Commun Nonlinear Sci Numer Simulat 145 (2025) 108734

Contents lists available at ScienceDirect

Communications in Nonlinear Science and Numerical Simulation

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Two accelerated double inertial algorithms for variational inequalities on Hadamard manifolds

Bing Tan [a], Hammed Anuoluwapo Abass [b,*], Songxiao Li [c], Olawale Kazeem Oyewole [d]

[a] School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
[b] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Science University, P.O. Box 94, Pretoria 0204, South Africa
[c] Department of Mathematics, Shantou University, Shantou 515821, China
[d] Department of Mathematics, Tshwane University of Technology, Arcadia, PMB 0007, Pretoria, South Africa

ARTICLE INFO

MSC:
47H06
47H09
47J25
65K15

Keywords:
Variational inequality problem
Double inertial
Proximal point algorithm
Uniformly continuous
Hadamard manifold

ABSTRACT

Two modified double inertial proximal point algorithms are proposed for solving variational inequality problems with a pseudomonotone vector field in the settings of a Hadamard manifold. Weak convergence of the proposed methods is attained without the requirement of Lipschitz continuity conditions. The convergence efficiency of the proposed algorithms is improved with the help of the double inertial technique and the non-monotonic self-adaptive step size rule. We present a numerical experiment to demonstrate the effectiveness of the proposed algorithm compared to several existing ones. The results extend and generalize many recent methods in the literature.

1. Introduction

The variational inequality problem (VIP for short) is one of the most important theories in applied mathematics. It serves as an effective tool for studying various problems, including equilibrium problems, PDE boundary value problems, and optimization problems (see, e.g. [1]). Such a problem in the setting of a real Hilbert space $\mathbb{H}$ is to find a point $w \in \mathbb{D}$ such that

$$\langle \mathsf{F}w, y - w\rangle \geq 0, \quad \forall y \in \mathbb{D}, \tag{1.1}$$

where $\mathsf{F} : \mathbb{D} \to \mathbb{H}$ is a mapping with $\mathbb{D} \subseteq \mathbb{H}$. To approximate the solution of (1.1), Korpelevich [2] proposed the following extragradient method (in short, EGM):

$$\begin{cases} z_k = Proj_{\mathbb{D}}(w_k - \lambda \mathsf{F} w_k), \\ w_{k+1} = Proj_{\mathbb{D}}(w_k - \lambda \mathsf{F} z_k), \end{cases}$$

where $\lambda \in (0, 1/L)$, $Proj_{\mathbb{D}}(w)$ represents the projection of w on $\mathbb{D}$, and the operator F is monotone and L-Lipschitz continuous. Note that the efficiency of the EGM can be affected by the complex structure of the feasible set $\mathbb{D}$ or the condition on the operator F. To

* Corresponding author.
E-mail addresses: bingtan72@gmail.com (B. Tan), hammed.abass@smu.ac.za (H.A. Abass), jyulsx@163.com (S. Li), Oyewoleolawalekazeem@gmail.com (O.K. Oyewole).

https://doi.org/10.1016/j.cnsns.2025.108734
Received 5 September 2024; Received in revised form 29 January 2025; Accepted 23 February 2025
Available online 3 March 2025

overcome this drawback, Censor et al. [3] proposed the following subgradient extragradient method:

$$\begin{cases} z_k = Proj_{\mathbb{D}}(w_k - \lambda \mathsf{F} w_k), \\ w_{k+1} = Proj_{T_k}(w_k - \lambda \mathsf{F} z_k), \end{cases}$$

where $T_k := \{x \in \mathbb{H} : \langle w_k - \lambda \mathsf{F} w_k - z_k, x - z_k\rangle \le 0\}$ and $\lambda \in (0, 1/L)$. They considered a projection onto a half space instead of the second projection in the EGM to accelerate the convergence of the method. Moreover, Tseng [4] proposed the following algorithm:

$$\begin{cases} z_k = Proj_{\mathbb{D}}(w_k - \lambda \mathsf{F} w_k), \\ w_{k+1} = z_k - \lambda(\mathsf{F} z_k - \mathsf{F} w_k), \end{cases}$$

where $\lambda \in (0, 1/L)$. Note that the Tseng extragradient method only needs to compute the projection once in each iteration. In addition, the projection and contraction method proposed by He [5] also only needs to calculate the projection on the feasible set once in each iteration. The iterative process of the algorithm is as follows:

$$\begin{cases} z_k = Proj_{\mathbb{D}}\left(w_k - \lambda \mathsf{F} w_k\right), \\ y_k = w_k - z_k - \lambda\left(\mathsf{F} w_k - \mathsf{F} z_k\right), \\ \Psi_k = \langle w_k - z_k, y_k\rangle / \|y_k\|^2, \\ w_{k+1} = w_k - \tau\lambda\Psi_k y_k, \end{cases}$$

where $\tau \in (0, 2)$ and $\lambda \in (0, 1/L)$. Under some suitable conditions, weak convergence theorems for the three algorithms in [3–7] are established.

In recent years, there has been an extension of various theories and techniques related to optimization and nonlinear analysis from linear spaces to manifolds; for instance, the concepts of variational inequalities and equilibrium problems have been studied and extended from linear spaces to manifolds (see, e.g. [8–10]). It is worth noting that extending the theory and results of linear spaces to Riemannian manifolds offers several significant advantages; such as, non-convex optimization problems can be transformed into convex problems and constrained problems can be reduced to unconstrained problems on Riemannian manifolds, and non-monotone vector fields can be transformed into monotone vector fields on manifolds by choosing a suitable Riemannian metric. In general, a manifold does not possess a linear structure and when we replace linear spaces with Riemannian manifolds (in particular, Hadamard manifolds), the line segment is replaced by a geodesic. Many known properties and techniques in the linear setting do not hold in the settings of manifolds. Therefore, the extension of the concepts, techniques, and results for various problems from linear spaces to nonlinear manifolds is natural and interesting.

Variational inequality problems on Hadamard manifolds were first introduced and established by Németh [8] for single-valued vector fields. In the settings of a Hadamard manifold, the VIP is to find $w \in \mathbb{D}$ such that

$$\langle \mathsf{F} w, \exp_w^{-1} y\rangle \ge 0, \quad \forall y \in \mathbb{D}, \tag{1.2}$$

where $\mathbb{D}$ is a nonempty, closed, and geodesic convex subset of a Hadamard manifold $\mathcal{B}$, $\mathsf{F} : \mathbb{D} \to T\mathcal{B}$ is a vector field, $T\mathcal{B}$ is the tangent bundle of $\mathcal{B}$, and $\exp^{-1} : \mathcal{B} \to T\mathcal{B}$ is the inverse of the exponential map. We denote the solution set of (1.2) by VIP($\mathbb{D}, \mathsf{F}$).

Since the inception of VIP on Hadamard manifolds, several authors have proposed different iterative algorithms for solving the VIP (1.2). There are two types of methods to solve VIP (1.2) on Hadamard manifolds: one is based on proximal point algorithms, and the other is based on extragradient-type methods. In 2009, Li et al. [9] introduced the proximal point algorithm for the singularity of a maximal monotone vector field on Hadamard manifolds. They proved that the proposed algorithm is also valid for variational inequality problem (1.2) whenever the vector field F is monotone. It is known that many non-monotone problems are encountered in the real world and it is therefore of great importance to investigate the theory and methods for non-monotone (such as pseudomonotone) problems. In 2013, Tang et al. [11] introduced a proximal point algorithm to solve pseudomonotone variational inequality problems on Hadamard manifolds. Recently, Ansari and Uddin [12] proposed two adaptive algorithms for solving the monotone inclusion problem in Hadamard manifolds. The first algorithm uses an Armijo-type step size, while the second one uses a non-monotonic step size criterion. The robustness and good convergence characteristics of the algorithms in [12] have been shown by both theoretical analysis and numerical experiments. On the other hand, Ferreira et al. [13] proposed an extragradient algorithm with an Armijo-type step size to solve the VIP (1.2). Subsequently, Tang and Huang [14] introduced a modified extragradient algorithm to find solutions for pseudomonotone variational inequality problems on Hadamard manifolds. In addition, Tang et al. [15] proposed a projection-based extragradient algorithm to solve variational inequality problems with pseudomonotone and continuous vector fields. Recently, Sahu et al. [16] explored an iterative scheme based on the extragradient algorithm to solve VIP (1.2). Under some suitable conditions, they proved that the proposed algorithm could be applied to monotone or non-monotone variational inequality problems. Furthermore, they demonstrated the computational advantages of the proposed algorithm compared to the algorithms in [14,15] through a numerical experiment. For more algorithms used to solve VIP (1.2), one can refer to [17,18]. On the other hand, it is worth noting that the convergence rate of algorithms is crucial when solving optimization problems. To accelerate the convergence speed of iterative algorithms, Polyak [19] proposed an inertial extrapolation method based on the discrete form of second-order dissipative dynamical systems, which takes the form $w_k + \beta_k(w_k - w_{k-1})$, where β_k is called the inertial parameter. In [20], it is mentioned through an example that a single-step inertial extrapolation might not provide acceleration. Recently, some methods based on double inertial techniques have been proposed to address variational inequality problems in real Hilbert spaces; see, e.g., [21–24]. These methods have been validated in numerical experiments to be more computationally efficient compared to some single-step inertial methods.

Note that the results on pseudomonotone variational inequalities are limited due to the nonlinearity of manifolds and also the stepsizes used in most of these results depend on the Lipschitz constant of the vector field F. In this paper, we consider the uniform continuity condition on the vector field F, which is a weaker notion than the Lipschitz continuity condition, since it is noted that the latter can be difficult to compute. To the best of our knowledge, the proximal point algorithm has not been employed for solving variational inequalities involving a pseudomonotone and uniformly continuous vector field in Hadamard manifolds.

Motivated by the results described above, we propose two modified proximal point methods together with a double inertial technique for solving variational inequalities associated with a pseudomonotone and uniformly continuous vector field in Hadamard manifolds. Unlike many of the existing results in the literature, our proposed methods do not require the Lipschitz continuity assumption of the vector field. We incorporated our proposed methods with a suitable self-adaptive step size technique that generates a nonmonotonic sequence and is independent of the Lipschitz continuity modulus of F. The results discussed in this paper generalize the algorithms of Ansari and Uddin [12], Ferreira et al. [13], Tang and Huang [14], Tang et al. [15], Sahu et al. [16], and many related methods in the literature.

Our paper is organized as follows. Some important definitions and preliminary results for further use are given in Section 2. In Section 3, we discuss the convergence analysis of our proposed algorithms. In Section 4, some numerical results are given to illustrate the performance of our methods. Finally, we summarize the paper in Section 5.

2. Preliminaries

Let $\mathcal{B}$ be a Riemannian manifold of finite dimension, $T_p\mathcal{B}$ be the tangent space of $\mathcal{B}$ at the point $p \in \mathcal{B}$, and $T\mathcal{B}$ be the tangent bundle of $\mathcal{B}$ defined as $T\mathcal{B} = \bigcup_{p\in\mathcal{B}} T_p\mathcal{B}$. We denote the Riemannian metric $\langle \cdot, \cdot \rangle_p$ on $T_p\mathcal{B}$ and the corresponding norm $\|\cdot\|_p$. We can omit subscript p if no confusion occurs. Given a piecewise smooth curve $\gamma : [r, s] \to \mathcal{B}$ joining p to q (that is, $\gamma(r) = p$ and $\gamma(s) = q$), we define the length of the curve γ as $\ell(\gamma) := \int_r^s \|\dot\gamma(t)\| \,\mathrm{d}t$, where $\dot\gamma(t)$ denotes the tangent vector at $\gamma(t)$ in the tangent space $T_{\gamma(t)}\mathcal{B}$. The Riemannian distance between p and q is denoted by $d(p, q)$, which is the minimal length over the set of all such curves joining p and q.

Let ∇ be a Levi-Civita connection associated with the Riemannian metric. For a smooth curve γ, a tangent vector X along γ is said to be parallel if $\nabla_{\dot\gamma} X = \mathbf{0}$, where $\mathbf{0}$ denotes the zero tangent vector. If $\dot\gamma$ itself is parallel along γ, then γ is called a geodesic and $\|\dot\gamma\|$ is a constant. A geodesic joining p to q in $\mathcal{B}$ is called a minimizing geodesic if its length equals $d(p, q)$. If for any p in a Riemannian manifold $\mathcal{B}$, all geodesic emanating from p are defined for all $t \in \mathbb{R}$, then the Riemannian manifold $\mathcal{B}$ is said to be complete.

For a Riemannian manifold $\mathcal{B}$, the exponential map $\exp_p : T_p\mathcal{B} \to \mathcal{B}$ at $p \in \mathcal{B}$ is defined by $\exp_p v := \gamma_v(1, p)$, $\forall v \in T_p\mathcal{B}$, where $\gamma_v(\cdot, p)$ is the geodesic starting from p with velocity v, i.e., $\gamma_v(\mathbf{0}, p) = p$ and $\dot\gamma_v(\mathbf{0}, p) = v$. Note that the exponential map $\exp_p$ is differentiable on $T_p\mathcal{B}$ for any $p \in \mathcal{B}$. The inverse of the exponential map exists, denoted by $\exp_p^{-1} : \mathcal{B} \to T_p\mathcal{B}$. For any $p, q \in \mathcal{B}$, we have $d(p, q) = \|\exp_p^{-1} q\| = \|\exp_q^{-1} p\|$.

To compare the tangent vectors of different tangent spaces, we employ the parallel transport $P_{\gamma,\gamma(s)\gamma(r)} : T_{\gamma(r)}\mathcal{B} \to T_{\gamma(s)}\mathcal{B}$ with respect to ∇, defined by

$$P_{\gamma,\gamma(s),\gamma(r)} v = \mathsf{F}(\gamma(s)), \quad \forall r, s \in \mathbb{R}, \forall v \in T_{\gamma(r)}\mathcal{B},$$

where F is a vector field such that $\nabla_{\dot\gamma(t)}\mathsf{F} = \mathbf{0}$ for all $t \in [r, s]$ and $\mathsf{F}(\gamma(r)) = v$. When γ is a minimal geodesic joining $\gamma(r)$ to $\gamma(s)$, we write $P_{\gamma(s),\gamma(r)}$ instead of $P_{\gamma,\gamma(s),\gamma(r)}$. For every $r, s, u \in \mathbb{R}$, we have $P_{\gamma(s),\gamma(u)} \circ P_{\gamma(u),\gamma(r)} = P_{\gamma(s),\gamma(r)}$. Note that $P_{\gamma(s),\gamma(r)}$ is an isometry from $T_{\gamma(r)}\mathcal{B}$ to $T_{\gamma(s)}\mathcal{B}$.

A Hadamard manifold is a complete, simply connected Riemannian manifold with nonpositive sectional curvature. In the following, we always use $\mathcal{B}$ to represent a Hadamard manifold. Next, we review some definitions, properties, and lemmas that will be used in the subsequent convergence analysis.

Definition 2.1 (*[25]*). Let $\mathsf{F} : \mathbb{D} \to T\mathcal{B}$ be such that $\mathsf{F}p \in T_p\mathcal{B}$ for each $p \in \mathbb{D}$. A vector field F is said to be:

(i) monotone, if

$$\langle \mathsf{F}p, \exp_p^{-1} q \rangle \le \langle \mathsf{F}q, -\exp_q^{-1} p \rangle, \quad \forall p, q \in \mathbb{D}.$$

(ii) pseudomonotone, if

$$\langle \mathsf{F}p, \exp_p^{-1} q \rangle \ge 0 \Rightarrow \langle \mathsf{F}q, \exp_q^{-1} p \rangle \le 0, \quad \forall p, q \in \mathbb{D}.$$

(iii) L-Lipschitz continuous, if there exists $L > 0$ such that

$$\|P_{p,q}\mathsf{F}q - \mathsf{F}p\| \le L d(p, q), \quad \forall p, q \in \mathbb{D}.$$

(iv) uniformly continuous, if for all $p, q \in \mathbb{D}$, there exists $\epsilon > 0$ and $\delta = \delta(\epsilon) > 0$ such that

$$\|P_{p,q}\mathsf{F}q - \mathsf{F}p\| \le \epsilon, \quad \text{whenever } d(p, q) < \delta.$$

Definition 2.2 (*[26]*). A subset $\mathbb{D}$ of a Hadamard manifold $\mathcal{B}$ is said to be (geodesic) convex if for any two points $p, q \in \mathbb{D}$, the geodesic $\gamma : [r, s] \to \mathcal{B}$ satisfies $\gamma(rt + (1 - t)s) \in \mathbb{D}$ for all $t \in [0, 1]$ and $r, s \in \mathbb{R}$, where $p = \gamma(r)$ and $q = \gamma(s)$.

Proposition 2.1 ([26]). *Let $p \in \mathcal{B}$. The exponential mapping $\exp_p : T_p\mathcal{B} \to \mathcal{B}$ is a diffeomorphism. For any two points $p, q \in \mathcal{B}$, there exists a unique normalized geodesic joining p to q, which is given by $\gamma(t) = \exp_p t \exp_p^{-1} q$ for all $t \in [0,1]$.*

A geodesic triangle $\Delta(p,q,r)$ of a Hadamard manifold $\mathcal{B}$ is a set containing three points p, q, r and three minimizing geodesics joining these points.

Proposition 2.2 ([26]). *Let $\Delta(p,q,r)$ be a geodesic triangle in $\mathcal{B}$. Then*

$$d^2(p,q) + d^2(q,r) - 2\langle \exp_q^{-1} p, \exp_q^{-1} r\rangle \le d^2(r,q),$$

and

$$d^2(p,q) \le \langle \exp_p^{-1} r, \exp_p^{-1} q\rangle + \langle \exp_q^{-1} r, \exp_q^{-1} p\rangle.$$

Moreover, if θ is the angle at p, then

$$\langle \exp_p^{-1} q, \exp_p^{-1} r\rangle = d(q,p)d(p,r)\cos\theta. \tag{2.1}$$

Note that

$$\|\exp_p^{-1} q\|^2 = \langle \exp_p^{-1} q, \exp_p^{-1} q\rangle = d^2(p,q). \tag{2.2}$$

Proposition 2.3 ([9]). *If $p, q \in \mathcal{B}$ and $v \in T_q\mathcal{B}$, then*

$$\langle v, -\exp_q^{-1} p\rangle = \langle v, P_{q,p}\exp_p^{-1} q\rangle = \langle P_{p,q} v, \exp_p^{-1} q\rangle.$$

The following lemmas are very useful in our convergence analysis.

Lemma 2.1 ([27]). *Let $\mathbb{D}$ be a nonempty, closed, and convex subset of a Hadamard manifold $\mathcal{B}$, then $\mathsf{F} : \mathbb{D} \to T\mathcal{B}$ is said to be uniformly continuous if and only if, for every $\epsilon > 0$, there exists a constant $M < +\infty$ such that*

$$\|P_{q,p}\mathsf{F}p - \mathsf{F}q\| \le M d(p,q) + \epsilon, \quad \forall p,q \in \mathbb{D}.$$

Proof. Suppose that $\mathsf{F} : \mathbb{D} \to T\mathcal{B}$ is uniformly continuous and fix $\epsilon > 0$. Then, there exists a $\delta > 0$ such that $\|P_{p,q}\mathsf{F}q - \mathsf{F}p\| \le \epsilon$, whenever $d(p,q) < \delta$. Fix p and q in $\mathbb{D}$ and let

$$v_k = \exp_p(-k)\frac{\delta}{2}\frac{\exp_p^{-1} q}{d(q,p)}, \text{ for } k = 0,1,2\cdots,N,$$

where $N = \frac{\lfloor d(q,p)\rfloor}{\frac{\delta}{2}}$, and $\lfloor\cdot\rfloor$ denotes the greatest integer function. From the convexity of $\mathbb{D}$, we see that v_k belongs to $\mathbb{D}$. Also,

$$v_0 = p.$$

$$d(v_k, v_{k-1}) = \frac{\delta}{2},$$

and

$$d(q, v_N) < \frac{\delta}{2}.$$

The remaining part of the proof follows from Theorem 1 of [27], so we omit it. □

Lemma 2.2 ([28]). *Let $\Delta(w_1,w_2,w_3)$ be a geodesic triangle in $\mathcal{B}$. There exists a comparison triangle $\Delta(\bar{w}_1,\bar{w}_2,\bar{w}_3)$ corresponding to $\Delta(w_1,w_2,w_3)$ such that $d(w_i,w_{i+1}) = \|\bar{w}_i - \bar{w}_{i+1}\|$ with the indices taken modulo 3. The points $\bar{w}_1$, $\bar{w}_2$ and $\bar{w}_3$ are called comparison points to w_1, w_2 and w_3. This triangle is unique up to isometries of $\mathbb{R}^2$.*

Lemma 2.3 ([29]). *Let $\Delta(p,q,r)$ be a geodesic triangle in a Hadamard manifold $\mathcal{B}$ and $\Delta(p',q',r')$ be its comparison triangle. Let α,β,γ (resp., α',β',γ') be the angles of $\Delta(p,q,r)$ (resp., $\Delta(p',q',r')$) at the vertices p,q,r (resp., p',q',r'). Then $\alpha' \ge \alpha$, $\beta' \ge \beta$, and $\gamma' \ge \gamma$.*

Lemma 2.4. *Let $t_k, q_k, u_k, u_{k+1} \in \mathcal{B}$. Consider the geodesic triangle $\Delta(t_k,q_k,u_k)$ (resp., $\Delta(t_k,q_k,u_{k+1})$) and its comparison triangle $\Delta(t'_k,q'_k,u'_k)$ (resp., $\Delta(t'_k,q'_k,u'_{k+1})$). Then*

$$\left\langle u'_{k+1} - q'_k, t'_k - u'_k\right\rangle \le \left\langle P_{u_k,q_k}\exp_{q_k}^{-1} u_{k+1}, \exp_{u_k}^{-1} t_k\right\rangle.$$

Proof. Let $\Psi = \exp_{q_k}^{-1} u_{k+1}$ and $a = \exp_{u_k} P_{u_k,q_k} \exp_{q_k}^{-1} u_{k+1}$. The comparison point of a is $a' = u'_k + u'_{k+1} - q'_k$. Let β (resp., β') be the angle of $\Delta(a, u_k, t_k)$ (resp., $\Delta(a', u'_k, t'_k)$) at the vertice u_k (resp., u'_k). From Lemma 2.3, we have $\beta' \geq \beta$. since $\beta, \beta' \in (0, \pi)$, it follows that $\cos\beta' \leq \cos\beta$. By using Lemma 2.2 and (2.1), one obtains

$$\begin{aligned}\left\langle a' - u'_k, t'_k - u'_k \right\rangle &= \left\|a' - u'_k\right\| \left\|t'_k - u'_k\right\| \cos\beta' \\ &\leq d\left(a, u_k\right) d\left(t_k, u_k\right) \cos\beta \\ &= \left\langle \exp_{u_k}^{-1}\left(\exp_{u_k} P_{u_k,q_k} \exp_{q_k}^{-1} u_{k+1}\right), \exp_{u_k}^{-1} t_k \right\rangle. \\ &= \left\langle P_{u_k,q_k} \exp_{q_k}^{-1} u_{k+1}, \exp_{u_k}^{-1} t_k \right\rangle.\end{aligned}$$

This completes the proof. □

Lemma 2.5 ([30]). *Let $\{a_k\}, \{\varphi_k\}$, and $\{\beta_k\}$ be nonnegative sequences which satisfy*

$$a_{k+1} \leq (1+\beta_k)a_k + \varphi_k, \quad \forall k \geq 1.$$

If $\sum_{k=1}^{\infty} \beta_k < +\infty$ and $\sum_{k=1}^{\infty} \varphi_k < +\infty$, then $\lim_{k\to\infty} a_k$ exists.

Lemma 2.6 ([31]). *Let $\{v_k\}$ and $\{\delta_k\}$ be nonnegative sequences which satisfy*

$$v_{k+1} \leq (1+\delta_k)v_k + \delta_k v_{k-1}, \quad \forall k \geq 1.$$

Then $v_{k+1} \leq M \cdot \prod_{j=1}^{k}(1+2\delta_j)$, where $M = \max\{v_1, v_2\}$. Moreover, if $\sum_{k=1}^{\infty} \delta_k < +\infty$, then $\{v_k\}$ is bounded.

Lemma 2.7 ([9]). *Let $w_0 \in \mathcal{B}$ and $\{w_k\} \subset \mathcal{B}$ with $w_k \to w_0$ as $k \to \infty$. Then the following assertions hold:*

(i) For any $y \in \mathcal{B}$, we have $\exp_{w_k}^{-1} y \to \exp_{w_0}^{-1} y$ and $\exp_y^{-1} w_k \to \exp_y^{-1} w_0$;
(ii) Given $u_k, v_k \in T_{w_k}\mathcal{B}$ and $u_0, v_0 \in T_{w_0}\mathcal{B}$, if $u_k \to u_0$ and $v_k \to v_0$, then $\langle u_k, v_k\rangle \to \langle u_0, v_0\rangle$;
(iii) For any $u \in T_{w_0}\mathcal{B}$, the function $\mathsf{F} : \mathcal{B} \to T\mathcal{B}$, defined by $\mathsf{F}(x) = P_{x,w_0}u$ for each $x \in \mathcal{B}$ is continuous on $\mathcal{B}$.

Lemma 2.8 ([32]). *Let $\mathbb{D}$ be a nonempty, closed, and convex subset of $\mathcal{B}$ and $\{w_k\} \subset \mathcal{B}$. Assume that (i) for every $p \in \mathbb{D}$, $\lim_{k\to\infty} d(w_k, p)$ exists; and (ii) every cluster point of $\{w_k\}$ belongs to $\mathbb{D}$. Then $\{w_k\}$ converges to a point in $\mathbb{D}$.*

3. Main result

In this section, we present two modified proximal point algorithm with the double inertial method and the self-adaptive step size rule for solving variational inequality problem (1.2). Under certain suitable conditions, we established the convergence theorems of the suggested algorithms. The first proposed iterative scheme is shown in Algorithm 3.1.

To analyze the convergence of Algorithm 3.1, we assume that the following Assumption 3.1 holds.

Assumption 3.1.

(A1) The vector field $\mathsf{F} : \mathbb{D} \to T\mathcal{B}$ is pseudomonotone and uniformly continuous.
(A2) The solution set $\mathrm{VIP}(\mathbb{D}, \mathsf{F})$ is nonempty.
(A3) The sequences $\{\beta_k\}$, $\{\theta_k\}$, and $\{\delta_k\}$ satisfy $\sum_{k=1}^{\infty} \beta_k < +\infty$, $\sum_{k=1}^{\infty} \delta_k < +\infty$, and $\sum_{k=1}^{\infty} \theta_k < +\infty$.

The following two lemmas are crucial for the convergence analysis of the algorithm.

Lemma 3.1. *Let $\{\lambda_k\}$ be a sequence generated by Algorithm 3.1. Suppose that Assumption 3.1 holds. Then, $\{\lambda_k\}$ is well-defined and $\lim_{k\to\infty} \lambda_k = \lambda \in \big[\min\{\mu/M, \lambda_1\}, \lambda_1 + \sum_{k=1}^{\infty} \theta_k\big]$ for some $M > 0$.*

Proof. It is obvious that $\{\lambda_k\}$ is a non-monotonic sequence. Since F is uniformly continuous, it follows from Lemma 2.1 that for any $\epsilon > 0$, there exists $K < +\infty$ such that $\|P_{z_k,u_k}\mathsf{F}u_k - \mathsf{F}z_k\| \leq Kd(u_k, z_k) + \epsilon$. In the case of $\|P_{z_k,u_k}\mathsf{F}u_k - \mathsf{F}z_k\| \neq 0$, we obtain

$$\frac{\mu d(u_k, z_k)}{\|P_{z_k,u_k}\mathsf{F}u_k - \mathsf{F}z_k\|} \geq \frac{\mu d(u_k, z_k)}{Kd(u_k, z_k) + \epsilon} = \frac{\mu d(u_k, z_k)}{(K+\epsilon_1)d(u_k, z_k)} = \frac{\mu}{M},$$

where $\epsilon = \epsilon_1 d(u_k, z_k)$ and $M = K + \epsilon_1$. Thus, by the definition of λ_{k+1} in (3.5), the sequence $\{\lambda_k\}$ has lower bound $\{\mu/M, \lambda_1\}$ and upper bound $\lambda_1 + \sum_{k=1}^{\infty} \theta_k$. From Lemma 2.5, one obtains that $\lim_{k\to\infty} \lambda_k$ exists and $\lim_{k\to\infty} \lambda_k = \lambda$. Clearly, we have $\lambda \in \big[\min\{\mu/M, \lambda_1\}, \lambda_1 + \sum_{k=1}^{\infty} \theta_k\big]$. This completes the proof. □

Lemma 3.2. *If $u_k = z_k$ or $y_k = 0$ in Algorithm 3.1, then $u_k \in \mathrm{VIP}(\mathbb{D}, \mathsf{F})$.*

Algorithm 3.1 The first modified proximal point algorithm

Initialization: Choose $\lambda_1 > 0$, $\mu \in (0,1)$, $\tau \in (0,2)$, $\{\beta_k\}$, $\{\theta_k\}$, and $\{\delta_k\}$ are real positive sequences for all $k \in \mathbb{N}$. Let $w_0, w_1 \in \mathcal{B}$ be arbitrary.

Step 1: Given the current iterates w_{k-1}, w_k, compute

$$\begin{cases} v_k = \exp_{w_k}(-\beta_k \exp_{w_k}^{-1} w_{k-1}), \\ u_k = \exp_{v_k}(-\delta_k \exp_{v_k}^{-1} w_{k-1}). \end{cases} \tag{3.1}$$

Step 2: Compute $z_k \in \mathbb{D}$ such that

$$\left\langle \exp_{z_k}^{-1} u_k - \lambda_k P_{z_k,u_k} \mathsf{F} u_k, \exp_{z_k}^{-1} y \right\rangle \le 0, \quad \forall y \in \mathbb{D}. \tag{3.2}$$

If $u_k = z_k$, then stop and $u_k \in \mathrm{VIP}(\mathbb{D}, \mathsf{F})$. Otherwise, go to Step 3.

Step 3: Calculate

$$w_{k+1} = \exp_{u_k}(-\tau \Psi_k P_{u_k,z_k} y_k), \tag{3.3}$$

where $y_k = \exp_{z_k}^{-1} u_k - \lambda_k (P_{z_k,u_k} \mathsf{F} u_k - \mathsf{F} z_k)$ and

$$\Psi_k = \begin{cases} \dfrac{\langle \exp_{z_k}^{-1} u_k, y_k \rangle}{\|y_k\|^2}, & \text{if } y_k \neq 0, \\ 0, & \text{if } y_k = 0. \end{cases} \tag{3.4}$$

Update λ_{k+1} by

$$\lambda_{k+1} = \begin{cases} \min\left\{ \dfrac{\mu d(u_k, z_k)}{\|P_{z_k,u_k} \mathsf{F} u_k - \mathsf{F} z_k\|}, \lambda_k + \theta_k \right\}, & \text{if } \|P_{z_k,u_k} \mathsf{F} u_k - \mathsf{F} z_k\| \neq 0, \\ \lambda_k + \theta_k, & \text{otherwise.} \end{cases} \tag{3.5}$$

Set $k := k+1$ and return to Step 1.

Proof. By applying the definition of y_k in Algorithm 3.1 and (3.5), we have

$$\begin{aligned} \|y_k\| &= \| \exp_{z_k}^{-1} u_k - \lambda_k (P_{z_k,u_k} \mathsf{F} u_k - \mathsf{F} z_k)\| \\ &\ge \| \exp_{z_k}^{-1} u_k\| - \lambda_k \|P_{z_k,u_k} \mathsf{F} u_k - \mathsf{F} z_k\| \\ &\ge d(u_k, z_k) - \frac{\mu \lambda_k}{\lambda_{k+1}} d(u_k, z_k) \\ &= \left(1 - \mu \frac{\lambda_k}{\lambda_{k+1}}\right) d(u_k, z_k), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \|y_k\| &\le \| \exp_{z_k}^{-1} u_k\| + \lambda_k \|P_{z_k,u_k} \mathsf{F} u_k - \mathsf{F} z_k\| \\ &\le \left(1 + \mu \frac{\lambda_k}{\lambda_{k+1}}\right) d(u_k, z_k). \end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we deduce that

$$\left(1 - \mu \frac{\lambda_k}{\lambda_{k+1}}\right) d(u_k, z_k) \le \|y_k\| \le \left(1 + \mu \frac{\lambda_k}{\lambda_{k+1}}\right) d(u_k, z_k).$$

Thus, $y_k = 0$ if and only if $u_k = z_k$. If $u_k = z_k$, then by (3.2), one sees that u_k is a solution of Problem (1.2). □

Lemma 3.3. *Suppose that Assumption 3.1 holds. Let $\{u_k\}$, $\{z_k\}$ and $\{w_k\}$ be sequences generated by Algorithm 3.1. Then, for each $q \in \mathrm{VIP}(\mathbb{D}, \mathsf{F})$,*

$$d^2(w_{k+1}, q) \le d^2(u_k, q) - \tau(2-\tau) \frac{\left(1 - \mu \frac{\lambda_k}{\lambda_{k+1}}\right)^2}{\left(1 + \mu \frac{\lambda_k}{\lambda_{k+1}}\right)^2} d^2(u_k, z_k), \quad \forall k \ge 0.$$

$\lfloor x \rfloor$

Proof. Fix $q \in \mathrm{VIP}(\mathbb{D}, \mathsf{F})$. Consider the geodesic triangle $\Delta\left(q, w_{k+1}, u_k\right)$ and its comparison triangle $\Delta\left(q', w'_{k+1}, u'_k\right)$. It follows from Lemma 2.2 that

$$d\left(q, u_k\right) = \left\|q' - u'_k\right\|, \quad d\left(w_{k+1}, u_k\right) = \left\|w'_{k+1} - u'_k\right\|, \quad d\left(q, w_{k+1}\right) = \left\|q' - w'_{k+1}\right\|.$$

The comparison point for w_{k+1} is $w'_{k+1} = u'_k - \tau\Psi_k\left(u'_k - z'_k + \lambda_k(\mathsf{F}z'_k - \mathsf{F}u'_k)\right)$. For convenience, we let $\chi'_k = u'_k - z'_k + \lambda_k(\mathsf{F}z'_k - \mathsf{F}u'_k)$. From the definitions of w_{k+1} and w'_{k+1}, we have

$$\tau\Psi_k\left\|y_k\right\| = d\left(w_{k+1}, u_k\right) = \left\|w'_{k+1} - u'_k\right\| = \tau\Psi_k\left\|\chi'_k\right\|.$$

By the definition of w_{k+1} in (3.3), we obtain

$$\begin{aligned} d^2\left(w_{k+1}, q\right) &= \|w'_{k+1} - q'\|^2 \\ &= \|u'_k - \tau\Psi_k\chi'_k - q'\|^2 \\ &= \|u'_k - q'\|^2 - 2\tau\Psi_k\langle u'_k - q', \chi'_k\rangle + \tau^2\Psi_k^2\|\chi'_k\|^2 \\ &= d^2\left(u_k, q\right) + 2\tau\Psi_k\langle q' - u'_k, \chi'_k\rangle + \tau^2\Psi_k^2\|y_k\|^2. \end{aligned} \tag{3.8}$$

Let $a := \exp_{z_k} y_k$ and $b := \exp_{u_k} P_{u_k,z_k} y_k$. The comparison points of a and b are $a' = u'_k + \lambda_k\left(\mathsf{F}z'_k - \mathsf{F}u'_k\right)$ and $b' = 2u'_k - z'_k + \lambda_k\left(\mathsf{F}z'_k - \mathsf{F}u'_k\right)$, respectively. By the definition of χ'_k and Lemma 2.4, one has

$$\begin{aligned} \langle q' - u'_k, \chi'_k\rangle &= \langle z'_k - u'_k, \chi'_k\rangle + \langle q' - z'_k, \chi'_k\rangle \\ &= \langle z'_k - u'_k, b' - u'_k\rangle + \langle q' - z'_k, a' - z'_k\rangle \\ &\le \left\langle \exp^{-1}_{u_k} z_k, \exp^{-1}_{u_k} b\right\rangle + \left\langle \exp^{-1}_{z_k} q, \exp^{-1}_{z_k} a\right\rangle \\ &= \left\langle \exp^{-1}_{u_k} z_k, P_{u_k,z_k} y_k\right\rangle + \left\langle \exp^{-1}_{z_k} q, y_k\right\rangle. \end{aligned} \tag{3.9}$$

From Proposition 2.3 and (3.4), one obtains

$$\left\langle \exp^{-1}_{u_k} z_k, P_{u_k,z_k} y_k\right\rangle = -\left\langle \exp^{-1}_{z_k} u_k, y_k\right\rangle = -\Psi_k\left\|y_k\right\|^2. \tag{3.10}$$

By the definition of z_k and $q \in \mathbb{D}$, one sees that

$$\left\langle \exp^{-1}_{z_k} u_k - \lambda_k P_{z_k,u_k}\mathsf{F}u_k, \exp^{-1}_{z_k} q\right\rangle \le 0. \tag{3.11}$$

Since $q \in \mathrm{VIP}(\mathbb{D}, \mathsf{F})$ and $z_k \in \mathbb{D}$, we have $\left\langle \mathsf{F}q, \exp^{-1}_{q} z_k\right\rangle \ge 0$. It follows from the pseudomonotonicity of F that $\left\langle \mathsf{F}z_k, \exp^{-1}_{z_k} q\right\rangle \le 0$. This combined with (3.11) gives

$$\left\langle \exp^{-1}_{z_k} u_k + \lambda_k\left(\mathsf{F}z_k - P_{z_k,u_k}\mathsf{F}u_k\right), \exp^{-1}_{z_k} q\right\rangle \le 0. \tag{3.12}$$

Using (3.9), (3.10), and (3.12), we have

$$\langle q' - u'_k, \chi'_k\rangle \le -\Psi_k\left\|y_k\right\|^2. \tag{3.13}$$

It follows from (3.7) that

$$\frac{1}{\|y_k\|^2} \ge \frac{1}{\left(1 + \mu\frac{\lambda_k}{\lambda_{k+1}}\right)^2 d^2(u_k, z_k)}. \tag{3.14}$$

According to (3.4), (3.5), and Cauchy–Schwarz inequality, we have

$$\begin{aligned} \langle \exp^{-1}_{z_k} u_k, y_k\rangle &= \langle \exp^{-1}_{z_k} u_k, \exp^{-1}_{z_k} u_k - \lambda_k(P_{z_k,u_k}\mathsf{F}u_k - \mathsf{F}z_k)\rangle \\ &= \langle \exp^{-1}_{z_k} u_k, \exp^{-1}_{z_k} u_k\rangle - \lambda_k\langle \exp^{-1}_{z_k} u_k, P_{z_k,u_k}\mathsf{F}u_k - \mathsf{F}z_k\rangle \\ &\ge \|\exp^{-1}_{z_k} u_k\|^2 - \lambda_k\|\exp^{-1}_{z_k} u_k\|\,\|P_{z_k,u_k}\mathsf{F}u_k - \mathsf{F}z_k\| \\ &\ge \left(1 - \mu\frac{\lambda_k}{\lambda_{k+1}}\right)d^2(u_k, z_k). \end{aligned} \tag{3.15}$$

From (3.14) and (3.15), one has

$$\Psi_k = \frac{\langle \exp^{-1}_{z_k} u_k, y_k\rangle}{\|y_k\|^2} \ge \frac{\left(1 - \mu\frac{\lambda_k}{\lambda_{k+1}}\right)}{\left(1 + \mu\frac{\lambda_k}{\lambda_{k+1}}\right)^2}. \tag{3.16}$$

By (3.10), (3.15), and (3.16), we have

$$\Psi_k^2\|y_k\|^2 \ge \frac{\left(1 - \mu\frac{\lambda_k}{\lambda_{k+1}}\right)^2}{\left(1 + \mu\frac{\lambda_k}{\lambda_{k+1}}\right)^2} d^2(u_k, z_k).$$

This together with (3.8) and (3.13) yields

$$\begin{aligned} d^2\left(w_{k+1}, q\right) &\le d^2\left(u_k, q\right) - 2\tau\Psi_k^2\left\|y_k\right\|^2 + \tau^2\Psi_k^2\|y_k\|^2 \\ &\le d^2\left(u_k, q\right) - \tau\left(2 - \tau\right)\frac{\left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right)^2}{\left(1 + \frac{\mu\lambda_k}{\lambda_{k+1}}\right)^2} d^2\left(u_k, z_k\right). \end{aligned}$$

This completes the proof. □

Theorem 3.1. *Suppose that Assumption 3.1 holds and let $\{w_k\}$ be a sequence generated by Algorithm 3.1, then $\{w_k\}$ converges to an element of* $\mathrm{VIP}(\mathbb{D},\mathsf{F})$.

Proof. Fix $q \in \mathrm{VIP}(\mathbb{D},\mathsf{F})$. Consider the geodesic triangles $\Delta(v_k, w_k, q)$ and $\Delta(w_k, w_{k-1}, q)$ with their comparison triangles $\Delta(v'_k, w'_k, q')$ and $\Delta(w'_k, w'_{k-1}, q')$. By applying Lemma 2.2, we have

$$\begin{aligned}&d(v_k, w_k) = \|v'_k - w'_k\|, \quad d(v_k, q) = \|v'_k - q'\|, \quad d(w_k, w_{k-1}) = \|w'_k - w'_{k-1}\|,\\ &d(w_k, q) = \|w'_k - q'\|, \quad d(w_{k-1}, q) = \|w'_{k-1} - q'\|.\end{aligned}$$

In addition, consider the geodesic triangles $\Delta(w_{k-1}, v_k, q)$ and $\Delta(w_k, u_k, q)$ with their comparison triangles $\Delta(w'_{k-1}, v'_k, q')$ and $\Delta(w'_k, u'_k, q')$. It follows from Lemma 2.2 that

$$d(w_{k-1}, v_k) = \|w'_{k-1} - v'_k\|, \quad d(u_k, q) = \|u'_k - q'\|, \quad d(u_k, w_k) = \|u'_k - w'_k\|.$$

According to (3.1), the comparison points of v_k and u_k are $v'_k = w'_k + \beta_k(w'_k - w'_{k-1})$ and $u'_k = v'_k + \delta_k(v'_k - w'_{k-1})$, respectively. Therefore,

$$\begin{aligned}d(v_k, q) &= \|v'_k - q'\|\\ &\le \|w'_k - q'\| + \beta_k\|w'_k - w'_{k-1}\|\\ &= d(w_k, q) + \beta_k d(w_k, w_{k-1}),\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}d(v_k, w_{k-1}) &= \|v'_k - w'_{k-1}\|\\ &\le \|w'_k - w'_{k-1}\| + \beta_k\|w'_k - w'_{k-1}\|\\ &= (1+\beta_k) d(w_k, w_{k-1}).\end{aligned} \tag{3.18}$$

From the definition of u_k, (3.17), and (3.18), we obtain

$$\begin{aligned}d(u_k, q) &= \|u'_k - q'\|\\ &\le d(v_k, q) + \delta_k d(v_k, w_{k-1})\\ &\le d(w_k, q) + (\beta_k + \delta_k(1+\beta_k)) d(w_k, w_{k-1}).\end{aligned} \tag{3.19}$$

Since $\tau \in (0,2)$, $\lim_{k\to\infty} \lambda_k = \lambda > 0$, and $\mu \in (0,1)$, it follows from Lemma 3.3 and (3.19) that

$$d(w_{k+1}, q) \le d(w_k, q) + (\beta_k + \delta_k(1+\beta_k)) d(w_k, w_{k-1}). \tag{3.20}$$

This gives

$$\begin{aligned}d(w_{k+1}, q) &\le d(w_k, q) + (\beta_k + \delta_k(1+\beta_k))(d(w_k, q) + d(w_{k-1}, q))\\ &= (1 + \beta_k + \delta_k(1+\beta_k)) d(w_k, q) + (\beta_k + \delta_k(1+\beta_k)) d(w_{k-1}, q).\end{aligned}$$

By utilizing Lemma 2.6, we obtain that

$$d(w_{k+1}, q) \le M_1 \cdot \prod_{j=1}^{k} (1 + 2(\beta_j + \delta_j(1+\beta_j))), \tag{3.21}$$

where $M_1 = \max\{d(w_1, q), d(w_2, q)\}$. Since $\sum_{k=1}^{\infty} \beta_k < +\infty$ and $\sum_{k=1}^{\infty} \delta_k < +\infty$, by Lemma 2.6 and (3.21), the sequence $\{d(w_k, q)\}$ is bounded. This also implies that $\sum_{k=1}^{\infty} \beta_k d(w_k, w_{k-1}) < +\infty$ and $\sum_{k=1}^{\infty} \delta_k d(w_k, w_{k-1}) < +\infty$. By utilizing Lemma 2.5 in (3.20), we claim that $\lim_{k\to\infty} d(w_k, q)$ exists. It follows from Lemma 2.2 that

$$\begin{aligned}d^2(v_k, q) &= \|w'_k + \beta_k(w'_k - w'_{k-1}) - q'\|^2\\ &= \|(1+\beta_k)(w'_k - q') - \beta_k(w'_{k-1} - q')\|^2\\ &= (1+\beta_k) d^2(w_k, q) - \beta_k d^2(w_{k-1}, q)\\ &\quad + \beta_k(1+\beta_k) d^2(w_k, w_{k-1}).\end{aligned} \tag{3.22}$$

By utilizing (2.2), we have

$$\begin{aligned}d^2(v_k, w_{k-1}) &= \|w'_k + \beta_k(w'_k - w'_{k-1}) - w'_{k-1}\|^2\\ &= \|w'_k - w'_{k-1}\|^2 + 2\langle w'_k - w'_{k-1}, \beta_k(w'_k - w'_{k-1})\rangle\\ &\quad + \beta_k^2\|w'_k - w'_{k-1}\|^2\\ &= d^2(w_k, w_{k-1}) + 2\beta_k d^2(w_k, w_{k-1}) + \beta_k^2 d^2(w_k, w_{k-1})\\ &= (1+\beta_k)^2 d^2(w_k, w_{k-1}).\end{aligned} \tag{3.23}$$

We deduce from (3.22) and (3.23) that

$$\begin{aligned} d^2(u_k,q) &= \|v'_k + \delta_k(v'_k - w'_{k-1}) - q'\|^2 \\ &= \|(1+\delta_k)(v'_k - q') - \delta_k(w'_{k-1} - q')\|^2 \\ &= (1+\delta_k)d^2(v_k,q) - \delta_k d^2(w_{k-1},q) + \delta_k(1+\delta_k)d^2(v_k,w_{k-1}) \\ &= (1+\delta_k)\Big(d^2(w_k,q) + \beta_k\big(d^2(w_k,q) - d^2(w_{k-1},q)\big) \\ &\quad + \beta_k(1+\beta_k)d^2(w_k,w_{k-1})\Big) \\ &\quad - \delta_k d^2(w_{k-1},q) + \delta_k(1+\delta_k)(1+\beta_k)^2 d^2(w_k,w_{k-1}). \end{aligned}$$

Therefore

$$\begin{aligned} d^2(u_k,q) &= d^2(w_k,q) + \beta_k(1+\beta_k)(1+\delta_k)d^2(w_k,w_{k-1}) \\ &\quad + (\delta_k + (1+\delta_k)\beta_k)\big(d^2(w_k,q) - d^2(w_{k-1},q)\big) \\ &\quad + \delta_k(1+\delta_k)(1+\beta_k)^2 d^2(w_k,w_{k-1}). \end{aligned} \tag{3.24}$$

On substituting (3.24) into Lemma 3.3, we have

$$\begin{aligned} &\tau(2-\tau)\frac{\left(1-\mu\frac{\lambda_k}{\lambda_{k+1}}\right)^2}{\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)^2} d^2(u_k,z_k) \\ &\le d^2(w_k,q) - d^2(w_{k+1},q) + (\delta_k + (1+\delta_k)\beta_k)\big(d^2(w_k,q) - d^2(w_{k-1},q)\big) \\ &\quad + \beta_k(1+\beta_k)(1+\delta_k)d^2(w_k,w_{k-1}) + \delta_k(1+\delta_k)(1+\beta_k)^2 d^2(w_k,w_{k-1}). \end{aligned} \tag{3.25}$$

From $\sum_{k=1}^{\infty} \beta_k d(w_k,w_{k-1}) < +\infty$ and $\sum_{k=1}^{\infty} \delta_k d(w_k,w_{k-1}) < +\infty$, one has

$$\lim_{k\to\infty} \beta_k d(w_k,w_{k-1}) = 0, \quad \lim_{k\to\infty} \delta_k d(w_k,w_{k-1}) = 0. \tag{3.26}$$

Using the fact that $\lim_{k\to\infty} d(w_k,q)$ exists and (3.26), we can deduce from (3.25) that

$$\lim_{k\to\infty} d(u_k,z_k) = 0. \tag{3.27}$$

It follows from (3.26) that

$$\begin{aligned} d(v_k,w_k) &= \|w'_k + \beta_k(w'_k - w'_{k-1}) - w'_k\| \\ &= \beta_k d(w_k,w_{k-1}) \to 0, \text{ as } k\to\infty. \end{aligned} \tag{3.28}$$

By utilizing (3.26) and (3.28), we obtain

$$\begin{aligned} d(u_k,w_k) &= \|v'_k + \delta_k(v'_k - w'_{k-1}) - w'_k\| \\ &\le \|v'_k - w'_k\| + \delta_k\|w'_k + \beta_k(w'_k - w'_{k-1}) - w'_{k-1}\| \\ &\le d(v_k,w_k) + \delta_k d(w_k,w_{k-1}) + \beta_k\delta_k d(w_k,w_{k-1}) \to 0, \text{as } k\to\infty. \end{aligned} \tag{3.29}$$

From the fact that $\{d(w_k,q)\}$ is bounded, one obtains that $\{w_k\}$ is also bounded, and thus there exists a subsequence $\{w_{k_j}\}$ which converges to a cluster $\overline{q}$. From (3.27) and (3.29), one has $u_{k_j} \to \overline{q}$ and $z_{k_j} \to \overline{q}$. According to (3.2), one has

$$\Big\langle P_{z_{k_j},u_{k_j}} \mathsf{F}u_{k_j} - \frac{1}{\lambda_{k_j}}\exp^{-1}_{z_{k_j}} u_{k_j}, \exp^{-1}_{z_{k_j}} p\Big\rangle \ge 0, \quad \forall p\in\mathbb{D},$$

which implies that

$$\Big\langle P_{z_{k_j},u_{k_j}} \mathsf{F}u_{k_j}, \exp^{-1}_{z_{k_j}} p\Big\rangle \ge \frac{1}{\lambda_{k_j}}\Big\langle \exp^{-1}_{z_{k_j}} u_{k_j}, \exp^{-1}_{z_{k_j}} p\Big\rangle.$$

Consider the geodesic triangle $\Delta(z_{k_j},u_{k_j},p)$. Then, using Proposition 2.2, we obtain

$$\langle \exp^{-1}_{z_{k_j}} u_{k_j}, \exp^{-1}_{z_{k_j}} p\rangle \ge \frac{1}{2}\big(d^2(z_{k_j},u_{k_j}) + d^2(z_{k_j},p) - d^2(u_{k_j},p)\big).$$

Combined with the last two inequalities, we have

$$\langle P_{z_{k_j},u_{k_j}} \mathsf{F}u_{k_j}, \exp^{-1}_{z_{k_j}} p\rangle \ge \frac{1}{2\lambda_{k_j}}\big(d^2(z_{k_j},u_{k_j}) + d^2(z_{k_j},p) - d^2(u_{k_j},p)\big). \tag{3.30}$$

By passing the limit as $j\to\infty$ in (3.30) and using (3.27), Lemmas 2.7 and 3.1, we obtain

$$\langle \mathsf{F}\overline{q}, \exp^{-1}_{\overline{q}} p\rangle \ge 0, \quad \forall p\in\mathbb{D}.$$

This means that $\overline{q} \in \mathrm{VIP}(\mathbb{D},\mathsf{F})$. Lastly, by Lemma 2.8, we obtain that $\{w_k\}$ converges to an element in $\mathrm{VIP}(\mathbb{D},\mathsf{F})$. The proof is finished. □

Inspired by the Tseng extragradient algorithm, the second iterative scheme presented in this paper is shown in Algorithm 3.2.

Algorithm 3.2 The second modified proximal point algorithm

Initialization: Choose $\lambda_1 > 0$, $\mu \in (0,1)$, $\{\beta_k\}$, $\{\theta_k\}$, and $\{\delta_k\}$ are real positive sequences for all $k \in \mathbb{N}$. Let $w_0, w_1 \in \mathcal{B}$ be arbitrary.
Step 1: Given the current iterates w_{k-1}, w_k, compute v_k and u_k as (3.1).
Step 2: Compute $z_k \in \mathbb{D}$ such that (3.2) holds.
Step 3: Calculate $w_{k+1} = \exp_{z_k}\left(\lambda_k(P_{z_k,u_k}\mathsf{F}u_k - \mathsf{F}z_k)\right)$. Update λ_{k+1} by (3.5).
Set $k := k+1$ and return to Step 1.

Theorem 3.2. *Let $\{u_k\}$ be generated by Algorithm 3.2 and Assumption 3.1 holds. Then $\{w_k\}$ converges to a solution of VIP (1.2).*

Proof. Fix $q \in \mathrm{VIP}(\mathbb{D}, \mathsf{F})$. Consider the geodesic triangle $\Delta(u_k, z_k, q)$ and its comparison triangle $\Delta(u'_k, z'_k, q')$, and consider the geodesic triangle $\Delta(w_{k+1}, z_k, q)$ and its comparison triangle $\Delta\left(w'_{k+1}, z'_k, q'\right)$. It follows from Lemma 2.2 that

$$d(u_k, q) = \left\|u'_k - q'\right\|, \quad d(z_k, q) = \left\|z'_k - q'\right\|, \quad d(u_k, z_k) = \left\|u'_k - z'_k\right\|,$$
$$d(w_{k+1}, q) = \left\|w'_{k+1} - q'\right\|, \quad d(z_k, q) = \left\|z'_k - q'\right\|, \quad d(w_{k+1}, z_k) = \left\|w'_{k+1} - z'_k\right\|.$$

By the definition of w_{k+1} in Algorithm 3.2, one obtains that the comparison point of w_{k+1} is $w'_{k+1} = z'_k + \lambda_k\left(\mathsf{F}u'_k - \mathsf{F}z'_k\right)$. Moreover,

$$\left\|\exp_{z_k}^{-1} w_{k+1}\right\| = \left\|\exp_{z_k}^{-1}\exp_{z_k}\left(\lambda_k\left(P_{z_k,u_k}\mathsf{F}u_k - \mathsf{F}z_k\right)\right)\right\|.$$

Thus

$$\lambda_k\left\|P_{z_k,u_k}\mathsf{F}u_k - \mathsf{F}z_k\right\| = d(w_{k+1}, z_k) = \|w'_{k+1} - z'_k\| = \lambda_k\left\|\mathsf{F}z'_k - \mathsf{F}u'_k\right\|.$$

Let $a = \exp_{z_k}(\mathsf{F}z_k - P_{z_k,u_k}\mathsf{F}u_k)$. Then the comparison point of a is $a' = z'_k + \mathsf{F}z'_k - \mathsf{F}u'_k$. According to Lemma 2.4, we have

$$\left\langle a' - z'_k, q' - z'_k\right\rangle \le \left\langle \mathsf{F}z_k - P_{z_k,u_k}\mathsf{F}u_k, \exp_{z_k}^{-1} q\right\rangle.$$

Hence, we deduce that

$$\begin{aligned} d^2(w_{k+1}, q) &= \|w'_{k+1} - q'\|^2 \\ &= \left\|z'_k + \lambda_k\left(\mathsf{F}u'_k - \mathsf{F}z'_k\right) - q'\right\|^2 \\ &= \left\|z'_k - q'\right\|^2 + \lambda_k^2\left\|\mathsf{F}u'_k - \mathsf{F}z'_k\right\|^2 \\ &\quad + 2\lambda_k\left\langle \mathsf{F}z'_k - \mathsf{F}u'_k, q' - z'_k\right\rangle \\ &\le \left\|z'_k - q'\right\|^2 + \lambda_k^2\left\|P_{z_k,u_k}\mathsf{F}u_k - \mathsf{F}z_k\right\|^2 \\ &\quad + 2\lambda_k\left\langle \mathsf{F}z_k - P_{z_k,u_k}\mathsf{F}u_k, \exp_{z_k}^{-1} q\right\rangle. \end{aligned} \tag{3.31}$$

By using Lemma 2.4 again, one has

$$\begin{aligned} \left\|z'_k - q'\right\|^2 &= \left\|z'_k - u'_k\right\|^2 + \left\|u'_k - q'\right\|^2 + 2\left\langle z'_k - u'_k, u'_k - q'\right\rangle \\ &= \left\|u'_k - q'\right\|^2 + \left\|z'_k - u'_k\right\|^2 - 2\left\langle z'_k - u'_k, z'_k - u'_k\right\rangle \\ &\quad + 2\left\langle z'_k - u'_k, z'_k - q'\right\rangle \\ &= \left\|u'_k - q'\right\|^2 - \left\|z'_k - u'_k\right\|^2 + 2\left\langle u'_k - z'_k, q' - z'_k\right\rangle \\ &\le d^2(u_k, q) - d^2(u_k, z_k) + 2\left\langle \exp_{z_k}^{-1} u_k, \exp_{z_k}^{-1} q\right\rangle. \end{aligned} \tag{3.32}$$

From the definition of z_k and $q \in \mathbb{D}$, one obtains

$$\left\langle \exp_{z_k}^{-1} u_k - \lambda_k P_{z_k,u_k}\mathsf{F}u_k, \exp_{z_k}^{-1} q\right\rangle \le 0. \tag{3.33}$$

Since $q \in \mathrm{VIP}(\mathbb{D}, \mathsf{F})$ and $z_k \in \mathbb{D}$, we have $\left\langle \mathsf{F}q, \exp_q^{-1} z_k\right\rangle \ge 0$. It follows from the pseudomonotonicity of F that

$$\left\langle \mathsf{F}z_k, \exp_{z_k}^{-1} q\right\rangle \le 0. \tag{3.34}$$

Combining (3.5), (3.31), (3.32), (3.33), and (3.34), one concludes that

$$\begin{aligned} d^2(w_{k+1}, q) &\le d^2(u_k, q) - d^2(u_k, z_k) + \lambda_k^2\left\|P_{z_k,u_k}\mathsf{F}u_k - \mathsf{F}z_k\right\|^2 \\ &\le d^2(u_k, q) - \left(1 - \frac{\lambda_k^2}{\lambda_{k+1}^2}\right)d^2(u_k, z_k). \end{aligned}$$

Table 1
Numerical results of all algorithms.

Algorithms	$w_0 = 20$		$w_0 = 30$		$w_0 = 60$		$w_0 = 80$	
	Iter.	Time	Iter.	Time	Iter.	Time	Iter.	Time
Our Alg. 3.1	10	0.3316	10	0.3400	10	0.4337	10	0.3551
Our Alg. 3.2	34	0.4313	34	0.3983	35	0.7093	35	0.3745
SFS Alg. 2	100	27.2960	100	32.8638	100	38.0513	100	39.9684
TH Alg. 4.1	100	0.4111	100	0.2154	100	0.2249	100	0.5795
TWL Alg. 3.1	93	0.4002	90	0.3785	96	0.7257	97	0.6041

The remaining proof is similar to that of Theorem 3.1 and is therefore omitted. □

Remark 3.1. The results of Theorems 3.1 and 3.2 hold if F is pseudomonotone and L-Lipschitz continuous.

4. Numerical experiments

In this section, we give a numerical example to demonstrate the computational efficiency of the proposed algorithms compared with some known methods in the literature [14–16]. All codes are implemented on a personal computer in MATLAB R2023b.

Example 4.1. Let $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$, and define the Riemannian metric $\langle \cdot, \cdot \rangle$ by

$$\langle u, v \rangle := \frac{uv}{x^2}, \quad \forall u, v \in T_x\mathcal{B}, \ \forall x \in \mathcal{B}.$$

With this metric, $\mathcal{B} = \left(\mathbb{R}_{++}, \langle \cdot, \cdot \rangle\right)$ forms a Riemannian manifold. The Riemannian distance $d : \mathcal{B} \times \mathcal{B} \to \mathbb{R}_{++}$ between $x, y \in \mathcal{B}$ is defined as

$$d(x, y) = \left| \ln \left(\frac{x}{y} \right) \right|, \quad \forall x, y \in \mathcal{B}.$$

The exponential map is given by $\exp_x(tv) = xe^{\left(\frac{v}{x}\right)t}$ for all $x \in \mathcal{B}$, $t \in \mathbb{R}$, and $v \in T_x\mathcal{B}$. The inverse exponential map is expressed as

$$\exp_x^{-1}(y) = x \ln \left(\frac{y}{x} \right), \quad \forall x, y \in \mathcal{B}.$$

Consider the set $\mathbb{D} = [1, 100]$, which is a subset of $\mathbb{R}_{++}$. Define the single-valued vector field $\mathsf{F} : \mathbb{D} \to T\mathcal{B}$ by

$$\mathsf{F}x = x \ln x, \quad \forall x \in \mathbb{D}.$$

It is evident that F is both pseudo-monotone and monotone on $\mathbb{D}$. According to [33, Example 1], F is 1-Lipschitz continuous. Clearly, the VIP (1.2) associated with the above F and $\mathbb{D}$ has a unique solution $w^* = 1$.

Next, we use the proposed algorithms to solve Example 4.1 and compare them with the Algorithm 4.1 of Tang and Huang [14] (shortly, TH Alg. 4.1), the Algorithm 3.1 of Tang et al. [15] (shortly, TWL Alg. 3.1), and the Algorithm 2 of Sahu et al. [16] (shortly, SFS Alg. 2). The parameters of the algorithms are set as follows.

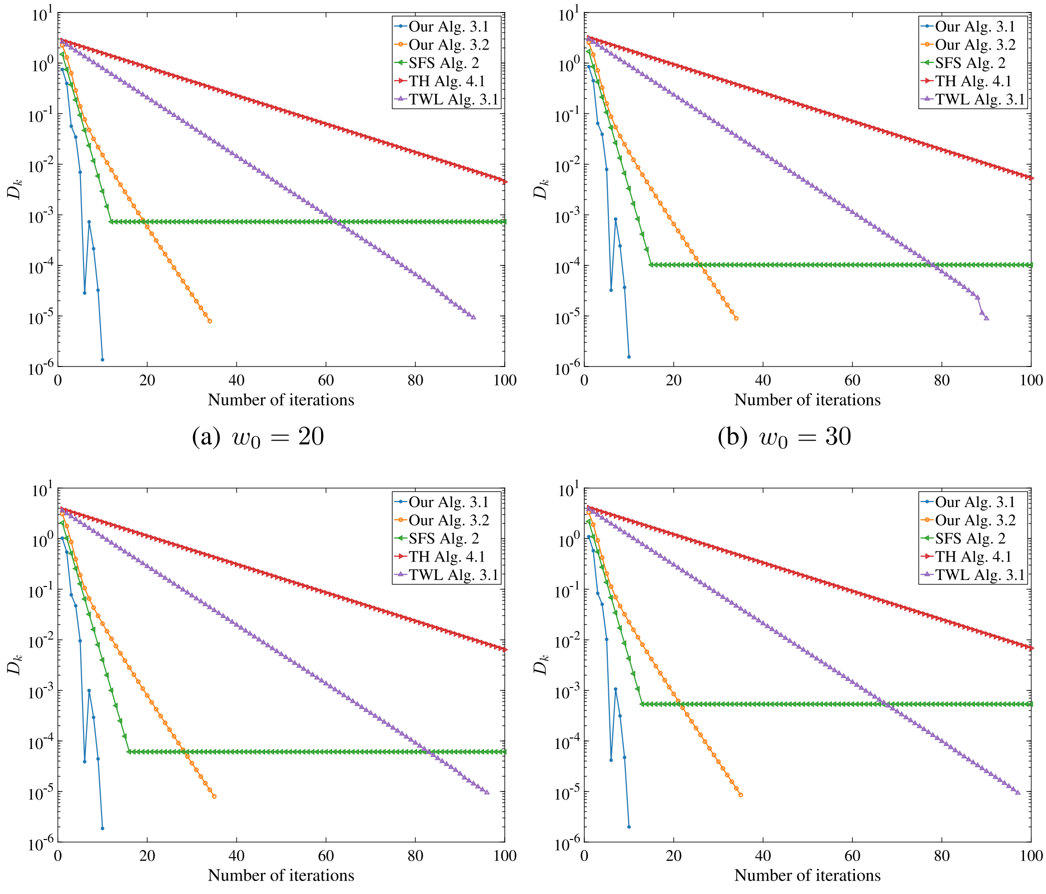
- Take $\mu = 0.5$, $\tau = 1.5$, $\beta_k = 1/(k+1)^2$, $\delta_k = 1/(k+1)^2$, $\lambda_1 = 0.5$, and $\theta_k = 0.1/(k+1)^2$ for the proposed Algorithms 3.1 and 3.2.
- Choose $\beta_k = 0.5$ and $\delta = 0.91$ for TH Alg. 4.1.
- Select $\sigma = 0.85$ and $\mu = 0.5$ for TWL Alg. 3.1.
- Set $\lambda = 1$, $\sigma = 0.85$, and $\eta = 0.5$ for SFS Alg. 2.

We represent the iteration error at step n of the algorithms as $D_k = d(w_k, w^*)$ and set a common stopping condition of $D_k < 10^{-5}$ or a maximum of 100 iterations. To evaluate the convergence performance of the proposed algorithm and those in [14–16] for Example 4.1, we tested four initial points with $w_0 = w_1$, as illustrated in Table 1 and Fig. 1. In Table 1, "Iter". represents the number of iterations, and "Time" indicates the execution time in seconds.

Remark 4.1. From Table 1 and Fig. 1, it can be observed that our algorithms are applicable to solving the variational inequality problem on the Hadamard manifold. Moreover, our algorithms require fewer iterations and shorter execution time compared to the algorithms in the literature [14–16] under the same stopping criteria, and these results are independent of the choice of initial values. Therefore, the algorithms proposed in this paper are both efficient and robust.

5. Conclusions

In this paper, we propose two accelerated iterative algorithms to solve variational inequality problems on Hadamard manifolds. The proposed algorithms are inspired by the double inertial method, the proximal point algorithm, the Tseng extragradient algorithm, and the projection and contraction method. Additionally, we introduce a non-monotonic step size criterion, enabling



(a) $w_0 = 20$ (b) $w_0 = 30$

(c) $w_0 = 60$ (d) $w_0 = 80$

Fig. 1. Numerical behavior of all algorithms.

the algorithms to operate adaptively. Under the assumption that the bifunction is pseudomonotone and uniformly continuous, we prove two convergence theorems for the proposed algorithms. Finally, we demonstrate the computational efficiency and advantages of the proposed algorithms compared to some known ones in the literature through a numerical example.

CRediT authorship contribution statement

Bing Tan: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Hammed Anuoluwapo Abass:** Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Conceptualization. **Songxiao Li:** Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Conceptualization. **Olawale Kazeem Oyewole:** Writing – review & editing, Writing – original draft, Validation, Methodology, Conceptualization.

Declaration of competing interest

The authors declare no competing interests.

Acknowledgments

B. Tan thanks the support of the Natural Science Foundation of Chongqing, China (No. CSTB2024NSCQ-MSX0354) and the support of the Fundamental Research Funds for the Central Universities, China (No. SWU-KQ24052). S. Li acknowledges the National Natural Science Foundation of China (No. 12371131), GuangDong Basic and Applied Basic Research Foundation (Nos. 2024A1515012404 and 2023A1515010614), and STU Scientific Research Initiation Grant (No. NTF23004). All authors have read and approved the final manuscript.

Data availability

No data was used for the research described in the article.

References

[1] Kinderlehrer D, Stampacchia G. An introduction to variational inequalities and their applications. Society for Industrial and Applied Mathematics; 2000.
[2] Korpelevich GM. The extragradient method for finding saddle points and other problems. Èkon I Mat Metod 1976;12:747–56.
[3] Censor Y, Gibali A, Reich S. The subgradient extragradient method for solving variational inequalities in Hilbert space. J Optim Theory Appl 2011;148:318–35.
[4] Tseng PA. A modified forward–backward splitting method for maximal monotone mappings. SIAM J Control Optim 2000;38:431–46.
[5] He BS. A class of projection and contraction methods for monotone variational inequalities. Appl Math Optim 1997;35:69–76.
[6] Censor Y, Gibali A, Reich S. Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space. Optimization 2012;61:1119–32.
[7] Censor Y, Gibali A, Reich S. Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space. Optim Methods Softw 2011;26:827–45.
[8] Németh SZ. Variational inequalities on Hadamard manifolds. Nonlinear Anal 2003;52:1491–8.
[9] Li C, López G, Martín-Márquez V. Monotone vector fields and the proximal point algorithm on Hadamard manifolds. J Lond Math Soc 2009;79:663–83.
[10] Oyewole OK, Reich S. An inertial subgradient extragradient method for approximating solutions to equilibrium problems in Hadamard manifolds. Axioms 2023;12:256.
[11] Tang GJ, Zhou LW, Huang NJ. The proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds. Optim Lett 2013;7:779–90.
[12] Ansari QH, Uddin M. Modified proximal point algorithms for variational inclusion problems on Hadamard manifolds. Soft Comput 2024;28:6595–606.
[13] Ferreira OP, Pérez LRL, Németh SZ. Singularities of monotone vector fields and an extragradient-type algorithm. J Global Optim 2005;31:133–51.
[14] Tang GJ, Huang NJ. Korpelevich's method for variational inequality problems on Hadamard manifolds. J Global Optim 2012;54:493–509.
[15] Tang GJ, Wang X, Liu HW. A projection-type method for variational inequalities on Hadamard manifolds and verification of solution existence. Optimization 2015;64:1081–96.
[16] Sahu DR, Feeroz B, Shikher S. A new self-adaptive iterative method for variational inclusion problems on Hadamard manifolds with applications. Numer Algorithms 2023;94:1435–60.
[17] Oyewole OK, Abass HA, Shehu Y. Iterative approximation of common solution to variational inequality problems in Hadamard manifold. Appl Anal 2024. http://dx.doi.org/10.1080/00036811.2024.2302964.
[18] Abass HA, Oyewole OK, Aremu KO, Jolaoso LO. Modified inertial tseng method for solving variational inclusion and fixed point problems on Hadamard manifolds. Appl Anal 2024;103:1604–21.
[19] Polyak BT. Some methods of speeding up the convergence of iterative methods. Zh Vychisl Mat Mat Fiz 1964;4:1–17.
[20] Poon C, Liang J. Geometry of first-order methods and adaptive acceleration. 2020, arXiv:2003.03910.
[21] Abass HA. Linear convergence of alternating inertial Tseng-type method for solving inclusion problems on Hadamard manifolds. Proceedings of the Edinburgh Mathematical Society; 2024, http://dx.doi.org/10.1017/S0013091524000543.
[22] Yao Y, Iyiola OS, Shehu Y. Subgradient extragradient method with double inertial steps for variational inequalities. J Sci Comput 2022;90:71.
[23] Suantai S, Inkrong P, Cholamjiak P. Forward-backward-forward algorithms involving two inertial terms for monotone inclusions. Comput Appl Math 2023;42:255.
[24] Thong DV, Reich S, Li XH, Tham PTH. An efficient algorithm with double inertial steps for solving split common fixed point problems and an application to signal processing. Comput Appl Math 2025;44. Paper No. 102.
[25] Wang JH, López G, Martín-Márquez V, Li C. Monotone and accretive vector fields on Riemannian manifolds. J Optim Theory Appl 2010;146:691–708.
[26] Sakai T. Riemannian geometry. Transl. math. monogr., vol. 149, Providence: Amer. Math. Soc.; 1996.
[27] Vanderbei RJ. Uniform continuity is almost Lipschitz continuity. Technical report SOR-91-11. Statistics and operations research series, Princeton: Princeton University; 1991.
[28] Bridson MR, Haefliger A. Metric spaces of non-positive curvature. In: Grundlehren math. wiss.. Vol. 319, Berlin: Springer; 1999.
[29] Li C, López G, Martín-Márquez V. Iterative algorithms for nonexpansive mappings on Hadamard manifolds. Taiwanese J Math 2010;14:541–59.
[30] Osilike MO, Aniagbosor SC. Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings. Math Comput Modelling 2000;32:1181–91.
[31] Hanjing A, Suantai S. A fast image restoration algorithm based on a fixed point and optimization method. Mathematics 2020;8:378.
[32] Ferreira OP, Oliveira PR. Proximal point algorithm on Riemannian manifolds. Optimization 2002;51:257–70.
[33] Ansari QH, Babu F. Proximal point algorithm for inclusion problems in Hadamard manifolds with applications. Optim Lett 2021;15:901–21.