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ON THE RESOLUTION OF VARIATIONAL INEQUALITY PROBLEMS WITH A DOUBLE-HIERARCHICAL STRUCTURE

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Abstract. In this paper, we discuss a pseudo-monotone variational inequality problem with a variational inequality constraint over a general, nonempty, closed and convex set, which is called the double-hierarchical constrained optimization problem. In addition, we propose an iterative algorithm by incorporating inertial terms in the extragradient algorithm. Strong convergence of the algorithm to the unique solution of the problem is guaranteed under certain assumptions.

1. Introduction-preliminaries

Problem 1.1. Let C be a nonempty closed convex subset of H and let $A : H \to H$ be a nonlinear operator,

$$\text{find } \ x^* \in VI(C, A) := \{x^* \in C : \langle z - x^*, A(x^*)\rangle \geq 0, \ \forall z \in C\}, \tag{1.1}$$

where H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$.

Variational inequalities serve as an important tool in studying a wide class of many related problems arising in mathematical problems, regional problems, physical problems. Also, variational inequalities can be viewed as a natural framework for unifying the treatment of equilibrium problems and fixed point problems. Indeed, (1.1) has many applications in the analysis of piece-wise-linear resistive circuits, economic equilibrium modeling, bimatrix equilibrium points, and traffic network equilibrium modeling, signal and image processing, and pattern recognition (see [12, 13, 28]). Recently, many authors employed various numerical methods for solving the variational inequality problem; see, for example, [3–5, 8, 10, 11, 19, 23, 27] and the references therein.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

Then P_C is called the metric projection of H onto C. It is known that the projection operator can be characterized by (i) $\|P_C x - P_C y\| \leq \|x - y\|, \ \forall x, y \in H$; (ii) $\langle x - P_C x, y - P_C x\rangle \leq 0, \ \forall x \in H, y \in C$; Furthermore, the property (ii) is equivalent to $\|x - P_C x\|^2 + \|y - P_C x\|^2 \leq \|x - y\|^2, \ \forall x \in H, y \in C$.

Remark 1.2. Indeed, we can turn the variational inequality problem into a fixed point problem, in other words, the variational inequality is equivalent to

$$x^* = P_C(I - \alpha A)x^*, \ \text{ for all } \alpha > 0.$$

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The above equivalence plays a significant role in a lot of convex optimization problems; see, e.g., [1,2,6,7,9,18,20–22,29,30] and the references therein.

An operator $A : H \to H$ is said to be (i) L-Lipschitz continuous on C if there exists $L > 0$ such that $\|Ax - Ay\| \le L\|x - y\|$, $\forall x, y \in C$; (ii) sequentially weakly continuous on C if for each sequence $\{x_k\}$, we have that $\{x_k\}$ converges weakly to x^* implies that $\{Ax_k\}$converges weakly to Ax^*. (iii) pseudomonotone on C if $\langle Ax, y - x\rangle \ge 0 \Rightarrow \langle Ay, y - x\rangle \ge 0$, $\forall x, y \in C$; (iv) monotone on C if $\langle Ax - Ay, x - y\rangle \ge 0$, $\forall x, y \in C$; (v)strongly monotone on C if there exists $\delta > 0$ such that $\langle Ax - Ay, x - y\rangle \ge \delta\|x - y\|^2$, $\forall x, y \in C$; (vi) nonexpansive if $\|Ax - Ay\| \le \|x - y\|$, $\forall x, y \in H$. Suppose that $g : H \to \mathbb{R}$ is convex and continuously Frechet differentiable. Then g is convex if and only if $\nabla g : H \to H$ is monotone. It is well known that ∇g is κ-strongly monotone (pseudo-monotone) if and only if g is κ-strongly convex (pseudo-convex).

The goal of this paper is to present an iterative algorithm for the following double-hierarchical constrained optimization problem.

Problem 1.3. (Variational inequality problem over solution set of variational inequality problem)

Assume that

(i) Let C be a nonempty, closed, convex set of a Hilbert space H;
(ii) $A_1 : H \to H$ is pseudo-monotone and Lipschitz continuous and sequentially weakly continuous;
(iii) $A_2 : H \to H$ is strongly monotone and Lipschitz continuous with $VI(VI(C, A_1), A_2) \neq \emptyset$.

Our objective is to

$$\text{find } x^* \in VI(VI(C, A_1), A_2) = \{x^* \in VI(C, A_1) : \langle z - x^*, A_2x^*\rangle \ge 0,\ \forall z \in VI(C, A_1)\}. \tag{1.2}$$

The simplest solution method solving variational inequality 1.1 is the classical projection method, which generates a sequence $\{x_k\}$ through the following iteration formula

$$x_{k+1} = P_C(I - \alpha A)x_k, \tag{1.3}$$

where $A : H \to H$ is δ-strongly monotone, ι-Lipschitz continuous and $\alpha \in \left(0, \frac{2\delta}{\iota^2}\right)$. It is worth mentioning that these hypotheses are quite strong. If the strong monotonicity assumption is relaxed to plain monotonicity, the situation becomes more complicated, and one may get a divergent sequence. In order to deal with this situation, Korpelevich in [14] proposed an algorithm for the Euclidean case, known as the extragradient method for solving variational inequalities when A is monotone and ι-Lipschitz continuous

$$\begin{cases} y_k = P_C(x_k - \alpha Ax_k), \\ x_{k+1} = P_C(x_k - \alpha Ay_k), \end{cases} \tag{1.4}$$

for each $k = 1, 2, \dots$, where $\alpha \in \left(0, \frac{1}{\iota}\right)$. Recently, the extragradient method was also applied to solve pseudo-monotone, Lipschitz continuous variational inequalities in the Hilbert space [31].

The sequence $\{x_k\}$ generated by (1.3) and (1.4) are weakly convergent under certain assumptions. By comparing the above modified iterative schemes, we find that the conditions of the underlying mapping were weaken, but the projection was still preserved during the calculation. So is there a way to avoid the calculation of projections and can it also solve the variational inequality problem?

In 2011, Yamada [33] introduced an algorithmic solution to solve the variational inequality problem, which was named the hybrid steepest descent method. Given the initial data $x_0 \in H$,

$$x_{k+1} = (I - \mu\alpha_k A)Tx_k, \quad \forall k \in \mathbb{N}, \tag{1.5}$$

where T is a nonexpansive mapping with $Fix(T) \neq \emptyset$ and $A : H \to H$ is a ι-Lipschitz continuous, κ-strongly monotone mapping. It does not rerquire to calculate P_C but requires a closed form expression of a nonexpansive mapping T. The sequence $\{x_k\}$ generated by (1.5) converges strongly to a point x^*, which is a unique solution of the hierarchical variational inequality $\langle Ax^*, x - x^*\rangle \geq 0$, $\forall x \in Fix(T)$. Recently, descent-like solution methods have extensively investigated for solving variational inequality problems; see, e.g., [15, 17, 24–26, 34]

To prove the main theorem of this paper, here we recall some known results.

Lemma 1.4 ([33]). *Let the operator $A : H \to H$ be ι-Lipschitz continuous and δ-strongly monotone with constants $\iota > 0$, $\delta > 0$. Assume that $\gamma \in \left(0, \frac{2\delta}{\iota^2}\right)$. For $\alpha \in (0, 1)$, define $T_\alpha := x - \alpha\gamma A(x)$. Then for all $x, y \in H$,*

$$\|T_\alpha(x) - T_\alpha(y)\| \leq (1 - \alpha\omega)\|x - y\|$$

holds, where $\omega := 1 - \sqrt{1 - \gamma(2\delta - \gamma\iota^2)} \in (0, 1)$.

Lemma 1.5 ([32]). *Let $\{a_k\}$ be a sequence of nonnegative real numbers such that*

$$a_{k+1} \leq (1 - \alpha_k)a_k + \alpha_k b_k,$$

where $\{\alpha_k\} \subset (0, 1)$ and $\{b_k\}$ is a sequence such that $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\limsup_{k\to\infty} b_k \leq 0$. Then $\lim_{k\to\infty} a_k = 0$.

Lemma 1.6 ([16]). *Let $\{a_k\}$ be a sequence of nonnegative real numbers such that there exists a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ such that $a_{k_j} < a_{k_j+1}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_i\} \subset \mathbb{N}$ such that $\lim_{i\to\infty} m_i = \infty$ and the following properties are satisfied by all (sufficiently large) number $i \in \mathbb{N}$:*

$$a_{m_i} \leq a_{m_i+1} \text{ and } a_i \leq a_{m_i+1}.$$

In fact, m_i is the largest number of k in the set $\{1, 2, \ldots, i\}$ such that $a_k \leq a_{k+1}$ holds.

In this paper, we introduce a variational inequality with variational inequality constraint over a nonempty, closed and convex set. Since this problem has a double structure, it is referred here as a double-hierarchical constrained optimization problem. The inertial algorithm can be regarded as a procedure of speeding up the convergence properties. To solve the above problem, we focus on an inertial extragradient algorithm by incorporating the inertial term in Algorithm (1.4). It is worth mentioning that the strong convergence of the inertial extragradient method in the setting of Hilbert spaces is still unexplored. Therefore, the goal here is to prove that our proposed algorithm strongly converges to a unique solution of (1.2), by combining two well-known methods, the inertial extragradient method and the hybrid steepest descent method.

2. Algorithm and convergence

We work in the following framework, which has been delineated in the previous section.

- The feasible set C is a nonempty, convex and closed set in a real Hilbert space H;
- The operator $A_1 : H \to H$ is pseudo-monotone, L-Lipschitz and sequentially weakly continuous;
- The operator $A_2 : H \to H$ is κ-strongly monotone, ι-Lipschitz continuous with its solution set $VI(VI(C, A_1), A_2) \neq \emptyset$;

Algorithm 1

Require: Input the algorithm parameters $(\alpha_i)_{i\in\mathbb{N}}$, $(\lambda_i)_{i\in\mathbb{N}}$ and $(\mu_i)_{i\in\mathbb{N}}$.
Ensure: Output x
Set $k \leftarrow 1$.
Initialize the data $x_0, x_1 \in H$.
while not converged **do**
  Update $w_k := x_k + \alpha_k(x_k - x_{k-1})$.
  Update $y_k := P_C(w_k - \lambda_k A_1 w_k)$.
  Update $x_{k+1} = (I - \chi\mu_k A_2)P_C(y_k - \lambda_k A_1 y_k)$.
  Set $k \leftarrow k + 1$.
end while
return $x = x_k$

We recall two important properties of the iterative sequence generated by Algorithm 2.1, which are extracted from [31].

Proposition 2.1. *Let the operator A_1 be pseudo-monotone and Lipschitz continuous on C with $VI(C, A_1) \neq \emptyset$. Let x^* be a solution of $VI(C, A_1)$. Setting $z_k = P_C(y_k - \lambda_k A_1 y_k)$, then for each $k \in \mathbb{N}$, we have*

$$\|z_k - x^*\|^2 \leq \|w_k - x^*\|^2 - (1 - \lambda_k^2 L^2)\|y_k - w_k\|^2.$$

Proposition 2.2. *Assume that A_1 is pseudo-monotone, L-Lipschitz continuous and sequentially weakly continuous on C. Additionally assume that $VI(C, A_1)$ is nonempty. If $\lim_{k\to\infty} \|y_k - w_k\| = 0$ and $\liminf_{k\to\infty} \lambda_k > 0$, then the sequence $\{w_k\}$ generated by Algorithm 2.1 converges weakly to a solution of $VI(C, A_1)$.*

We now in a position to establish the main result of this note.

Theorem 2.3. *Let $\{\lambda_k\}$, $\{\mu_k\}$ be two real sequences in $(0, 1)$ such that $0 < a \leq \lambda_n \leq b < \frac{1}{L}$ for some $a, b \in (0, 1)$, $\lim_{k\to\infty} \mu_k = 0$, $\sum_{k=1}^{\infty} \mu_k = \infty$. Assume that the extrapolation sequence $\{\alpha_k\}$ is chosen such that $\lim_{k\to\infty} \frac{\alpha_k}{\mu_k}\|x_k - x_{k-1}\| = 0$ and $\chi \in \left(0, \frac{2\kappa}{\iota^2}\right)$. Then the sequence $\{x_k\}$ generated by Algorithm 2.1 converges strongly to the unique solution $\hat{x} \in VI(VI(C, A_1), A_2)$.*

Proof. Let us stress the fact that $I-\chi A_2$ is a contractive mapping. Owning to $A_2 : H \to H$ is κ-strongly monotone and ι-Lipschitz continuous, we have

$$\begin{aligned}
&\|(I-\chi A_2)x-(I-\chi A_2)y\|^2\\
=&\|x-y\|^2-2\chi\langle x-y, A_2x-A_2y\rangle+\chi^2\|A_2x-A_2y\|^2\\
\leq&\|x-y\|^2-2\chi\kappa\|x-y\|^2+\chi^2\iota^2\|x-y\|^2\\
\leq&(1-\sigma)^2\|x-y\|^2,
\end{aligned}\tag{2.1}$$

where $\sigma=\frac{1}{2}\chi(2\kappa-\chi\iota^2)\in(0,1)$. By a classical result, we obtain that $I-\chi A_2 : H \to H$ is a contraction with constant $1-\sigma$. Hence the mapping $P_{VI(C,A_1)}(I-\chi A_2)$ is a contraction as well. By using Banach contraction principle, there exists a unique fixed point $x^*=P_{VI(C,A_1)}(I-\chi A_2)x^*$. By using the definition of $\{w_k\}$, it is immediate to check that

$$\|w_k-x^*\|=\|x_k+\alpha_k(x_k-x_{k-1})-x^*\|\leq\|x_k-x^*\|+\alpha_k\|x_k-x_{k-1}\|.\tag{2.2}$$

On the other hand,

$$\begin{aligned}
\|w_k-x^*\|^2=&\|x_k+\alpha_k(x_k-x_{k-1})-x^*\|^2\\
\leq&\|x_k-x^*\|^2+2\alpha_k\langle x_k-x_{k-1}, w_k-x^*\rangle\\
\leq&\|x_k-x^*\|^2+2\alpha_k\|x_k-x_{k-1}\|\|w_k-x^*\|.
\end{aligned}\tag{2.3}$$

To simplify the notation, let us set $z_k=P_C(y_k-\lambda_kA_1y_k)$ for every $k\geq 1$. By using successively Proposition 2.1, Lemma 1.4, (2.2), together with the definition of $\{x_k\}$, we have that

$$\begin{aligned}
\|x_{k+1}-x^*\|=&\|(I-\chi\mu_kA_2)z_k-x^*\|\\
=&\|(I-\chi\mu_kA_2)z_k-(I-\chi\mu_kA_2)x^*-\chi\mu_kA_2x^*\|\\
\leq&\|(I-\chi\mu_kA_2)z_k-(I-\chi\mu_kA_2)x^*\|+\chi\mu_k\|A_2x^*\|\\
\leq&(1-\tau\mu_k)\|z_k-x^*\|+\chi\mu_k\|A_2x^*\|\\
\leq&(1-\tau\mu_k)(\|x_k-x^*\|+\alpha_k\|x_k-x_{k-1}\|)+\chi\mu_k\|A_2x^*\|,
\end{aligned}\tag{2.4}$$

where $\tau=1-\sqrt{1-\chi(2\kappa-\chi\iota^2)}\in(0,1)$. The assumption $\lim_{k\to\infty}\frac{\alpha_k}{\mu_k}\|x_k-x_{k-1}\|=0$ yields that there exists a positive constant $M_1>0$ such that $\frac{\alpha_k}{\mu_k}\|x_k-x_{k-1}\|\leq M_1$. With the help of (2.4), an immediate recurrence shows that for every $k\geq 1$, we have

$$\begin{aligned}
\|x_{k+1}-x^*\|\leq&(1-\tau\mu_k)(\|x_k-x^*\|+\mu_kM_5)+\chi\mu_k\|A_2x^*\|\\
\leq&(1-\tau\mu_k)\|x_k-x^*\|+\tau\mu_k\left(\frac{M_5}{\tau}+\frac{\chi}{\tau}\|A_2x^*\|\right)\\
\leq&\max\left\{\|x_k-x^*\|,\frac{M_5}{\tau}+\frac{\chi}{\tau}\|A_2x^*\|\right\}\\
\leq&\cdots\leq\max\left\{\|x_0-x^*\|,\frac{M_5}{\tau}+\frac{\chi}{\tau}\|A_2x^*\|\right\}.
\end{aligned}\tag{2.5}$$

It ensues that $\{x_k\}$ is bounded. Therefore, we immediately find that sequences $\{y_k\}$, $\{w_k\}$ and $\{z_k\}$ are bounded as well. On the other hand, the definition of

$\{x_k\}$, with the help of (2.3) and Proposition 2.1, gives that

$$\begin{aligned}\|x_{k+1}-x^*\|^2 =&\|(I-\chi\mu_k A_2)z_k - x^*\|^2\\ =&\|\mu_k((I-\chi A_2)z_k - x^*)+(1-\mu_k)(z_k-x^*)\|^2\\ \le&\mu_k\|(I-\chi A_2)z_k - x^*\|^2+(1-\mu_k)\|z_k-x^*\|^2\\ \le&\mu_k\|(I-\chi A_2)z_k - x^*\|^2+\|x_k-x^*\|^2\\ &+2\alpha_k\|x_k-x_{k-1}\|\|w_k-x^*\|\\ &-(1-\mu_k)(1-\lambda_k^2L^2)\|y_k-w_k\|^2.\end{aligned} \tag{2.6}$$

In view of the definition of $\{x_k\}$, together with (2.1)-(2.3), Proposition 2.1, we find that

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &= \|(I-\chi\mu_k A_2)z_k - x^*\|^2\\ &= \|\mu_k((I-\chi A_2)z_k - x^*)+(1-\mu_k)(z_k-x^*)\|^2\\ &\le (1-\mu_k)^2\|z_k-x^*\|^2+2\mu_k\langle (I-\chi A_2)z_k-x^*, x_{k+1}-x^*\rangle\\ &\le (1-\mu_k)^2\|z_k-x^*\|^2\\ &\quad+2\mu_k\langle (I-\chi A_2)z_k-(I-\chi A_2)x^*, x_{k+1}-x^*\rangle\\ &\quad+2\mu_k\langle (I-\chi A_2)x^*-x^*, x_{k+1}-x^*\rangle\\ &\le (1-\mu_k)^2\|z_k-x^*\|^2+2\mu_k(I-\sigma)\|z_k-x^*\|\|x_{k+1}-x^*\|\\ &\quad+2\mu_k\langle (I-\chi A_2)x^*-x^*, x_{k+1}-x^*\rangle\\ &\le (1-\mu_k)^2(\|x_k-x^*\|^2+2\alpha_k\|x_k-x_{k-1}\|\|w_k-x^*\|)\\ &\quad+2\mu_k(I-\sigma)(\|x_k-x^*\|\\ &+\alpha_k\|x_k-x_{k-1}\|)\|x_{k+1}-x^*\|\\ &\quad+2\mu_k\langle (I-\chi A_2)x^*-x^*, x_{k+1}-x^*\rangle\\ &\le (1-\mu_k)^2\|x_k-x^*\|^2+2\mu_k(1-\sigma)\|x_k-x^*\|\|x_{k+1}-x^*\|\\ &\quad+2\alpha_k\|x_k-x_{k-1}\|(\|w_k-x^*\|\\ &+\mu_k\|x_{k+1}-x^*\|)+2\mu_k\langle (I-\chi A_2)x^*-x^*, x_{k+1}-x^*\rangle.\end{aligned} \tag{2.7}$$

Case 1: Suppose that there exists $K\in\mathbb{N}$ such that $\|x_{k+1}-x^*\|^2\le\|x_k-x^*\|^2$ for all $k\ge K$. It ensues that $\lim_{k\to\infty}\|x_k-x^*\|^2$ exists. On the other hand, the assumption $0<a\le\lambda_n\le b<\frac{1}{L}$ comes down to the fact that

$$0<1-b^2L^2\le 1-\lambda_k^2L^2\le 1-a^2L^2<1. \tag{2.8}$$

Recalling the assumptions that $\lim_{k\to\infty}\frac{\alpha_k}{\mu_k}\|x_k-x_{k-1}\|=0$ and $\lim_{n\to\infty}\mu_k=0$, we infer that $\lim_{k\to\infty}\alpha_k\|x_k-x_{k-1}\|=0$. Coming back to (2.6) and collecting the above results, with the help of the condition $\lim_{k\to\infty}\mu_k=0$ and (2.8), we obtain that

$$\lim_{k\to\infty}\|y_k-w_k\|=0. \tag{2.9}$$

From the nonexpansivity of P_C and the L-Lipschitz continuity of A_1, it follows that

$$\|z_k-y_k\|=\|P_C(y_k-\lambda_k A_1 y_k)-P_C(w_k-\lambda_k A_1 w_k)\|\le(1+\lambda_k L)\|y_k-w_k\|. \tag{2.10}$$

Combining (2.9) with (2.10), we find that

$$\lim_{k\to\infty} \|y_k - z_k\| = 0. \tag{2.11}$$

We deduce from the expression of $\{w_k\}$ that

$$\lim_{k\to\infty} \|w_k - x_k\| = \lim_{k\to\infty} \alpha_k \|x_k - x_{k-1}\| = 0. \tag{2.12}$$

By putting together (2.9), (2.11), (2.12), we immediately obtain that

$$\lim_{k\to\infty} \|z_k - x_k\| \le \lim_{k\to\infty} (\|z_k - y_k\| + \|y_k - w_k\| + \|w_k - x_k\|) = 0. \tag{2.13}$$

Recalling that $\{x_k\}$ is bounded, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j} \rightharpoonup \hat{x}$ as $j \to \infty$. From this, and by applying (2.12), it can be easily shown that $w_{k_j} \rightharpoonup \hat{x}$ as $j \to \infty$. Thus by using Proposition 2.2, we obtain directly that $\hat{x} \in VI(C, A_1)$. Invoking $x^* = P_{VI(C,A_1)}(I - \chi A_2)x^*$, we further obtain that

$$\limsup_{j\to\infty} \langle (I - \chi A_2)x^* - x^*, x_{k_j} - x^* \rangle = \langle (I - \chi A_2)x^* - x^*, \hat{x} - x^* \rangle \le 0. \tag{2.14}$$

From the expression of $\{x_k\}$, together with the condition $\lim_{k\to\infty} \mu_k = 0$ and (2.13), we infer

$$\|x_{k+1} - x_k\| = \|(I - \chi\mu_k A_2)z_k - x_k\| \le \|z_k - x_k\| + \chi\mu_k \|A_2 z_k\| \to 0,\ k \to \infty. \tag{2.15}$$

Thus combining (2.14) with (2.15), we have that

$$\limsup_{j\to\infty} \langle (I - \chi A_2)x^* - x^*, x_{k_j+1} - x^* \rangle \le 0. \tag{2.16}$$

Recalling that $\{x_k\}$ is bounded, we deduce the existence of $M_2 > 0$ such that $\|w_k - x^*\| + \mu_k \|x_{k+1} - x^*\| < M_2$. In this case, the consequence of (2.7) is that

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2 \\
&\le (1 - 2\mu_k\sigma)\|x_k - x^*\|^2 \\
&\quad + \mu_k^2 \|x_k - x^*\|^2 + 2\alpha_k \|x_k - x_{k-1}\| M_2 + 2\mu_k \langle (I - \chi A_2)x^* - x^*, x_{k+1} - x^* \rangle \\
&\le (1 - 2\mu_k\sigma)\|x_k - x^*\|^2 \\
&\quad + 2\mu_k\sigma \left(\frac{\mu_k}{2\sigma} \|x_k - x^*\|^2 + \frac{\alpha_k}{\mu_k\sigma} \|x_k - x_{k-1}\| M_2 + \frac{\langle (I - \chi A_2)x^* - x^*, x_{k+1} - x^* \rangle}{\sigma} \right).
\end{aligned} \tag{2.17}$$

Combining (2.16), (2.17) with the conditions $\lim_{k\to\infty} \frac{\alpha_k}{\mu_k} \|x_k - x_{k+1}\| = 0$, $\lim_{k\to\infty} \mu_k = 0$, $\sum_{k=1}^{\infty} \mu_k = \infty$, $\sigma \in (0, 1)$, it follows from Lemma 1.5 that $\lim_{k\to\infty} \|x_k - x^*\|^2 = 0$. In other words, we find that $\lim_{k\to\infty} x_k = x^*$.

Case 2: Let us restrict ourselves to the case that there is no $k_0 \in \mathbb{N}$ such that $\{\|x_k - p\|^2\}_{k=k_0}^{\infty}$ is monotonically decreasing. According to Lemma 1.6, we define a mapping $\varsigma : \mathbb{N} \to \mathbb{N}$ by

$$\varsigma(k) := \max\{i \in \mathbb{N} : i \le k, \|x_i - x^*\|^2 < \|x_{i+1} - x^*\|^2\},\ \ \forall k \ge k_0,$$

i.e., $\varsigma(k)$ is the largest number i in $\{1, 2, \ldots, k\}$ such that $\{\|x_i - p\|^2\}$ is monotonically decreasing at $i = \varsigma(k)$. In this case, observe that $\varsigma(k)$ is well-defined for all sufficiently large k. Furthermore, $\varsigma(\cdot)$ is a nondecreasing mapping such that $\varsigma(k)\to\infty$ as $k\to\infty$. Thus it ensues that for any $k\geq k_0$,

$$\|x_{\varsigma(k)}-p\|^2\leq\|x_{\varsigma(k)+1}-p\|^2,\quad \|x_k-p\|^2\leq\|x_{\varsigma(k)+1}-p\|^2. \tag{2.18}$$

Coming back to (2.6) and rearranging the terms, we have that

$$\begin{aligned}&(1-\mu_{\varsigma(k)})(1-\lambda_{\varsigma(k)}^2L^2)\|y_{\varsigma(k)}-w_{\varsigma(k)}\|^2\\ &\leq\|x_{\varsigma(k)}-x^*\|^2-\|x_{\varsigma(k)+1}-x^*\|^2\\ &\quad+2\alpha_{n_i}\|x_{\varsigma(k)}-x_{n_i-1}\|\|x_{\varsigma(k)}-x^*\|+\mu_{\varsigma(k)}\|(I-\chi A_2)z_{\varsigma(k)}-x^*\|^2.\end{aligned} \tag{2.19}$$

Combining the boundedness of $\{z_{\varsigma(k)}\}$ with conditions $\lim_{k\to\infty}\alpha_{\varsigma(k)}\|x_{\varsigma(k)}-x_{\varsigma(k)-1}\|=0$, $\lim_{k\to\infty}\mu_{\varsigma(k)}=0$, it follows from (2.18) and (2.19) that

$$\lim_{k\to\infty}\|y_{\varsigma(k)}-w_{\varsigma(k)}\|=0. \tag{2.20}$$

Proceeding as in the proof of **Case 1**, we easily see that

$$\lim_{k\to\infty}\|w_{\varsigma(k)}-x_{\varsigma(k)}\|=\lim_{k\to\infty}\|x_{\varsigma(k)+1}-x_{\varsigma(k)}\|=0, \tag{2.21}$$

and

$$\limsup_{k\to\infty}\langle(I-\chi A_2)x^*-x^*,x_{\varsigma(k)+1}-x^*\rangle\leq 0. \tag{2.22}$$

Another consequence of (2.7) is that

$$\begin{aligned}\|x_{\varsigma(k)+1}-x^*\|^2\leq(1-2\mu_{\varsigma(k)}\sigma)\|x_{\varsigma(k)+1}-x^*\|^2+2\mu_{\varsigma(k)}\sigma\Bigg(&\frac{\mu_{\varsigma(k)}}{2\sigma}\|x_{\varsigma(k)+1}-x^*\|^2\\ &+\frac{\alpha_{\varsigma(k)}}{\mu_{\varsigma(k)}\sigma}\|x_{\varsigma(k)}-x_{\varsigma(k)-1}\|M_2+\frac{\langle(I-\chi A_2)x^*-x^*,x_{\varsigma(k)+1}-x^*\rangle}{\sigma}\Bigg).\end{aligned}$$

It entails that

$$\begin{aligned}\|x_{\varsigma(k)+1}-x^*\|^2\leq&\frac{\mu_{\varsigma(k)}}{2\sigma}\|x_{\varsigma(k)+1}-x^*\|^2\\ &+\frac{\alpha_{\varsigma(k)}}{\mu_{\varsigma(k)}\sigma}\|x_{\varsigma(k)}-x_{\varsigma(k)-1}\|M_2\\ &+\frac{\langle(I-\chi A_2)x^*-x^*,x_{\varsigma(k)+1}-x^*\rangle}{\sigma}.\end{aligned} \tag{2.23}$$

The boundedness of $\{x_k\}$, together with the conditions $\lim_{k\to\infty}\frac{\alpha_{\varsigma(k)}}{\mu_{\varsigma(k)}}\|x_{\varsigma(k)}-x_{\varsigma(k)-1}\|=0$, $\lim_{k\to\infty}\mu_{\varsigma(k)}=0$, in light of (2.22), (2.23), we have that $\limsup_{k\to\infty}\|x_{\varsigma(k)+1}-x^*\|^2\leq 0$. From this, with the help of (2.18), we conclude that $x_k\to x^*$ as $k\to\infty$.

Regarding the property of the distance projection, we observe that

$$\begin{aligned}x^*=P_{VI(C,A_1)}(I-\chi A_2)x^*&\Leftrightarrow\langle(I-\chi A_2)x^*-x^*,z-x^*\rangle\leq 0\\ &\Leftrightarrow\langle A_2x^*,z-x^*\rangle\geq 0,\forall z\in VI(C,A_1).\end{aligned}$$

This means that $x^*\in VI(VI(C,A_1),A_2)$. This completes the proof. □

3. Conclusion

In this manuscript, we presented an iterative algorithm for solving the double-hierarchical constrained optimization problem with a pseudo-monotone, Lipschitz continuous and sequentially weakly continuous operator in real Hilbert spaces. Under certain assumptions, we established the convergence theorem of the proposed algorithm. Our results extend and generalize some existing results in the literature.

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