Commun Nonlinear Sci Numer Simulat 148 (2025) 108851

Contents lists available at ScienceDirect

Communications in Nonlinear Science and Numerical Simulation

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Convergence analysis of subgradient extragradient method with inertial technique for solving variational inequalities and fixed point problems

Danni Guo[a,*], Gang Cai[b], Bing Tan[c]

[a] School of Mathematics, East China University of Science and Technology, Shanghai 200237, China
[b] School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China
[c] School of Mathematics and Statistics, Southwest University, Chongqing 400715, China

ARTICLE INFO

MSC:
47H05
47H07
47H10
54H25

Keywords:
Strong convergence
Subgradient extragradient method
Fixed point
Pseudomonotone operator
Quasinonexpansive mapping

ABSTRACT

The paper presents a new iterative algorithm based on Mann-type subgradient extragradient method to solve pseudomonotone variational inequalities and fixed point problems of quasi-nonexpansive mappings in real Hilbert spaces. Our algorithm, employing inertial technique in each iteration, significantly enhances its convergence. We prove a strong convergence theorem under suitable conditions imposed on the operators and parameters, without prior knowledge of the Lipschitz constant. The efficacy and validity of the proposed method are confirmed through several numerical experiments.

1. Introduction

In this work, let $\mathcal{H}$ be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Assume that C is a nonempty closed convex subset of $\mathcal{H}$. The main purpose of this paper is to construct a new iterative algorithm which has a faster convergence rate than other existing related algorithms to find a common solution of variational inequality problems and fixed point problems. Next, recall that the variational inequality problem (for short, VIP) is defined as finding $x^* \in C$ such that

$$\langle Fx^*, x - x^*\rangle \geq 0, \quad \forall x \in C, \tag{1}$$

where $F : C \to \mathcal{H}$ is a nonlinear operator. Let $\mathrm{VI}(C,F)$ denote the set of all solutions of VIP. In recent years, numerical methods of variational inequalities have attracted extensive attention, many numerical iterative methods have been constructed for solving variational inequality problems and related optimization problems(see [1–14]and references therein). It is clear that a simple method for solving VIP is the following gradient projection method:

$$x_0 \in \mathcal{H}, \ \ x_{n+1} = P_C(x_n - \tau F x_n), \quad \forall n \geq 0, \tag{2}$$

where τ is a parameter that satisfies certain conditions and P_C is the metric projection from $\mathcal{H}$ onto C. However, the operator F is strongly monotone and Lipschitz continuous for guaranteeing the convergence of the sequence generated by (2).

* Corresponding author.
E-mail addresses: Danniguomath@163.com (D. Guo), caigang-aaaa@163.com (G. Cai), bingtan@swu.edu.cn (B. Tan).

https://doi.org/10.1016/j.cnsns.2025.108851
Received 29 November 2024; Received in revised form 18 February 2025; Accepted 6 April 2025
Available online 17 April 2025

In order to weaken the stronger assumption of the operator mentioned above, Korpelevich [15] proposed the following extragradient method in the Euclidean space:

$$\begin{cases} x_0 \in C, \\ t_n = P_C(x_n - \tau F x_n), \\ x_{n+1} = P_C(x_n - \tau F t_n), \end{cases}$$

where $\tau \in (0, \frac{1}{L})$, the associated mapping F is monotone and L-Lipschitz continuous. More precisely, the sequence $\{x_n\}$ generated by the above algorithm converges weakly to an element of $\mathrm{VI}(C, F)$. It is noted that this method needs us to compute twice the projections onto C in each iteration, and this will take a lot of computation time in numerical experiments.

Afterwards, many authors have created many methods to overcome this disadvantage (see [16–22] and references therein). For instance, Censor et al. [23] proposed subgradient extragradient method as follows:

$$\begin{cases} t_n = P_C(x_n - \tau F x_n), \\ T_n = \{x \in \mathcal{H} \mid \langle x_n - \tau F x_n - t_n, x - t_n \rangle \le 0\}, \\ x_{n+1} = P_{T_n}(x_n - \tau F t_n), \end{cases}$$

where $\tau \in (0, \frac{1}{L})$ and the algorithm achieved a weakly convergent conclusion. In addition, it should be noted that the closed convex set C is replaced by a specific constructible half-space in the second projection of the above algorithm. In this way, this method significantly reduces the difficulty of calculations in numerical experiments. Furthermore, this method was further extended to equilibrium problems and other optimization problems(see, for examples, [24–29]).

Let $T : \mathcal{H} \to \mathcal{H}$ be a nonlinear mapping. A point $x \in \mathcal{H}$ is called a fixed point of mapping $\mathcal{H}$ if $Tx = x$. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$:

$$\mathrm{Fix}(T) := \{x \in \mathcal{H} \mid Tx = x\}.$$

It is well known that variational inequality problems can be transformed into fixed point problems in Hilbert space. Very recently, many iterative methods have been proposed for finding a common element of VIP and fixed point problem in Hilbert spaces(see [30–34]and references therein). For example, Thong and Hieu [35] introduced a algorithm 1 as follows:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ t_n = P_C(w_n - \tau_n F w_n), \\ z_n = P_{T_n}(w_n - \tau_n F t_n), \\ T_n := \{x \in \mathcal{H} \mid \langle w_n - \tau_n F w_n - t_n, x - t_n \rangle \le 0\}, \\ x_{n+1} = (1 - \beta_n) w_n + \beta_n T z_n, \end{cases}$$

where τ_n is chosen to be the largest $\tau \in \{\lambda, \lambda l, \lambda l^2, \dots\}$ satisfying

$$\tau \|F w_n - F t_n\| \le \mu \|w_n - t_n\|.$$

The sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to $z \in \mathrm{VI}(C, F) \cap \mathrm{Fix}(T)$ when $F : \mathcal{H} \to \mathcal{H}$ is monotone and L-Lipschitz continuous mapping. Note that two identical stepsizes are used in Algorithm 1, which may affect the convergence speed of the algorithm. In order to ensure the convergence of Algorithm 1, its correlation mapping is required to be monotonic. It is well known that the monotonicity can deduce the pseudo-monotone, but not vice versa.

Inspired by the above ideas, this paper prove a strong convergence theorem under several suitable conditions imposed on the operators and parameters. It should be emphasized that the proposed algorithm adopts two linearly related step sizes to improve the convergence performance of the algorithm. Meanwhile, we also give several practical numerical examples to illustrate the efficiency of the presented algorithm.

2. Preliminaries

In a real Hilbert space $\mathcal{H}$, it is clear that

$$\begin{aligned} \|\alpha x + \beta t + \gamma z\|^2 = {} & \alpha\|x\|^2 + \beta\|t\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - t\|^2 \\ & - \alpha\gamma\|x - z\|^2 - \beta\gamma\|t - z\|^2 \end{aligned}$$

and

$$\|x + t\|^2 \le \|x\|^2 + 2\langle t, x + t\rangle$$

for every $x,\ t,\ z \in \mathcal{H}$ and $\alpha,\ \beta,\ \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$.

For every point $x \in \mathcal{H}$, it is obvious that there exists a unique nearest point in C, which is denoted by $P_C x$ satisfying

$$\|x - P_C x\| \le \|x - t\|, \quad \forall t \in C.$$

We call P_C the metric projection.

Now we recall some useful definitions and facts. A mapping $T : \mathcal{H} \to \mathcal{H}$ is called to be:
L-Lipschitz continuous with a constant $L > 0$ if

$$\|Tx - Tt\| \leq L\|x - t\|, \quad \forall x,\ t \in \mathcal{H}.$$

if $L \in (0, 1)$, then T is called a contraction. nonexpansive if

$$\|Tx - Tt\| \leq \|x - t\|, \quad \forall x,\ t \in \mathcal{H}.$$

quasi-nonexpansive if $\mathrm{Fix}(T) \neq \emptyset$,

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in \mathcal{H},\ p \in \mathrm{Fix}(T).$$

monotone if

$$\langle Tx - Tt, x - t\rangle \geq 0, \quad \forall x,\ t \in \mathcal{H}.$$

pseudo-monotone if

$$\langle Tx, t - x\rangle \geq 0 \Rightarrow \langle Tt, t - x\rangle \geq 0, \quad \forall x,\ t \in \mathcal{H}.$$

We say that T is sequentially weakly continuous if for every sequence $\{x_n\}$ satisfying $x_n \rightharpoonup x$, then we obtain $Tx_n \rightharpoonup Tx$. The following lemmas are very important for proving our main results.

Lemma 1 ([36]). *Let $\mathcal{H}$ be a real Hilbert space and C be a nonempty closed convex subset of $\mathcal{H}$. Let $z \in C$ and $x \in \mathcal{H}$, we have*

$$z = P_C x \Leftrightarrow \langle x - z, z - t\rangle \geq 0, \quad \forall t \in C.$$

Lemma 2 ([36]). *Suppose that C is a closed and convex subset in a real Hilbert space $\mathcal{H}$ and given $x \in \mathcal{H}$. We get*
(1) $\|P_C x - P_C t\|^2 \leq \langle P_C x - P_C t, x - t\rangle, \quad \forall t \in \mathcal{H}$,
(2) $\|P_C x - t\|^2 \leq \|x - t\|^2 - \|x - P_C x\|^2, \quad \forall t \in C$.

Lemma 3 ([37]). *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $F : C \to \mathcal{H}$ be a pseudo-monotone and continuous operator. Then, $t^* \in VI(C, F)$ if and only if*

$$\langle Ft, t - t^*\rangle \geq 0, \quad \forall t \in C.$$

Lemma 4 ([38]). *Let $\{b_n\}$ be a sequence of positive real numbers such that there exists a subsequence $\{b_{n_i}\}$ of $\{b_n\}$ satisfying $b_{n_i} < b_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_k\}$ of $\mathbb{N}$ satisfying $\lim_{k\to\infty} m_k = \infty$, at the same time, for all sufficiently large number $k \in \mathbb{N}$, we have*

$$b_{m_k} \leq b_{m_k+1} \text{ and } b_k \leq b_{m_k+1}.$$

Lemma 5 ([39]). *Let $\{x_n\}$ be a nonnegative real sequence such that*

$$x_{n+1} \leq (1 - \alpha_n)x_n + \alpha_n\beta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:
(a) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(b) $\limsup_{n\to\infty} \beta_n \leq 0$.
Then $\lim_{n\to\infty} x_n = 0$.

Lemma 6 ([40]). *Assume that $F : \mathcal{H} \to \mathcal{H}$ is a monotone and L-Lipschitz continuous operator. Let $T = P_C(I - \rho F)$, $\rho > 0$. Suppose that $\{x_n\}$ is a sequence in $\mathcal{H}$ such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x \in \mathrm{VI}(C, F) = \mathrm{Fix}(T)$.*

3. Main results

In this section, we introduce a modified Mann-type subgradient extragradient method for solving the problem (VIP) and the fixed point problem of a nonexpansive mapping. Now we provide the following assumptions.

Condition 1. C is a nonempty closed and convex subset of a real Hilbert space $\mathcal{H}$.

Condition 2. The mapping $F : \mathcal{H} \to \mathcal{H}$ is pseudo-monotone, L-Lipschitz continuous on $\mathcal{H}$ and sequentially weakly continuous on bounded subsets of C.

Condition 3. The solution set $\mathrm{VI}(C, F) \cap \mathrm{Fix}(T) \neq \emptyset$.

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0,1)$ satisfying $\{\beta_n\} \subset (a,b) \subset (0, 1-\alpha_n)$ for some $a, b > 0$ and assume that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Let $\{\theta_n\} \in (0,\theta)$ from some $\theta > 0$ such that $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Next we introduce our algorithm.

Algorithm 2

Initialization: Take $\tau_0 > 0$, $\mu \in (0,1)$, $\eta \in (\mu, 1]$. Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.
Iterative Steps: Calculate x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and $x_n (n \geq 1)$. Set

$$w_n = x_n + \theta_n (x_n - x_{n-1}).$$

Step 2. Compute

$$t_n = P_C(w_n - \frac{\tau_n}{\eta} F w_n),$$
$$z_n = P_{T_n}(w_n - \tau_n F t_n),$$

where $T_n := \{x \in \mathcal{H} \mid \langle w_n - \frac{\tau_n}{\eta} F w_n - t_n, x - t_n\rangle \leq 0\}$ and

$$\tau_{n+1} = \begin{cases} \min\left\{ \mu \dfrac{\|w_n - t_n\|^2 + \|z_n - t_n\|^2}{2\langle F w_n - F t_n, z_n - t_n\rangle}, \tau_n \right\}, & \text{if } \langle F w_n - F t_n, z_n - t_n\rangle > 0, \\ \tau_n, & \text{otherwise}. \end{cases} \tag{3}$$

Step 3. Compute

$$x_{n+1} = (1 - \alpha_n - \beta_n) w_n + \beta_n T z_n.$$

Set $n := n + 1$ and go to **Step 1**.

The following lemmas are quite helpful to obtain the convergence of our proposed algorithm.

Lemma 7. *Assume that Conditions 1–3 hold. The sequence $\{\tau_n\}$ generated by (3) is a non-increasing sequence and*

$$\lim_{n\to\infty} \tau_n = \tau \geq \min\{\tau_0, \frac{\mu}{L}\}.$$

Proof. From (3), we have $\tau_n \geq \min\{\tau_0, \frac{\mu}{L}\}$ for all $n \in \mathbb{N}$. In fact, since F is L-Lipschitz continuous on $\mathcal{H}$, we obtain $\|F w_n - F t_n\| \leq L\|w_n - t_n\|$. Hence, if $\langle F w_n - F t_n, z_n - t_n\rangle > 0$, we get

$$\begin{aligned} \mu \frac{\|w_n - t_n\|^2 + \|z_n - t_n\|^2}{2\langle F w_n - F t_n, z_n - t_n\rangle} &\geq \mu \frac{\|w_n - t_n\|\|z_n - t_n\|}{\|F w_n - F t_n\|\|z_n - t_n\|} \\ &= \mu \frac{\|w_n - t_n\|}{\|F w_n - F t_n\|} \\ &\geq \frac{\mu}{L}, \end{aligned}$$

which together with (3) yields

$$\tau_n \geq \min\{\tau_0, \frac{\mu}{L}\}. \tag{4}$$

Therefore, it is obvious that the sequence $\{\tau_n\}$ is non-increasing and lower bounded. Therefore $\lim_{n\to\infty} \tau_n = \tau \geq \min\{\tau_0, \frac{\mu}{L}\}$. This completes the proof. $\square$

Lemma 8. *Assume that Conditions 1–3 hold. Let $\{z_n\}$, $\{w_n\}$ and $\{t_n\}$ be the sequences generated by Algorithm 2. Then*

$$\begin{aligned} &\|z_n - p\|^2 \\ &\leq \|w_n - p\|^2 - (\eta - \frac{\mu\tau_n}{\tau_{n+1}})\|w_n - t_n\|^2 - (\eta - \frac{\mu\tau_n}{\tau_{n+1}})\|t_n - z_n\|^2 \\ &\quad - (1-\eta)\|z_n - w_n\|^2, \quad \forall p \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T). \end{aligned} \tag{5}$$

Proof. By Lemma 2, we obtain

$$
\begin{aligned}
&\|z_n - p\|^2 \\
&= \|P_{T_n}(w_n - \tau_n F t_n) - p\|^2 \\
&\le \|w_n - \tau_n F t_n - p\|^2 - \|w_n - \tau_n F t_n - z_n\|^2 \\
&= \|w_n - p\|^2 + \tau_n^2 \|F t_n\|^2 - 2\tau_n \langle w_n - p, F t_n\rangle - \|w_n - z_n\|^2 \\
&\quad - \tau_n^2 \|F t_n\|^2 + 2\tau_n \langle w_n - z_n, F t_n\rangle \\
&= \|w_n - p\|^2 - \|w_n - z_n\|^2 - 2\tau_n \langle F t_n, z_n - p\rangle \\
&= \|w_n - p\|^2 - \|w_n - z_n\|^2 - 2\tau_n \langle F t_n, z_n - t_n + t_n - p\rangle \\
&= \|w_n - p\|^2 - \|w_n - z_n\|^2 - 2\tau_n \langle F t_n, z_n - t_n\rangle - 2\tau_n \langle F t_n, t_n - p\rangle.
\end{aligned}
\tag{6}
$$

Since $t_n \in C$ and $p \in \mathrm{VI}(C, F)$, we obtain

$$\langle F p, t_n - p\rangle \ge 0.$$

Since F is a pseudo-monotone operator, we have

$$\langle F t_n, t_n - p\rangle \ge 0. \tag{7}$$

Hence, we obtain

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \|w_n - z_n\|^2 - 2\tau_n \langle F t_n, z_n - t_n\rangle. \tag{8}$$

For $\|w_n - z_n\|^2 + 2\tau_n \langle F t_n, z_n - t_n\rangle$, we have

$$
\begin{aligned}
&\|w_n - z_n\|^2 + 2\tau_n \langle F t_n, z_n - t_n\rangle \\
&= \|w_n - t_n + t_n - z_n\|^2 + 2\tau_n \langle F t_n, z_n - t_n\rangle \\
&= \|w_n - t_n\|^2 + \|t_n - z_n\|^2 + 2\langle w_n - t_n, t_n - z_n\rangle + 2\tau_n \langle F t_n, z_n - t_n\rangle \\
&= \|w_n - t_n\|^2 + \|t_n - z_n\|^2 + 2\langle t_n - w_n + \tau_n F t_n, z_n - t_n\rangle \\
&= 2\langle t_n - w_n + \frac{\tau_n}{\eta} F w_n - \frac{\tau_n}{\eta} F w_n + \frac{\tau_n}{\eta} F t_n - \frac{\tau_n}{\eta} F t_n + \tau_n F t_n, z_n - t_n\rangle \\
&\quad + \|w_n - t_n\|^2 + \|t_n - z_n\|^2 \\
&= \|w_n - t_n\|^2 + \|t_n - z_n\|^2 + 2\langle t_n - w_n + \frac{\tau_n}{\eta} F w_n, z_n - t_n\rangle \\
&\quad - \frac{2\tau_n}{\eta} \langle F w_n - F t_n, z_n - t_n\rangle + 2\tau_n (1 - \frac{1}{\eta}) \langle F t_n, z_n - t_n\rangle.
\end{aligned}
\tag{9}
$$

Since $t_n = P_C(w_n - \frac{\tau_n}{\eta} F w_n)$ and $z_n \in T_n$, we obtain

$$\langle w_n - \frac{\tau_n}{\eta} F w_n - t_n, z_n - t_n\rangle \le 0.$$

It follows that

$$
\begin{aligned}
&\|w_n - z_n\|^2 + 2\tau_n \langle F t_n, z_n - t_n\rangle \\
&\ge \|w_n - t_n\|^2 + \|t_n - z_n\|^2 - \frac{2\tau_n}{\eta} \langle F w_n - F t_n, z_n - t_n\rangle \\
&\quad + 2\tau_n (1 - \frac{1}{\eta}) \langle F t_n, z_n - t_n\rangle.
\end{aligned}
\tag{10}
$$

Thus, we have

$$
\begin{aligned}
&\frac{2\tau_n}{\eta} \langle F t_n, z_n - t_n\rangle \\
&\ge \|w_n - t_n\|^2 + \|t_n - z_n\|^2 - \|w_n - z_n\|^2 - \frac{2\tau_n}{\eta} \langle F w_n - F t_n, z_n - t_n\rangle,
\end{aligned}
$$

that is,

$$
\begin{aligned}
&2\tau_n \langle F t_n, z_n - t_n\rangle \\
&\ge \eta \|w_n - t_n\|^2 + \eta \|t_n - z_n\|^2 - \eta \|w_n - z_n\|^2 - 2\tau_n \langle F w_n - F t_n, z_n - t_n\rangle.
\end{aligned}
\tag{11}
$$

Substituting (11) into (8), we have

$$
\begin{aligned}
&\|z_n - p\|^2 \\
&\le \|w_n - p\|^2 - (1 - \eta) \|w_n - z_n\|^2 - \eta \|w_n - t_n\|^2
\end{aligned}
$$

$$-\eta\|z_n - t_n\|^2 + 2\tau_n\langle Fw_n - Ft_n, z_n - t_n\rangle. \tag{12}$$

By the definition of $\{\tau_n\}$, it is clear that

$$2\langle Fw_n - Ft_n, z_n - t_n\rangle \le \frac{\mu}{\tau_{n+1}}\|w_n - t_n\|^2 + \frac{\mu}{\tau_{n+1}}\|z_n - t_n\|^2. \tag{13}$$

Indeed, if $\langle Fw_n - Ft_n, z_n - t_n\rangle \le 0$, then the inequality (13) holds. Otherwise, from (3), we get

$$\tau_{n+1} \le \mu\frac{\|w_n - t_n\|^2 + \|z_n - t_n\|^2}{2\langle Fw_n - Ft_n, z_n - t_n\rangle},$$

which implies

$$2\langle Fw_n - Ft_n, z_n - t_n\rangle \le \frac{\mu}{\tau_{n+1}}\|w_n - t_n\|^2 + \frac{\mu}{\tau_{n+1}}\|z_n - t_n\|^2.$$

Therefore, the inequality (13) holds. Combining (12) and (13), we have

$$\begin{aligned}
&\|z_n - p\|^2 \\
&\le \|w_n - p\|^2 - (\eta - \frac{\mu\tau_n}{\tau_{n+1}})\|w_n - t_n\|^2 \\
&\quad - (\eta - \frac{\mu\tau_n}{\tau_{n+1}})\|t_n - z_n\|^2 - (1-\eta)\|z_n - w_n\|^2.
\end{aligned}$$

This completes the proof. □

By using Lemmas 6 and 8, we can obtain the following results. Its proof is similar to Lemma 3.3 of Thong et al. [41], we omit the details.

Lemma 9 ([41]). *Assume that Conditions 1–3 hold. Let $\{w_n\}$ be a sequence generated by Algorithm 2. If there exists a subsequence $\{w_{n_k}\}$ converges weakly to $z \in \mathcal{H}$ and $\lim_{k\to\infty}\|w_{n_k} - t_{n_k}\| = 0$, then $z \in \mathrm{VI}(C,F)$.*

Now we state the main results of this section.

Theorem 1. *Assume that Conditions 1–3 hold and the sequence $\{\theta_n\}$ is chosen such that*

$$\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0.$$

Then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to an element $p \in \mathrm{VI}(C,F)\cap\mathrm{Fix}(T)$, where $\|p\| = \min\{\|z\| : z \in \mathrm{VI}(C,F)\cap\mathrm{Fix}(T)\}$.

Proof. We divide the proof into four claims.

Claim 1. *We prove that the sequence $\{x_n\}$ is bounded. Indeed, by Lemma 9, there exists a $N_1 \in \mathbb{N}$ such that*

$$\|z_n - p\| \le \|w_n - p\|, \quad \forall n \ge N_1 \text{ and } p \in \mathrm{VI}(C,F)\cap\mathrm{Fix}(T). \tag{14}$$

It follows that for all $n \geqslant N_1$

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n-\beta_n)w_n + \beta_n Tz_n - p\| \\
&= \|(1-\alpha_n-\beta_n)(w_n - p) + \beta_n(Tz_n - p) - \alpha_n p\| \\
&\le \|(1-\alpha_n-\beta_n)(w_n - p) + \beta_n(Tz_n - p)\| + \alpha_n\|p\| \\
&\le (1-\alpha_n-\beta_n)\|w_n - p\| + \beta_n\|Tz_n - p\| + \alpha_n\|p\| \\
&\le (1-\alpha_n-\beta_n)\|w_n - p\| + \beta_n\|z_n - p\| + \alpha_n\|p\| \\
&\le (1-\alpha_n-\beta_n)\|w_n - p\| + \beta_n\|w_n - p\| + \alpha_n\|p\| \\
&\le (1-\alpha_n)\|w_n - p\| + \alpha_n\|p\|.
\end{aligned} \tag{15}$$

It follows from the definition of $\{w_n\}$ that

$$\begin{aligned}
\|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\
&\le \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \\
&\le \|x_n - p\| + \alpha_n\cdot\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|.
\end{aligned} \tag{16}$$

Since

$$\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0, \tag{17}$$

there exist constants $M_1 > 0$ *and* $N_2 > 0$ *such that* $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \le M_1$ *for all* $n \ge N_2$*. By (16), we have*

$$\|w_n - p\| \le \|x_n - p\| + \alpha_n M_1. \tag{18}$$

Let $N = \max\{N_1, N_2\}$ *and for any* $n \ge N$*, substituting (18) into (15), we obtain*

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\le (1-\alpha_n)\|x_n - p\| + (1-\alpha_n)\alpha_n M_1 + \alpha_n\|p\| \\
&\le (1-\alpha_n)\|x_n - p\| + \alpha_n M_2 \\
&\le \max\{\|x_n - p\|, M_2\} \\
&\le \cdots \\
&\le \max\{\|x_N - p\|, M_2\},
\end{aligned}$$

where $M_2 = M_1 + \|p\|$*. This implies that the sequence* $\{x_n\}$ *is bounded. Hence, the sequences* $\{z_n\}$ *and* $\{w_n\}$ *are also bounded. In the rest of the proof, we may assume, without loss of generality,* $N = 1$*.*

Claim 2.

$$\begin{aligned}
&\beta_n\Big(\eta - \frac{\mu\tau_n}{\tau_{n+1}}\Big)\|w_n - t_n\|^2 + \beta_n\Big(\eta - \frac{\mu\tau_n}{\tau_{n+1}}\Big)\|t_n - z_n\|^2 \\
&\quad + \beta_n(1-\eta)\|z_n - w_n\|^2 + \beta_n(1-\alpha_n-\beta_n)\|w_n - Tz_n\|^2 \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4,
\end{aligned} \tag{19}$$

for some $M_4 > 0$*. Indeed, since* T *is quasi-nonexpansive mapping, we have*

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|(1-\alpha_n-\beta_n)w_n + \beta_n Tz_n - p\|^2 \\
&= \|(1-\alpha_n-\beta_n)(w_n - p) + \beta_n(Tz_n - p) - \alpha_n p\|^2 \\
&\le (1-\alpha_n-\beta_n)\|w_n - p\|^2 + \beta_n\|Tz_n - p\|^2 \\
&\quad + \alpha_n\|p\|^2 - \beta_n(1-\alpha_n-\beta_n)\|w_n - Tz_n\|^2 \\
&\le (1-\alpha_n-\beta_n)\|w_n - p\|^2 + \beta_n\|z_n - p\|^2 \\
&\quad + \alpha_n\|p\|^2 - \beta_n(1-\alpha_n-\beta_n)\|w_n - Tz_n\|^2.
\end{aligned} \tag{20}$$

From (18), we have

$$\begin{aligned}
\|w_n - p\|^2 &\le (\|x_n - p\| + \alpha_n M_1)^2 \\
&= \|x_n - p\|^2 + \alpha_n(2M_1\|x_n - p\| + \alpha_n M_1^2) \\
&\le \|x_n - p\|^2 + \alpha_n M_3,
\end{aligned} \tag{21}$$

where $M_3 = \sup_{n\ge 1}(2M_1\|x_n - p\| + \alpha_n M_1^2)$*. Combining (5), (20) and (21), we obtain*

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le (1-\alpha_n-\beta_n)\|w_n - p\|^2 + \beta_n\|w_n - p\|^2 - \beta_n(1-\alpha_n-\beta_n)\|w_n - Tz_n\|^2 \\
&\quad - \beta_n\Big(\eta - \frac{\mu\tau_n}{\tau_{n+1}}\Big)\|t_n - z_n\|^2 - \beta_n(1-\eta)\|z_n - w_n\|^2 \\
&\quad - \beta_n\Big(\eta - \frac{\mu\tau_n}{\tau_{n+1}}\Big)\|w_n - t_n\|^2 + \alpha_n\|p\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 - \beta_n\Big(\eta - \frac{\mu\tau_n}{\tau_{n+1}}\Big)\|w_n - t_n\|^2 \\
&\quad - \beta_n(1-\alpha_n-\beta_n)\|w_n - Tz_n\|^2 - \beta_n\Big(\eta - \frac{\mu\tau_n}{\tau_{n+1}}\Big)\|t_n - z_n\|^2 \\
&\quad - \beta_n(1-\eta)\|z_n - w_n\|^2 + \alpha_n\|p\|^2 + \alpha_n M_3,
\end{aligned}$$

which implies

$$\begin{aligned}
&\beta_n\Big(\eta - \frac{\mu\tau_n}{\tau_{n+1}}\Big)\|w_n - t_n\|^2 + \beta_n\Big(\eta - \frac{\mu\tau_n}{\tau_{n+1}}\Big)\|t_n - z_n\|^2 \\
&\quad + \beta_n(1-\eta)\|z_n - w_n\|^2 + \beta_n(1-\alpha_n-\beta_n)\|w_n - Tz_n\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|p\|^2 + \alpha_n M_3 \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4,
\end{aligned}$$

where $M_4 = M_3 + \|p\|^2$*.*

Claim 3.

$$\begin{aligned}&\|x_{n+1}-p\|^2\\&\le(1-\alpha_n)\|x_n-p\|^2\\&\quad+\alpha_n[2\beta_n\|w_n-Tz_n\|\|x_{n+1}-p\|+\frac{M\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+2\langle p,p-x_{n+1}\rangle],\end{aligned}\tag{22}$$

where $M=\sup_{n\ge1}(2\|w_n-p\|)$. *Indeed, by the definition of* $\{w_n\}$, *we get*

$$\begin{aligned}\|w_n-p\|^2&=\|x_n+\theta_n(x_n-x_{n-1})-p\|^2\\&\le\|x_n-p\|^2+2\theta_n\langle x_n-x_{n-1},w_n-p\rangle\\&\le\|x_n-p\|^2+2\theta_n\|x_n-x_{n-1}\|\|w_n-p\|\\&\le\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|M.\end{aligned}\tag{23}$$

It follows that

$$\begin{aligned}&\|x_{n+1}-p\|^2\\&=\|(1-\alpha_n-\beta_n)w_n+\beta_nTz_n-p\|^2\\&=\|(1-\beta_n)w_n+\beta_nTz_n-\alpha_nw_n-p\|^2\\&=\|(1-\alpha_n)[(1-\beta_n)w_n+\beta_nTz_n-p]-\alpha_n\beta_n(w_n-Tz_n)-\alpha_np\|^2\\&\le(1-\alpha_n)^2\|(1-\beta_n)w_n+\beta_nTz_n-p\|^2\\&\quad+2\langle\alpha_n\beta_n(w_n-Tz_n)+\alpha_np,p-x_{n+1}\rangle\\&=(1-\alpha_n)^2\|(1-\beta_n)(w_n-p)+\beta_n(Tz_n-p)\|^2+2\alpha_n\langle p,p-x_{n+1}\rangle\\&\quad+2\alpha_n\langle\beta_n(w_n-Tz_n),p-x_{n+1}\rangle\\&\le(1-\alpha_n)[(1-\beta_n)\|w_n-p\|+\beta_n\|Tz_n-p\|]^2+2\alpha_n\langle p,p-x_{n+1}\rangle\\&\quad+2\alpha_n\beta_n\|w_n-Tz_n\|\|p-x_{n+1}\|\\&\le(1-\alpha_n)[(1-\beta_n)\|w_n-p\|^2+\beta_n\|z_n-p\|^2]\\&\quad+\alpha_n[2\beta_n\|w_n-Tz_n\|\|x_{n+1}-p\|+2\langle p,p-x_{n+1}\rangle]\\&\le(1-\alpha_n)\|w_n-p\|^2+\alpha_n[2\beta_n\|w_n-Tz_n\|\|x_{n+1}-p\|+2\langle p,p-x_{n+1}\rangle].\end{aligned}\tag{24}$$

By (23) *and* (24), *we have*

$$\begin{aligned}&\|x_{n+1}-p\|^2\\&\le(1-\alpha_n)\|x_n-p\|^2+\alpha_n[2\beta_n\|w_n-Tz_n\|\|x_{n+1}-p\|\\&\quad+\frac{M\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+2\langle p,p-x_{n+1}\rangle].\end{aligned}$$

Claim 4. *Finally, we prove that the sequence* $x_n\to p$ *as* $n\to\infty$. *In fact, we consider two possible cases.*

Case 1 *There exists a* $n_0\in\mathbb{N}$ *satisfying* $\{\|x_{n+1}-p\|^2\le\|x_n-p\|^2\},\forall n\ge n_0$. *Therefore,* $\lim_{n\to\infty}\|x_n-p\|$ *exists. Noticing the conditions* $\lim_{n\to\infty}\alpha_n=0$, $\{\beta_n\}\subset(a,b)\subset(0,1-\alpha_n)$ *and* $\lim_{n\to\infty}(\eta-\frac{\mu\tau_n}{\tau_{n+1}})=\eta-\mu>0$, *from* (19) *we obtain*

$$\lim_{n\to\infty}\|w_n-t_n\|=0,\tag{25}$$

$$\lim_{n\to\infty}\|t_n-z_n\|=0,\tag{26}$$

and

$$\lim_{n\to\infty}\|w_n-Tz_n\|=0.\tag{27}$$

Combining (25) *and* (26), *we have*

$$\|w_n-z_n\|\le\|w_n-t_n\|+\|t_n-z_n\|\to0,\tag{28}$$

as $n\to\infty$. *By* (27) *and* (28), *we get*

$$\|Tz_n-z_n\|\le\|Tz_n-w_n\|+\|w_n-z_n\|\to0,\tag{29}$$

as $n\to\infty$. *By the definition of* $\{w_n\}$, *we get*

$$\|w_n-x_n\|=\theta_n\|x_n-x_{n-1}\|=\alpha_n\cdot\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\to0,\tag{30}$$

as $n \to \infty$. *Using* (28) *and* (30), *we obtain*

$$\|z_n - x_n\| \le \|z_n - w_n\| + \|w_n - x_n\| \to 0. \tag{31}$$

On the other hand, by $\lim_{n\to\infty} \alpha_n = 0$, *we have*

$$\|x_{n+1} - w_n\| \le \beta_n\|Tz_n - w_n\| + \alpha_n\|w_n\| \to 0, \tag{32}$$

as $n \to \infty$. *Combining* (30) *and* (32), *we obtain*

$$\|x_{n+1} - x_n\| \le \|x_{n+1} - w_n\| + \|w_n - x_n\| \to 0, \tag{33}$$

as $n \to \infty$. *Since* $\{x_n\}$ *is bounded, we assume that there exists a subsequence* $\{x_{n_i}\}$ *of* $\{x_n\}$ *such that* $x_{n_i} \rightharpoonup q$ *and*

$$\limsup_{n\to\infty}\langle p, p - x_n\rangle = \limsup_{i\to\infty}\langle p, p - x_{n_i}\rangle = \langle p, p - q\rangle.$$

It follows from Lemma 9 *and* (25) *that* $q \in \mathrm{VI}(C, F)$. *From* (31), *we have* $z_{n_i} \rightharpoonup q$. *By* Lemma 6 *and* (31), *we obtain* $q \in \mathrm{Fix}(T)$. *Hence we have* $q \in \mathrm{VI}(C, F) \cap \mathrm{Fix}(T)$. *Since* $q \in \mathrm{VI}(C, F) \cap \mathrm{Fix}(T)$ *and* $\|p\| = \min\{\|z\| : z \in \mathrm{VI}(C, F) \cap \mathrm{Fix}(T)\}$, *that is* $p = P_{\mathrm{VI}(C,F)\cap\mathrm{Fix}(T)}0$, *we have*

$$\limsup_{n\to\infty}\langle p, p - x_n\rangle = \langle p, p - q\rangle \le 0.$$

By (33), *we obtain*

$$\limsup_{n\to\infty}\langle p, p - x_{n+1}\rangle \le 0.$$

Hence, by Claim 3 *and* Lemma 5, *we have* $\lim_{n\to\infty}\|x_n - p\|^2 = 0$; *that is* $\lim_{k\to\infty} x_n = p$.

Case 2 *There exists a subsequence* $\{\|x_{n_i} - p\|^2\}$ *of* $\{\|x_n - p\|^2\}$ *satisfying* $\|x_{n_i} - p\|^2 < \|x_{n_{i+1}} - p\|^2, \forall\, i \in \mathbb{N}$. *By* Lemma 4, *there exists a non-decreasing sequence* $\{m_k\}$ *of* $\mathbb{N}$ *such that* $\lim_{k\to\infty} m_k = \infty$ *and we have*

$$\|x_{m_k} - p\|^2 \le \|x_{m_{k+1}} - p\|^2 \text{ and } \|x_k - p\|^2 \le \|x_{m_{k+1}} - p\|^2.$$

By Claim 2, *we have*

$$\begin{aligned}
&\beta_{m_k}(\eta - \frac{\mu\tau_{m_k}}{\tau_{m_k+1}})\|w_{m_k} - t_{m_k}\|^2 + \beta_{m_k}(\eta - \frac{\mu\tau_{m_k}}{\tau_{m_k+1}})\|t_{m_k} - z_{m_k}\|^2\\
&\quad + \beta_{m_k}(1-\eta)\|z_{m_k} - w_{m_k}\|^2 + \beta_{m_k}(1 - \alpha_{m_k} - \beta_{m_k})\|w_{m_k} - Tz_{m_k}\|^2\\
&\le \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + \alpha_{m_k}M_4\\
&\le \alpha_{m_k}M_4.
\end{aligned}$$

Thus, we obtain

$$\lim_{k\to\infty}\|w_{m_k} - t_{m_k}\| = 0,$$

$$\lim_{k\to\infty}\|t_{m_k} - z_{m_k}\| = 0,$$

and

$$\lim_{k\to\infty}\|w_{m_k} - Tz_{m_k}\| = 0.$$

In a similar way, we have

$$\limsup_{k\to\infty}\|x_{m_k+1} - x_{m_k}\| = 0,$$

and

$$\limsup_{k\to\infty}\langle p, p - x_{m_k+1}\rangle \le 0.$$

According to Claim 3, *we obtain*

$$\begin{aligned}
&\|x_{m_k+1} - p\|^2\\
&\le (1-\alpha_{m_k})\|x_{m_k} - p\|^2 + \alpha_{m_k}[2\beta_{m_k}\|w_{m_k} - Tz_{m_k}\|\|x_{m_k+1} - p\|\\
&\quad + \frac{M\theta_{m_k}}{\alpha_{m_k}}\|x_{m_k} - x_{m_k-1}\| + 2\langle p, p - x_{m_k+1}\rangle]\\
&\le (1-\alpha_{m_k})\|x_{m_k+1} - p\|^2 + \alpha_{m_k}[2\beta_{m_k}\|w_{m_k} - Tz_{m_k}\|\|x_{m_k+1} - p\|\\
&\quad + \frac{M\theta_{m_k}}{\alpha_{m_k}}\|x_{m_k} - x_{m_k-1}\| + 2\langle p, p - x_{m_k+1}\rangle].
\end{aligned}$$

This implies that

$$\|x_k - p\|^2$$

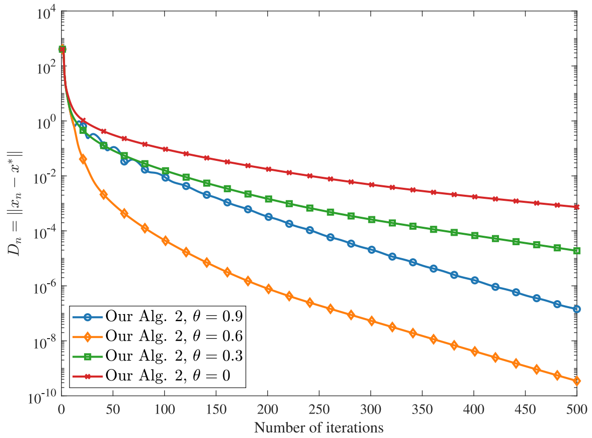


Fig. 1. Numerical results for Example 1.

$$
\begin{aligned}
&\leq \|x_{m_k+1} - p\|^2 \\
&\leq 2\beta_{m_k}\|w_{m_k} - Tz_{m_k}\|\|x_{m_k+1} - p\| + \frac{M\theta_{m_k}}{\alpha_{m_k}}\|x_{m_k} - x_{m_k-1}\| \\
&\quad + 2\langle p, p - x_{m_k+1}\rangle.
\end{aligned}
$$

Hence, we have $\lim_{k\to\infty}\|x_k - p\|^2 = 0$*; that is* $\lim_{k\to\infty} x_k = p$*. This completes the proof.* □

Remark 1. We note that the condition in our theorem is implemented easily in the numerical computation since the value of $\|x_n - x_{n-1}\|$ is known before choosing θ_n. In fact, the parameter θ_n can be chosen such that

$$
\theta_n = \begin{cases} \min\{\frac{\epsilon_n}{\|x_n - x_{n-1}\|}, \theta\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise}, \end{cases} \tag{34}
$$

where θ is a constant such that $0 < \theta < 1$ and $\{\epsilon_n\}$ is a positive sequence such that $\lim_{n\to\infty}\frac{\epsilon_n}{\alpha_n} = 0$.

4. Numerical experiments

In this section, we provide several numerical examples to demonstrate the efficiency of our algorithm compared to some known ones. All the programs were implemented in MATLAB 2018a on a Intel(R) Core(TM) i5-8250S CPU @1.60 GHz computer with RAM 8.00 GB. We apply the formula (34) to select the inertial parameter $\{\theta_n\}$ in the proposed Algorithm 2.

Example 1. Define the linear operator $F : \mathbb{R}^m \to \mathbb{R}^m$ $(m = 20)$ by $F(x) = Mx + q$, here $q \in \mathbb{R}^m$ and $M = NN^\top + Q + D$, N is an $m\times m$ matrix, Q is a $m\times m$ skew-symmetric matrix, and D is a $m\times m$ diagonal matrix with its diagonal entries being nonnegative (therefore M is positive symmetric definite). The feasible set C is defined by $C = \{x \in \mathbb{R}^m : -2 \leq x_i \leq 5, i = 1, \ldots, m\}$. It can by seen easily that F is monotone and Lipschitz continuous with constant $L = \|M\|$. Now all entries of N, Q are generated randomly in $[-2, 2]$, D is generated randomly in $[0, 2]$ and $q = \mathbf{0}$. It is easy to check that the solution of the variational inequality problem is $x^* = \{\mathbf{0}\}$. Take $\theta = \{0.9, 0.6, 0.3, 0\}$, $\epsilon_n = 100/(n+1)^2$, $\tau_0 = 1$, $\mu = 0.4$, $\eta = 0.9$, $\alpha_n = 1/(n+1)$, $\beta_n = 0.9(1-\alpha_n)$ and $T = I$ for our Algorithm 2. The maximum number of iterations 500 is used as a stopping criterion. Fig. 1 shows the numerical behavior $D_n = \|x_n - x^*\|$ of our Algorithm 2 with different parameter θ.

Example 2. Next, let us consider the variational inequality problem with

$$
F(x) = \begin{pmatrix} \left(x_1^2 + (x_2 - 1)^2\right)(1 + x_2) \\ -x_1^3 - x_1(x_2 - 1)^2 \end{pmatrix}
$$

and $C := \{x \in \mathbb{R}^2 : -10 \leq x_i \leq 10, i = 1, 2\}$. This problem has a unique solution $x^* = (0, -1)^\top$. Note that the mapping F is pseudo-monotone rather than monotone (see [42, Example 6.7]). Take $\theta = 0.6$, $\epsilon_n = 1/(10n+1)^2$, $\tau_0 = 0.1$, $\mu = 0.1$, $\eta \in \{0.4, 0.6, 0.8, 1\}$, $\alpha_n = 1/(10n+1)$, $\beta_n = 0.9(1-\alpha_n)$ and $T = I$ for our Algorithm 2. The maximum number of iterations 500 is used as a stopping criterion. Fig. 1 shows the numerical behavior $D_n = \|x_n - x^*\|$ of our Algorithm 2 with different parameter η.

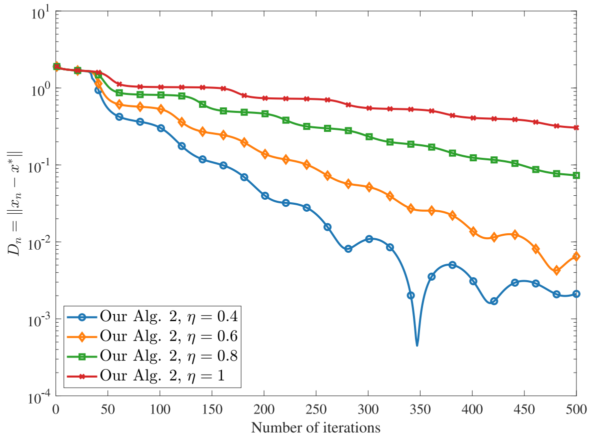


Fig. 2. Numerical results for Example 2.

Example 3. In this example, let $\mathcal{H} = L^2([0,1])$ with inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)\mathrm{d}t$ and norm $\|x\| := (\int_0^1 |x(t)|^2\mathrm{d}t)^{1/2}, \forall x, y \in \mathcal{H}$. Define $C := \{x \in \mathcal{H} : \|x\| \le 1\}$. Let $F : C \to \mathcal{H}$ be defined by

$$(Fx)(t) = \int_0^1 (x(t) - G(t,s)g(x(s)))\,\mathrm{d}s + h(t), \quad t \in [0,1],\ x \in C,$$

where

$$G(t,s) = \frac{2tse^{t+s}}{e\sqrt{e^2-1}}, \quad g(x) = \cos x, \quad h(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

It is obvious that F is monotone and L-Lipschitz continuous with $L = 2$. The projection on C is inherently explicit, that is,

$$P_C(x) = \begin{cases} \frac{x}{\|x\|}, & \text{if } \|x\| > 1; \\ x, & \text{if } \|x\| \le 1. \end{cases}$$

Through a straightforward calculation, we know that the solution of the variational inequality problem is $x^*(t) = 0$. We compare the proposed Algorithm 2 with some known algorithms in the literature [42–44]. The parameters of all algorithms are set as follows.

- In our Algorithm 2, we set $\alpha_n = 1/(n+1)$, $\beta_n = 0.9(1-\alpha_n)$, $\mu = 0.4$, $\tau_0 = 1$, $\theta = 0.3$, $\epsilon_n = 100/(n+1)^2$, $\eta = 0.9$ and $T = I$.
- In the Algorithm 3.1 proposed by Thong and Hieu [43], we take $\alpha_n = 1/(n+1)$, $\beta_n = 0.9(1-\alpha_n)$, $\mu = 0.4$ and $\tau_0 = 1$.
- In the Algorithm 3.1 introduced by Thong and Gibali [44], we choose $\alpha_n = 1/(n+1)$, $\beta_n = 0.9(1-\alpha_n)$, $\lambda = 0.5$, $l = 0.5$, $\mu = 0.4$ and $\gamma = 1.5$.
- In the Algorithm 4.3 offered by Shehu et al. [42], we select $\alpha_n = 1/(n+1)$, $\lambda_n = 0.5/L$ and $\gamma = 1.5$.

The maximum number of iterations 50 is used as a common stopping criterion. With four types of initial points, the numerical behavior $D_n = \|x_n(t) - x^*(t)\|$ of all algorithms is described in Fig. 3.

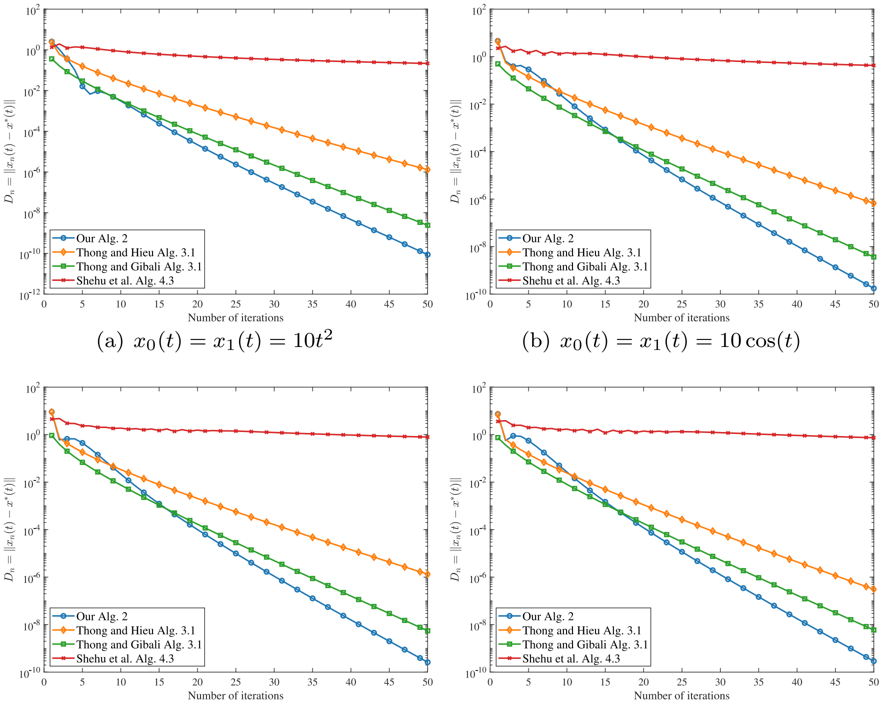
Example 4. Let $\mathcal{H} = L^2([0,1])$ be an infinite-dimensional Hilbert space with inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)\mathrm{d}t$ and norm $\|x\| := (\int_0^1 |x(t)|^2\mathrm{d}t)^{1/2}$. Let r, R be two positive real numbers such that $R/(k+1) < r/k < r < R$ for some $k > 1$. Take the feasible set as $C = \{x \in \mathcal{H} : \|x\| \le r\}$. The operator $F : \mathcal{H} \to \mathcal{H}$ is given by

$$F(x) = (R - \|x\|)x, \quad \forall x \in \mathcal{H}.$$

Note that the operator F is pseudo-monotone rather than monotone (see [45, Example 4.2]). For the experiment, we choose $R = 1.5$, $r = 1$, $k = 1.1$. The solution of this variational inequality problem is $x^*(t) = 0$. We compare the proposed Algorithm 2 with the Algorithm 2 presented by Thong and Vuong [46]. The parameters of our Algorithm 2 are the same as in Example 3. For Thong and Vuong's Algorithm 2, we take $\alpha_n = 1/(n+1)$, $\beta_n = 0.9(1-\alpha_n)$, $\gamma = 0.5$, $l = 0.5$ and $\mu = 0.4$. The maximum number of iterations 50 is used as a common stopping criterion. The numerical behavior $D_n = \|x_n(t) - x^*(t)\|$ of all algorithms with two different initial points is shown in Fig. 4.

Example 5. Assume that all images have $d := m \times n$ pixels and each pixel value is known to be within the range $[0, 255]$. We define $C = [0,255]^{m\times n} (\subseteq \mathbb{R}^{m\times n})$, i.e., C is the set of all $m \times n$ matrixs whose entries belong to $[0, 255]$. The image restoration problem can be modeled as follows:

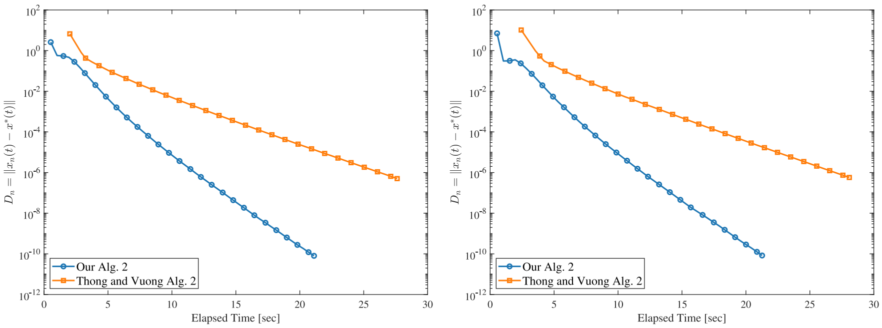
$$y = B\bar{x} + \varepsilon, \tag{35}$$



(a) $x_0(t) = x_1(t) = 10t^2$ (b) $x_0(t) = x_1(t) = 10\cos(t)$

(c) $x_0(t) = x_1(t) = 10e^t$ (d) $x_0(t) = x_1(t) = 10\log(t)$

Fig. 3. Numerical results for Example 3.



(a) $x_0(t) = x_1(t) = 100\sin(t)$ (b) $x_0(t) = x_1(t) = 100\log(t)$

Fig. 4. Numerical results for Example 4.

where $y \in \mathbb{R}^{m\times 1}$ is the observed image, $B \in C$ is the blurring matrix, ϵ is a noise term and $\bar{x} \in C' = [0, 255]^{n\times 1} (\subseteq \mathbb{R}^{n\times 1})$ is an original image. To solve problem (35), we aim to approximate the original image by transforming (35) to the following least squares (LS) problem:

$$\min_{x} \frac{1}{2}\|Bx - y\|_2^2, \tag{36}$$

where $\|\cdot\|_2$ is the Euclidean norm and $\{x \in C' \mid \|x\|_2 \leq 255\sqrt{n}\}$. The minimization (36) can be expressed as a variational inequality problem by setting $F := B^T(Bx - y)$, where the operator F is monotone and Lipschitz continuous with $L = \|B^T B\|$. To measure the quality of restored images, we use the signal-to-noise ratio (SNR) in decibels (dB) as follows:

$$\text{SNR} = 20\log_{10}\frac{\|\overline{x}\|_2}{\|x - \overline{x}\|_2},$$

Clearly, a large SNR value means that we have restored a better image.

In this example, we make comparison of Algorithm 2 (shortly, Our Alg. 2) with Algorithm 3.1 proposed by Thong and Hieu [43] (shortly, TH Alg. 3.1) and Algorithm 2 proposed by Thong and Vuong [46] (shortly, TV Alg. 2). We use the grey test images Cameraman (256×256) and Pout (291×240), which are degraded by Gaussian 7×7 blur kernel with standard deviation 4.

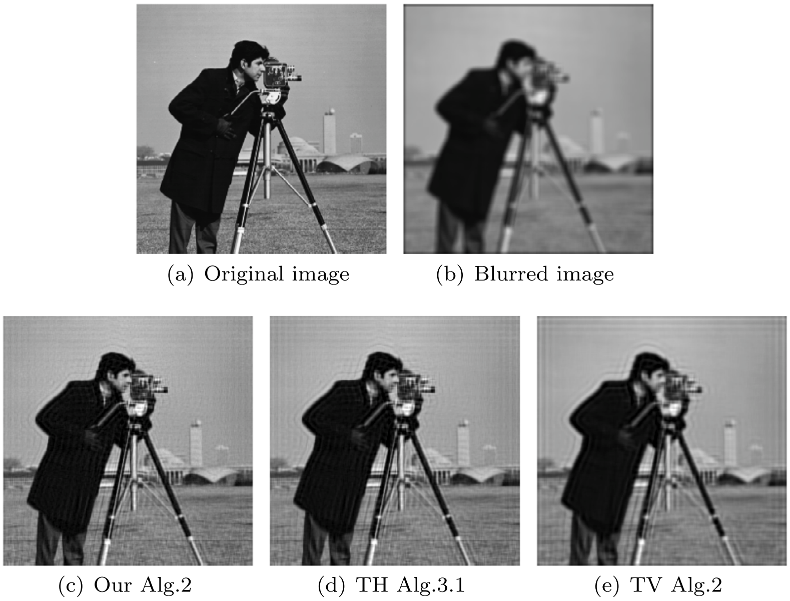


Fig. 5. Example 5. Cameraman: top left: original image; top right: blurred image; bottom left: restored image by Our Alg. 2 with SNR = 33.5640; bottom middle: restored image by TH Alg. 3.1 with SNR = 32.3471; bottom right: restored image by TV Alg. 2 with SNR = 29.5493.

Table 1
Numerical comparison of SNR (dB) values for Example 5.

Image	n	Our Alg. 2 SNR (dB)	TH Alg. 3.1 SNR (dB)	TV Alg. 2 SNR (dB)
Cameraman	100	30.7119	29.2645	25.8182
	500	32.7899	31.4801	28.6072
	1000	33.5640	32.3471	29.5493
Pout	100	36.6234	34.3334	28.4976
	500	39.2881	37.4563	32.6913
	1000	40.1085	38.5739	33.8318

The parameters of all algorithms are set as follows.

– In Our Alg. 2, we set $\alpha_n = 1/(3600n+1)$, $\beta_n = 0.999 - \alpha_n$, $\mu = 0.4$, $\tau_0 = 0.57$, $\theta = 0.606$, $\epsilon_n = 9500/(n^2+1)$, $\eta = 1$ and $T = I$.
– In TH Alg. 3.1, we take $\alpha_n = 1/(3600n+1)$, $\beta_n = 0.999 - \alpha_n$, $\mu = 0.4$ and $\tau_0 = 0.57$.
– In TV Alg. 2, we choose $\alpha_n = 1/(3600n+1)$, $\beta_n = 0.999 - \alpha_n$, $l = 0.1$, $\mu = 0.4$ and $\gamma = 0.57$.

Figs. 5 and 6 show the original, blurred and restored images by using Our Alg. 2, TH Alg. 3.1 and TV Alg. 2. Also, Fig. 7 shows the graph of SNR against number of iterations for each test image using the algorithms. Moreover, we report the SNR values for each algorithms in Table 1.

Example 6. In this example, we consider the recovery of original signal from a noisy signal. The model for signal processing is shown below:

$$y = Bx + \varepsilon,$$

where $B \in \mathbb{R}^{m\times n}$ is a bounded linear operator, $x \in \mathbb{R}^n$ is the original signal with k non-zero elements, $y \in \mathbb{R}^m$ is the obtained noisy observation and ε is the noisy data. We can convert this model into a variational inequality problem by setting $F = B^T(Bx - y)$ and $C = \{x \in \mathbb{R}^n \mid \|x\|_1 \le t\}$. In our numerical experiments, the matrix B and ε are randomly generated by the MATLAB function $B = \mathrm{randn}(m,n)$ and $\varepsilon = 10^{-3}\mathrm{randn}(m,1)$, respectively. The original signal x contains k ($k \ll n$) non-zero elements randomly created ± 1 spikes. We use the mean square error defined as $\mathrm{MSE} = \frac{1}{n}\|x^* - x\|^2$ to measure the precision of the error between the signal x^* recovered by the algorithm and the original signal x. The recovery procedure for all algorithms starts with the initial signal $x_0 = x_1 = \mathbf{0}$ and stops iterating when the MSE $< 10^{-6}$ is satisfied. In our test, we set $n = 1024, m = 512, k = \{40, 60, 80, 100\}$ and choose $t = k$.

In this test, we compare the proposed Algorithm 2 with the Algorithm 3.1 of Tan et al. [47] (shortly, TLC Alg. 3.1) and the Algorithm 3.11 of Jolaoso [48] (shortly, Jolaoso Alg. 3.11). The parameters of these algorithms are set as follows.


(a) Original image

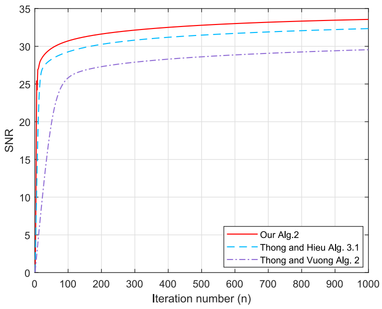

(b) Blurred image


(c) Our Alg.2

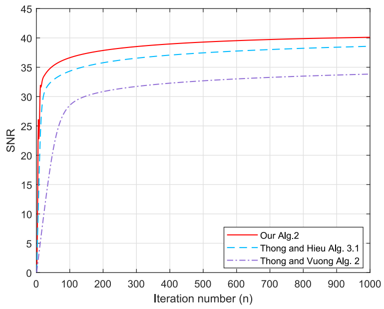

(d) TH Alg.3.1


(e) TV Alg.2

Fig. 6. Example 5. Pout: top left: original image; top right: blurred image; bottom left: restored image by Our Alg. 2 with SNR = 40.1085; bottom middle: restored image by TH Alg. 3.1 with SNR = 38.5739; bottom right: restored image by TV Alg. 2 with SNR = 33.8318.



(a) SNR for Cameraman



(b) SNR for Pout

Fig. 7. SNR for Example 5.

Table 2
Numerical results for all algorithms at different sparsity k in Example 6 ($n = 2014, m = 512$).

Algorithms	$k = 40$		$k = 60$		$k = 80$		$k = 100$	
	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.
Our Alg. 2	0.1557	151	0.2415	263	0.3916	388	0.8721	948
Jolaoso Alg. 3.11	1.3877	156	2.9914	294	4.6115	479	11.1500	1181
TLC Alg. 3.1	1.6788	168	3.0333	310	4.5972	467	11.3533	1220

– In Our Alg. 2, we set $\alpha_n = 0.01/(n+1)$, $\beta_n = 0.99(1-\alpha_n)$, $\mu = 0.9$, $\tau_0 = 0.006$, $\theta = 0.3$, $\epsilon_n = 100/(n+1)^2$, $\eta = 1$ and $T = I$.
– In TLC Alg. 3.1, we take $\alpha_n = 0.01/(n+1)$, $\beta_n = 0.8(1-\alpha_n)$, $\theta = 0.3$, $\epsilon_n = 100/(n+1)^2$, $\delta = 1.5$, $l = 0.5$, $\mu = 0.6$ and $\gamma = 2$.
– In Jolaoso Alg. 3.11, we choose $\alpha_n = 0.5/(n+1)$, $\theta = 0.3$, $\epsilon_n = 100/(n+1)^2$, $\delta = 1.5$, $f(x) = 0.9x$, $l = 0.5$, $\mu = 0.6$ and $\gamma = 2$.

Fig. 8 shows the recovery results of our Algorithm 2 for different sparsity signals. Table 2 lists the computation time (in seconds) and the number of iterations required by the proposed algorithm and the compared methods under different sparsity conditions.

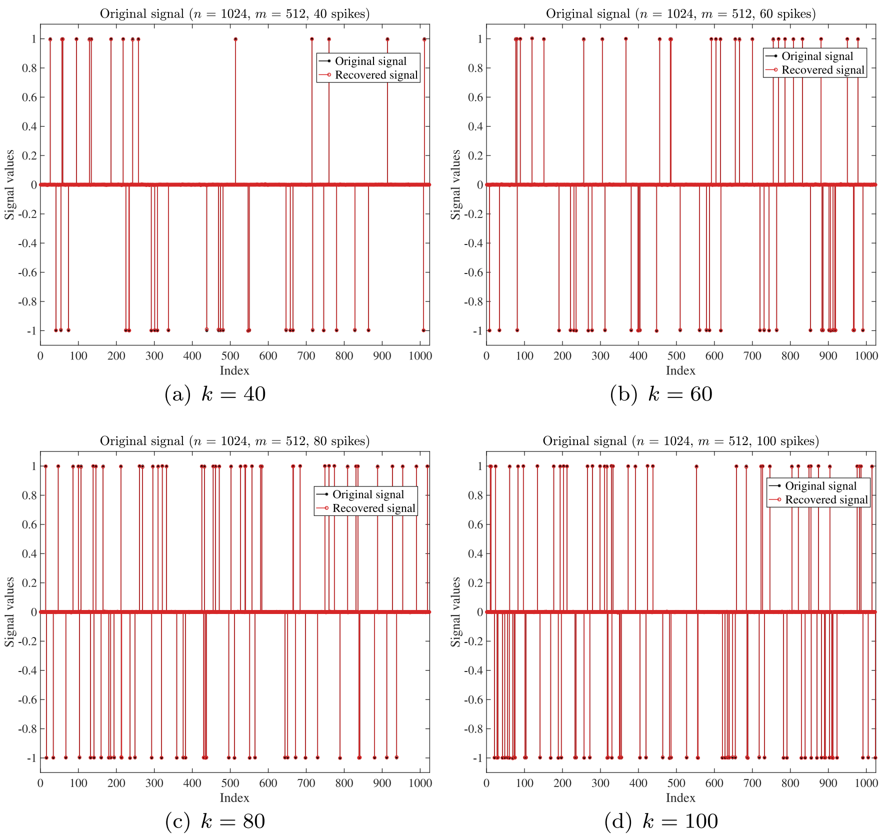


Fig. 8. Signals with different sparsity recovered by our Algorithm 2 in Example 6.

Remark 2. We have the following observations for Examples 1–6.

1. As can be seen in Fig. 1, our proposed algorithm with inertial terms converges faster than our algorithm without inertial. Moreover, our algorithm applies two different stepsizes in each iteration, which converges faster than the algorithm that uses two identical stepsizes (see Fig. 2).
2. It can be seen from Figs. 3 and 4 that our proposed algorithm has a faster convergence speed than some known algorithms in the literature [42–44,46] and that these results are not related to the choice of initial values. Moreover, as shown in Fig. 4, our proposed algorithm has higher accuracy and less execution time than the Armijo-type algorithm proposed by Thong and Vuong [46]. Therefore, the algorithm proposed in this paper is efficient and robust.
3. It is worth noting that the operator F in Example 4 is pseudo-monotone rather than monotone. In this case, the algorithms introduced in the literature [43,44] for solving monotone variational inequality problems will not be available. On the other hand, the Lipschitz constant of the operator F in Examples 2 and 4 are both unknown. In these cases, the fixed-step Algorithm 4.3 suggested by Shehu et al. [42] will not be available due to the fact that the algorithm requires the prior information of the Lipschitz constant of the mapping. Thus, the adaptive algorithm presented in this paper has a broader range of applications.
4. Table 1 shows that our Algorithm 2 is more efficient for restoring the degraded image than other comparison algorithms. Meanwhile, Table 2 also demonstrates that the algorithm proposed in this paper outperforms the results in [47,48].

CRediT authorship contribution statement

Danni Guo: Writing – original draft. **Gang Cai:** Writing – review & editing, Funding acquisition. **Bing Tan:** Data curation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

This work was supported by the NSF of China (Grant No. 12171062) and the Natural Science Foundation of Chongqing, China (Grant No. CSTB2022NSCQ-JQX0004).

Data availability

Data will be made available on request.

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