed a new methodology for tackling the GSVI and CFPP within the context of bilevel optimization. This has led to the derivation of iterative algorithms that can efficiently solve the generalized variational inequality (GSVI), variational inequality problem (VIP), and split feasibility problem (SFP), thus extending the scope of applicability of extragradient methods to a wider range of equilibrium problems. Through rigorous theoretical analysis, we have established strong convergence results for the proposed algorithms under suitable conditions.

The current approaches primarily address monotone bilevel split equilibrium problems. However, future work could focus on extending the proposed algorithms to non-monotone settings, where the equilibrium conditions are not guaranteed to satisfy monotonicity. This would require the development of new convergence analysis techniques and the exploration of alternative strategies for non-monotone equilibrium problems. Potential strategies include adapting the algorithms to handle weakly monotone or quasimonotone operators, which could be achieved by modifying the projection steps or incorporating relaxed conditions for convergence. Furthermore, dual inertial methods or other alternative minimization techniques could be integrated to improve convergence in the absence of strict monotonicity.

In conclusion, the methods presented in this work lay a strong foundation for solving complex bilevel equilibrium problems. Future advancements could further expand their applicability and efficiency, particularly in challenging real-world contexts. These include the potential adaptation to non-monotone problems and the incorporation of additional techniques to handle a broader range of equilibrium conditions.

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Disclosure statement

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Author contributions

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Data availability statement

Data sharing is not applicable for this article as no datasets were generated or analyzed during the current study.

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