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SELF-ADAPTIVE INERTIAL SHRINKING PROJECTION ALGORITHMS FOR SOLVING PSEUDOMONOTONE VARIATIONAL INEQUALITIES

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Abstract. In this paper, we construct two fast iterative methods to solve pseudomonotone variational inequalities in real Hilbert spaces. The advantage of the suggested iterative schemes is that they can adaptively update the iterative step size through some previously known information without performing any line search process. Strong convergence theorems of the proposed algorithms are established under some relaxed conditions imposed on the parameters. Finally, several numerical tests are given to show the advantages and efficiency of the proposed approaches compared with the existing results.

1. Introduction

Throughout this paper, C is assumed to be a convex and closed nonempty set in a Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Recall that the classical variational inequality problem (shortly, VIP) is formed as follows.

$$\text{(VIP)} \qquad \text{Find } x^* \in C \text{ such that } \langle Ax^*, z - x^* \rangle \geq 0, \quad \forall z \in C\,,$$

where $A : H \to H$ is a nonlinear mapping. The solution set of (VIP) is represented as $\mathrm{VI}(C, A)$.

Variational inequality is an essential tool for studying many fields of mathematics and applied science (such as physics, regional, social, engineering, and other issues); see, for example, [1, 2, 9, 12, 17, 19, 23]. The theories and methods of variational inequalities have been implemented in numerous areas of science and have proven to be successful and creative. In the past few decades, researchers have been very interested in developing effective and robust numerical approaches for solving variational inequality problems; see, e.g., [7, 18, 24] and the references therein.

The simplest and oldest projection method is the projected-gradient method (PGM): $x_{n+1} = P_C\left(x_n - \lambda A x_n\right), \forall n \geq 1$, where λ is a positive constant and P_C represents the metric projection from H onto C (see definition in Section 2). It is known that the iteration sequence defined by PGM converges to an element of $\mathrm{VI}(C, A)$ when A is L-Lipschitz continuous and α-strongly monotone and $\lambda \in \left(0, 2\alpha/L^2\right)$. This condition is very strong and limits the implementation of related algorithms. For avoiding the use of such assumptions, the extragradient method (EGM) [14] has been proposed for a monotone and L-Lipschitz continuous mapping A. The

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algorithm is described as follows:

$$
\begin{cases} y_n = P_C\left(x_n - \lambda A x_n\right) , \\ x_{n+1} = P_C\left(x_n - \lambda A y_n\right) , \quad \forall n \geq 0, \end{cases} \tag{1.1}
$$

where $\lambda \in (0, 1/L)$. The algorithm defined by (1.1) converges to an element of $\mathrm{VI}(C, A)$ provided that $\mathrm{VI}(C, A)$ is nonempty. Note that Algorithm (1.1) needs to calculate two projections from H onto the feasibility set C. If C is a general convex-closed set, this might require a prohibitive amount of computation time. To overcome this computational drawback, many authors have modified this method in various ways. Next, we introduce two modifications of the EGM.

The first approach is the Tseng's extragradient method (also known as the forward-backward-forward method) proposed by Tseng [32]. The advantage of this method is that the projection on the feasible set only needs to be calculated once in each iteration. More precisely, the method is stated as follows:

$$
\begin{cases} y_n = P_C\left(x_n - \lambda A x_n\right) , \\ x_{n+1} = y_n - \lambda\left(A y_n - A x_n\right) , \quad \forall n \geq 0, \end{cases} \tag{1.2}
$$

where mapping A is L-Lipschitz continuous monotone and $\lambda \in (0, 1/L)$. In 2011, Censor, Gibali and Reich [5] modified the extragradient algorithm by replacing the second projection onto the convex and closed subset with the one onto a subgradient half-space. The algorithm is now called subgradient extragradient algorithm (SEGM) and its form is expressed as follows:

$$
\begin{cases} y_n = P_C\left(x_n - \lambda A x_n\right) , \\ T_n = \{x \in H \mid \langle x_n - \lambda A x_n - y_n, x - y_n \rangle \leq 0\} , \\ x_{n+1} = P_{T_n}\left(x_n - \lambda A y_n\right) , \quad \forall n \geq 0 , \end{cases} \tag{1.3}
$$

where mapping A is L-Lipschitz continuous monotone and $\lambda \in (0, 1/L)$. Note that the projection on the half-space T_n can be calculated with a clear formula so that SEGM only needs to evaluate the projection on the feasible set C once in each iteration. We point out here that Algorithm (1.2) and Algorithm (1.3) only obtain weak convergence in an infinite-dimensional Hilbert space. Some practical problems that occur in the fields of image processing, quantum mechanics, medical imaging and machine learning need to be modeled and analyzed in infinite-dimensional space. Therefore, strong convergence results are preferable to weak convergence results in infinite-dimensional spaces. In 2011, inspired by the hybrid projection method suggested in [20] and Algorithm (1.3), Censor, Gibali and Reich [6] proposed a hybrid subgradient extragradient algorithm to solve the monotone variational inequality problem in real Hilbert spaces. The algorithm is stated as follows:

$$
\begin{cases} y_n = P_C\left(x_n - \lambda A x_n\right) , \\ z_n = \alpha_n x_n + \left(1 - \alpha_n\right) P_{T_n}\left(x_n - \lambda A y_n\right) , \\ C_n = \{p \in H : \|z_n - p\| \leq \|x_n - p\|\} , \\ Q_n = \{p \in H : \langle x_n - p, x_n - x_0 \rangle \leq 0\} , \\ x_{n+1} = P_{C_n \cap Q_n} x_0 , \quad \forall n \geq 0 , \end{cases} \tag{1.4}
$$

where mapping A is L-Lipschitz continuous monotone, the half-space T_n is as in (1.3), $\alpha_n \subset [0, \alpha]$ for some $\alpha \in [0, 1)$ and $\lambda \in (0, 1/L)$. They proved that the iterative sequence $\{x_n\}$ provided by (1.4) converges strongly to an element z of $\mathrm{VI}(C, A)$, where $z = P_{\mathrm{VI}(C,A)} x_0$. In addition, Takahashi, Takeuchi and Kubota [26] also presented a new projection-based method and obtained the strong convergence of the method. This method is now referred to as the shrinking projection algorithm. For the projection-based methods for solving various problems, readers can refer to [22, 27, 29] and the references therein.

On the other hand, we notice that the above algorithms are based on the condition that the prior information of the Lipschitz constant of the cost operator is known. This implies that the Lipschitz constant of the operator must be input to the algorithms as a priori parameter. However, the Lipschitz constant is difficult to obtain in practical nonlinear problems, which will further affect the feasibility of the algorithms used. To handle the case where the Lipschitz constant of the operator A is unknown, Armijo-type search methods are used by many scholars in the literature. However, this method requires multiple evaluations of the value of the operator in each iteration, which increases the calculation time of the algorithms. Recently, Yang and Liu [33] introduced the following self-adaptive step size algorithm for solving (VIP) in Hilbert spaces.

$$\begin{cases} y_n = P_C\left(x_n - \lambda A x_n\right), \\ z_n = y_n - \lambda\left(A y_n - A x_n\right), \\ x_{n+1} = \alpha_n f\left(x_n\right) + \left(1 - \alpha_n\right) z_n, \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$, $f : H \to H$ is a contraction mapping and step size λ_n is updated by the following:

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\left\|x_n - y_n\right\|}{\left\|A x_n - A y_n\right\|}, \lambda_n\right\}, & \text{if } A x_n - A y_n \neq 0; \\ \lambda_n, & \text{otherwise,} \end{cases}$$

where $\mu \in (0, 1)$ and $\lambda_0 > 0$. This criterion only needs to use some previously known information for a simple calculation to complete the step size update in each iteration.

In recent years, the development of fast iterative algorithms has attracted enormous interest, especially for inertial technology, which is based on discrete versions of a second-order dissipative dynamic system. Many researchers have constructed various fast iterative algorithms by using inertial technology, see, e.g., [10, 11, 21, 25, 28, 30, 34] and the references therein. One of the common features of these algorithms is that the next iteration depends on the combination of the previous two iterations. Note that this minor change greatly improves the performance of the algorithms. Recently, Liu and Qin [15] combined the inertial method, the hybrid projection method and the Tseng's extragradient method with an Armijo-like line search rule and proposed a new iterative algorithm for solving the pseudomonotone variational inequality problem in real Hilbert spaces. More precisely, the form of their algorithm is as follows:

$$
\begin{cases}
w_n = x_n + \theta_n (x_n - x_{n-1}) \,, \\
y_n = P_C (x_n - \lambda_n A w_n) \,, \\
z_n = y_n - \lambda_n (A y_n - A w_n) \,, \\
C_n = \big\{ p \in H : \|z_n - p\|^2 \le \|x_n - p\|^2 - \left(1 - 2\mu^2\right) \|x_n - y_n\|^2 \\
\qquad\qquad\qquad\qquad + 2\mu^2 \theta_n^2 \|x_{n-1} - x_n\|^2 \big\} \,, \\
Q_n = \{ p \in H : \langle x_n - p, x_n - x_0 \rangle \le 0 \} \,, \\
x_{n+1} = P_{C_n \cap Q_n} x_0 \,, \quad \forall n \ge 1 \,,
\end{cases}
\tag{1.5}
$$

where the mapping $A : H \to H$ is pseudomonotone, L-Lipschitz continuous, sequentially weakly continuous on C, $\theta_n \in (0, +\infty)$, $\mu \in (0, 1/\sqrt{2})$, $\lambda_n := \gamma \ell^{m_n} (\gamma, \ell \in (0,1))$ and m_n is the smallest non-negative integer m satisfying $\gamma \ell^m \|A y_n - A w_n\| \le \mu \|y_n - w_n\|$. Under some conditions, the algorithm defined by (1.5) converges strongly to $z = P_{\mathrm{VI}(C,A)} x_0$. Very recently, based on the inertial method, the subgradient extragradient algorithm and the hybrid projection algorithm, Thong et al. [31] introduced a new Armijo-type iterative scheme to solve the pseudomonotone variational inequality problem in a Hilbert space and established the strong convergence theorem of the proposed algorithm.

Motivated and stimulated by results as mentioned above, in this study, we propose two inertial projection-type extragradient algorithms with adaptive step size for solving the pseudomonotone variational problem in real Hilbert spaces. These stepsizes can be updated in each iteration through some prior information without performing any line search process. Furthermore, these iterative schemes are embedded with inertial terms to accelerate the convergence speed of the algorithms. Under reasonable assumptions about the parameters, the strong convergence theorems of the suggested algorithms are obtained. Finally, we give several numerical examples to support the theoretical results.

The remainder of this paper is organized as follows. In Section 2, we recall some preliminary results and lemmas for further use. Section 3 analyzes the convergence of the proposed algorithms. In Section 4, some numerical examples are provided to illustrate the numerical behavior of the proposed algorithms and compare them with other ones. Finally, we give some conclusion remarks in the last section.

2. Preliminaries

Let C be a nonempty closed and convex subset in a real Hilbert space H. The weak convergence and strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x are represented by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. Let $\omega_w(x_n)$ denote the set of all weak limits of $\{x_n\}$, i.e., $\omega_w(x_n) := \{x \in H : x_{n_j} \rightharpoonup x$ for some subsequence $\{n_j\}$ of $\{n\}\}$.

Let us review some nonlinear mappings in functional analysis for further use. For any elements $p, q \in H$, recall that a mapping $A : H \to H$ is said to be:

(1) *η-strongly monotone* if there is a positive number η such that $\langle Ap - Aq, p - q \rangle \ge \eta \|p - q\|$.
(2) *monotone* if $\langle Ap - Aq, p - q \rangle \ge 0$.
(3) *pseudomonotone* if $\langle Ap, q - p \rangle \ge 0 \Longrightarrow \langle Aq, q - p \rangle \ge 0$.

(4) *L-Lipschitz continuous* if there is $L > 0$ such that $\|Ap - Aq\| \leq L\|p - q\|$.

(5) *sequentially weakly continuous*, if for any sequence $\{p_n\}$ weakly converges to a point $p \in H$, then $\{Ap_n\}$ weakly converges to Ap.

Recall that a mapping $P_C : H \to C$ is called the metric projection from H onto C, if for all $x \in H$, there is a unique nearest point in C, which is represented by $P_C(x)$, such that $P_C x := \operatorname{argmin}\{\|x - y\|, y \in C\}$. It is known that P_C is nonexpansive and $P_C x$ has the following basic properties (2.1)–(2.3):

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \forall y \in C \tag{2.1}$$

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \forall y \in H \tag{2.2}$$

$$\|x - P_C(x)\|^2 \leq \|x - y\|^2 - \|y - P_C(x)\|^2, \forall y \in C \tag{2.3}$$

We give some projection calculation formulas that need to be used in numerical experiments. For more calculations on projections on specific sets, see [3].

(1) The projection of x onto a half-space $H_{u,v} = \{x : \langle u, x\rangle \leq v\}$ is computed by
$$P_{H_{u,v}}(x) = x - \max\{[\langle u, x\rangle - v]/\|u\|^2, 0\}u.$$

(2) The projection of x onto a box $\mathrm{Box}[a, b] = \{x : a \leq x \leq b\}$ is computed by
$$P_{\mathrm{Box}[a,b]}(x)_i = \min\{b_i, \max\{x_i, a_i\}\}.$$

(3) The projection of x onto the intersection of a hyperplane and a box $C = H_{u,v} \cap \mathrm{Box}[a, b] = \left\{x \in \mathbb{R}^n : u^\mathsf{T} x = v, a \leq x \leq b\right\}$ is computed by
$$P_C(x) = P_{\mathrm{Box}[a,b]}(x - \mu^* u),$$
where μ^* is a solution of the equation $\varphi(\mu) = u^\mathsf{T} P_{\mathrm{Box}[a,b]}(x - \mu u) - v$.

To prove the convergence of the proposed algorithms, we need the following lemmas.

Lemma 2.1 ([8])**.** *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : C \to H$ be a continuous and pseudomonotone operator. Then, x^* is a solution of* $\mathrm{VI}(C, A)$ *if and only if $\langle Ax, x - x^*\rangle \geq 0$, $\forall x \in C$.*

Lemma 2.2 ([13])**.** *Let C be a nonempty closed and convex subset of a real Hilbert space H. Given $x, y, z \in H$ and $a \in R$. $\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$ is convex and closed.*

Lemma 2.3 ([16])**.** *Let C be a closed convex subset of H, $\{x_n\} \subset H$ and $u \in H$. Let $q = P_C u$. If $\omega_w(x_n) \subset C$ and satisfies the condition $\|x_n - u\| \leq \|u - q\|$, $\forall n \in N$. Then $x_n \to q$.*

3. Main results

In this section, we introduce two inertial shrinking projection extragradient methods to approximate the solution of (VIP). The advantage of our suggested algorithms is that no prior knowledge of the Lipschitz constants of the variational inequality mapping is required. The strong convergence theorems of these two iterative schemes are established under some standard and mild conditions. Before starting to state our main results, assume that our algorithms meet the following two conditions.

(C1) The feasible set C is nonempty closed and convex, and the solution set $\mathrm{VI}(C, A)$ is nonempty.

(C2) The mapping $A : H \to H$ is pseudomonotone and L-Lipschitz continuous on H, and sequentially weakly continuous on C.

3.1. **The self-adaptive inertial shrinking subgradient extragradient algorithm.** The first algorithm is described as follows.

Algorithm 3.1 The self-adaptive inertial shrinking subgradient extragradient algorithm

Initialization: Set $\theta_n \in [-\theta, \theta]$ for some $\theta > 0$, $\lambda_1 > 0$, $\mu \in (0,1)$, $C_1 = H$. Let $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$. Set $w_n = x_n + \theta_n (x_n - x_{n-1})$.

Step 2. Compute $y_n = P_C (w_n - \lambda_n A w_n)$. If $y_n = w_n$ then stop and y_n is a solution of (VIP). Otherwise, update the step size λ_{n+1} of the next iteration in the following way.

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu \|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n\right\}, & \text{if } Aw_n - Ay_n \neq 0; \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 3. Compute $z_n = P_{T_n} (w_n - \lambda_n A y_n)$, where the half-space T_n is defined by $T_n := \{x \in H \mid \langle w_n - \lambda_n A w_n - y_n, x - y_n\rangle \leq 0\}$.

Step 4. Compute $x_{n+1} = P_{C_{n+1}} x_0$, where C_{n+1} is defined by

$$C_{n+1} := \{p \in C_n : \|z_n - p\|^2 \leq \|w_n - p\|^2 - \Big(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\Big)(\|y_n - w_n\|^2 + \|z_n - y_n\|^2)\}.$$

Set $n := n+1$ and go to **Step 1**.

The following lemmas are quite helpful to analyze the convergence of the algorithm.

Lemma 3.1. *The sequence* $\{\lambda_n\}$ *generated by* (3.1) *is nonincreasing and*

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \min\left\{\lambda_1, \frac{\mu}{L}\right\}.$$

Proof. It follows from (3.1) that $\lambda_{n+1} \leq \lambda_n$ for all $n \in \mathbb{N}$. Hence, $\{\lambda_n\}$ is nonincreasing. On the other hand, we get $\|Aw_n - Ay_n\| \leq L \|w_n - y_n\|$ since A is L-Lipschitz continuous. Thus,

$$\mu \frac{\|w_n - y_n\|}{\|Aw_n - Ay_n\|} \geq \frac{\mu}{L}, \quad \text{if } \ Aw_n \neq Ay_n\,,$$

which together with (3.1) implies that $\lambda_n \geq \min\{\lambda_1, \frac{\mu}{L}\}$. Therefore, $\lim_{n\to\infty} \lambda_n = \lambda \geq \min\left\{\lambda_1, \frac{\mu}{L}\right\}$ since the sequence $\{\lambda_n\}$ is nonincreasing and lower bounded. $\square$

Lemma 3.2. *Assume that Conditions (C1) and (C2) hold. Let $\{z_n\}$ be a sequence generated by Algorithm 3.1. Then, for all $p \in \mathrm{VI}(C, A)$,*

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \Big(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\Big)(\|y_n - w_n\|^2 + \|z_n - y_n\|^2)\,.$$

Proof. First, using the definition of $\{\lambda_n\}$, one obtains

$$\|Aw_n - Ay_n\| \leq \frac{\mu}{\lambda_{n+1}}\|w_n - y_n\|\,, \quad \forall n\,. \tag{3.2}$$

Indeed, if $Aw_n = Ay_n$ then the inequality (3.2) holds. Otherwise, it implies from (3.1) that

$$\lambda_{n+1} = \min\left\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n\right\} \leq \frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}\,.$$

Consequently, $\|Aw_n - Ay_n\| \leq \mu/\lambda_{n+1}\|w_n - y_n\|$. Therefore, the inequality (3.2) holds when $Aw_n = Ay_n$ and $Aw_n \neq Ay_n$. By the definition of z_n and (2.2), one sees that

$$\begin{aligned}
2\|z_n - p\|^2 \leq& 2\langle z_n - p, w_n - \lambda_n Ay_n - p\rangle \\
=& \|z_n - p\|^2 + \|w_n - \lambda_n Ay_n - p\|^2 - \|z_n - w_n + \lambda_n Ay_n\|^2 \\
=& \|z_n - p\|^2 + \|w_n - p\|^2 + \lambda_n^2\|Ay_n\|^2 - 2\langle w_n - p, \lambda_n Ay_n\rangle \\
&- \|z_n - w_n\|^2 - \lambda_n^2\|Ay_n\|^2 - 2\langle z_n - w_n, \lambda_n Ay_n\rangle \\
=& \|z_n - p\|^2 + \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - p, \lambda_n Ay_n\rangle\,.
\end{aligned}$$

This implies that

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - p, \lambda_n Ay_n\rangle\,. \tag{3.3}$$

Since p is the solution of (VIP), we have $\langle Ap, x - p\rangle \geq 0$ for all $x \in C$. By the pseudomontonicity of A on H, we get $\langle Ax, x - p\rangle \geq 0$ for all $x \in C$. Taking $x = y_n \in C$, one infers that $\langle Ay_n, p - y_n\rangle \leq 0$. Consequently,

$$\langle Ay_n, p - z_n\rangle = \langle Ay_n, p - y_n\rangle + \langle Ay_n, y_n - z_n\rangle \leq \langle Ay_n, y_n - z_n\rangle\,. \tag{3.4}$$

Combining (3.3) and (3.4), one obtains

$$\begin{aligned}
\|z_n - p\|^2 \leq& \|w_n - p\|^2 - \|z_n - w_n\|^2 + 2\lambda_n\langle Ay_n, y_n - z_n\rangle \\
=& \|w_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - w_n\|^2 \\
&- 2\langle z_n - y_n, y_n - w_n\rangle + 2\lambda_n\langle Ay_n, y_n - z_n\rangle \\
=& \|w_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - w_n\|^2 \\
&+ 2\langle z_n - y_n, w_n - \lambda_n Ay_n - y_n\rangle\,.
\end{aligned} \tag{3.5}$$

Using $z_n \in T_n$ and (3.2), we have

$$\begin{aligned}
&2\langle w_n - \lambda_n A y_n - y_n, z_n - y_n\rangle \\
&= 2\langle w_n - \lambda_n A w_n - y_n, z_n - y_n\rangle + 2\lambda_n \langle A w_n - A y_n, z_n - y_n\rangle \\
&\leq 2\lambda_n \|A y_n - A w_n\| \|y_n - z_n\| \\
&\leq 2\mu \frac{\lambda_n}{\lambda_{n+1}} \|w_n - y_n\| \|y_n - z_n\| \\
&\leq \mu \frac{\lambda_n}{\lambda_{n+1}} \|w_n - y_n\|^2 + \mu \frac{\lambda_n}{\lambda_{n+1}} \|y_n - z_n\|^2 .
\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we obtain

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \Big(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\Big)(\|y_n - w_n\|^2 + \|z_n - y_n\|^2) .$$

This completes the proof of the lemma. □

Remark 3.3. From Lemma 3.1 and $\mu \in (0,1)$, one sees that $1 - \mu\lambda_n/\lambda_{n+1} > \epsilon > 0$ for all $n \geq n_0$. For all $p \in \mathrm{VI}(C,A)$ and $n \geq n_0$, it follows from Lemma 3.2 that

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \epsilon(\|y_n - w_n\|^2 + \|z_n - y_n\|^2) .$$

Theorem 3.4. *Assume that Conditions (C1) and (C2) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges to $q \in \mathrm{VI}(C,A)$ in norm, where $q = P_{\mathrm{VI}(C,A)} x_0$.*

Proof. First, we show that $\mathrm{VI}(C,A) \subset C_{n+1}$ for all n. Indeed, by Lemma 2.2, it follows that C_{n+1} is convex and closed. Moreover, one has $p \in C_{n+1}$ by means of Lemma 3.2. Thus, we get $\mathrm{VI}(C,A) \subset C_{n+1} \subset C_n$ for all n and thus $x_{n+1} = P_{C_{n+1}} x_0$ is well defined. Next, we show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty} \|y_n - w_n\| = 0$. From $x_n = P_{C_n} x_0$ and $\mathrm{VI}(C,A) \subset C_n$, we have $\|x_n - x_0\| \leq \|z - x_0\|, \forall z \in \mathrm{VI}(C,A)$. In particular, one has

$$\|x_n - x_0\| \leq \|q - x_0\| , \quad \text{where } q = P_{\mathrm{VI}(C,A)} x_0 . \tag{3.7}$$

This indicates that sequence $\{x_n\}$ is bounded and thus sequences $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ are also bounded. Since $x_n = P_{C_n} x_0$ and $x_{n+1} \subset C_n$, one obtains $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$, which implies that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Using (2.3), one has

$$\|x_n - x_{n+1}\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 , \quad \forall n \geq 0 . \tag{3.8}$$

Combining (3.7) and (3.8), we can show that

$$\sum_{n=1}^{N} \|x_{n+1} - x_n\|^2 \leq \sum_{n=1}^{N} \Big(\|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2\Big) \leq \|q - x_0\|^2 - \|x_1 - x_0\|^2 ,$$

which implies that $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2$ is convergent and hence $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. From the definition of w_n, it follows that $\|w_n - x_n\| = |\theta_n| \|x_n - x_{n-1}\| \leq |\theta| \|x_n - x_{n-1}\| \to 0$. Therefore, we get $\lim_{n\to\infty} \|w_n - x_{n+1}\| = 0$. Since $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1}$, by the definiton of C_{n+1}, one infers that $\|z_n - x_{n+1}\|^2 \leq \|w_n - x_{n+1}\|^2$, which implies that $\lim_{n\to\infty} \|z_n - x_{n+1}\| = 0$ and $\|z_n - w_n\| = 0$. Moreover, from Lemma 3.2 and Remark 3.3, it follows that $\epsilon(\|y_n - w_n\|^2 + \|z_n - y_n\|^2) \leq (\|z_n - p\| + \|w_n - p\|)\|z_n - w_n\| \to 0$. Thus, one gets

$\lim_{n\to\infty} \|y_n - w_n\| = 0$ and $\lim_{n\to\infty} \|z_n - y_n\| = 0$. Finally, we easily see that each sequential weak cluster point of $\{x_n\}$ is in $\mathrm{VI}(C, A)$, that is, $\omega_w(x_n) \subset \mathrm{VI}(C, A)$. This together with (3.7), in the light of Lemma 2.3, yields that $x_n \to q$. The proof is completed. □

3.2. **The self-adaptive inertial shrinking Tseng extragradient algorithm.** Next, we introduce the second self-adaptive iterative scheme, which combines the inertial method, the Tseng's extragradient method and the shrinking projection method. More precisely, the method is stated in Algorithm 3.2.

Algorithm 3.2 The self-adaptive inertial shrinking Tseng extragradient algorithm

Initialization: Set $\theta_n \in [-\theta, \theta]$ for some $\theta > 0$, $\lambda_1 > 0$, $\mu \in (0, 1)$, $C_1 = H$. Let $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Calculate the next iteration point x_{n+1} as follows:

$$\begin{cases} w_n = x_n + \theta_n (x_n - x_{n-1}) \,, \\ y_n = P_C (w_n - \lambda_n A w_n) \,, \\ z_n = y_n - \lambda_n (A y_n - A w_n) \,, \\ x_{n+1} = P_{C_{n+1}} x_0 \,, \end{cases}$$

where the step size λ_n is updated by (3.1), and C_{n+1} is defined as follows

$$C_{n+1} := \{p \in C_n : \|z_n - p\|^2 \le \|w_n - p\|^2 - \Big(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\Big) \|w_n - y_n\|^2\}.$$

The following lemma plays an important role in the convergence analysis of Algorithm 3.2.

Lemma 3.5. *Assume that Conditions (C1) and (C2) hold. Let $\{z_n\}$ be a sequence generated by Algorithm 3.2. Then,*

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \Big(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\Big) \|w_n - y_n\|^2 \,, \quad \forall p \in \mathrm{VI}(C, A) \,,$$

and $\|z_n - y_n\| \le \mu \frac{\lambda_n}{\lambda_{n+1}} \|w_n - y_n\|$.

Proof. First, using the definition of $\{\lambda_n\}$, one obtains

$$\|A w_n - A y_n\| \le \frac{\mu}{\lambda_{n+1}} \|w_n - y_n\| \,, \quad \forall n \,. \tag{3.9}$$

By the definition of z_n, one sees that

$$\begin{aligned} \|z_n - p\|^2 &= \|w_n - p\|^2 + \|y_n - w_n\|^2 + 2 \langle y_n - w_n, w_n - p\rangle \\ &\quad + \lambda_n^2 \|A y_n - A w_n\|^2 - 2\lambda_n \langle y_n - p, A y_n - A w_n\rangle \\ &= \|w_n - p\|^2 + \|y_n - w_n\|^2 - 2 \langle y_n - w_n, y_n - w_n\rangle \\ &\quad + 2 \langle y_n - w_n, y_n - p\rangle + \lambda_n^2 \|A y_n - A w_n\|^2 \\ &\quad - 2\lambda_n \langle y_n - p, A y_n - A w_n\rangle \\ &= \|w_n - p\|^2 - \|y_n - w_n\|^2 + 2 \langle y_n - w_n, y_n - p\rangle \end{aligned} \tag{3.10}$$

$$+\lambda_n^2 \|Ay_n - Aw_n\|^2 - 2\lambda_n \langle y_n - p, Ay_n - Aw_n\rangle .$$

Since $y_n = P_C(w_n - \lambda_n Aw_n)$, using the property of projection (2.1), we obtain $\langle y_n - w_n + \lambda_n Aw_n, y_n - p\rangle \le 0$, or equivalently

$$\langle y_n - w_n, y_n - p\rangle \le -\lambda_n \langle Aw_n, y_n - p\rangle . \tag{3.11}$$

From (3.9), (3.10) and (3.11), we have

$$\begin{aligned}\|z_n - p\|^2 &\le \|w_n - p\|^2 - \|y_n - w_n\|^2 - 2\lambda_n \langle Aw_n, y_n - p\rangle \\ &\quad + \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} \|w_n - y_n\|^2 - 2\lambda_n \langle y_n - p, Ay_n - Aw_n\rangle \\ &\le \|w_n - p\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|w_n - y_n\|^2 - 2\lambda_n \langle y_n - p, Ay_n\rangle .\end{aligned} \tag{3.12}$$

Since $p \in \mathrm{VI}(C, A)$, we have $\langle Ap, y_n - p\rangle \ge 0$. From the pseudomonotonicity of A, we get

$$\langle Ay_n, y_n - p\rangle \ge 0 . \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|w_n - y_n\|^2 .$$

From the definition of z_n and (3.2), we have $\|z_n - y_n\| \le \mu \frac{\lambda_n}{\lambda_{n+1}} \|w_n - y_n\|$. The proof of the lemma is now complete. □

Theorem 3.6. *Assume that Conditions (C1) and (C2) hold. Then the sequence $\{x_n\}$ created by Algorithm 3.2 converges to $q \in \mathrm{VI}(C, A)$ in norm, where $q = P_{\mathrm{VI}(C,A)}x_0$.*

Proof. The proof of the theorem is very similar to Theorem 3.4, so we omit it here. □

Remark 3.7. We comment on algorithms 3.1 and 3.2 as follows.

(1) According to the definition of C_{n+1}, we can easily see that C_{n+1} is the intersection of a series of half-spaces. In fact, we have the following observations:

$$H = C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots$$

Therefore, this is why iterative schemes 3.1 and 3.2 are called the "shrinking projection algorithms".

(2) Note that the inertial parameters $\{\theta_n\}$ in Algorithm 3.1 are located in $[-\theta, \theta]$ for some $\theta > 0$ and other additional conditions are not required. They are weaker than some known algorithms in the literature [7, 11, 25, 34].

(3) Note that our suggested methods use a new step size criterion without any line search process, which enables them to work without the prior information of the Lipschitz constant of the variational inequality mapping. In addition, we comment here that the Armijo-type line search methods [15, 31] require multiple evaluation of the value of operator A, which further increases the execution time of the algorithms used.

(4) It should be noted that the mapping related to (VIP) is pseudomonotone, which is a broader set of mappings than monotone mappings. Therefore, the two iterative algorithms proposed in this paper have more extensive applications in practical problems.

4. Numerical examples

In this section, we perform some numerical examples to demonstrate the computational performance of the proposed algorithms and compare them with some known strongly convergent algorithms in the literature, including the Algorithm 1 suggested by Liu and Qin [15] (shortly, LQ Alg. 1) and the Algorithm 3.2 proposed by Thong et al. [31] (shortly, TSIT Alg. 3.2). We use the FOM Solver [4] to effectively calculate the projections onto $C_n \cap Q_n$. All the programs were implemented in Matlab 2018a on a Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz computer with RAM 8.00 GB.

Example 4.1. Our first test example is the nonlinear complementarity problem (NCP) considered by many researchers. Recall that the NCP is described as follows:

$$\text{find } x^* \in C \text{ such that } x^* \geq 0,\ Ax^* \geq 0 \text{ and } \langle x^*, Ax^* \rangle = 0\,.$$

It should be noted that NCP is a special case of (VIP) when the constraint of (VIP) is non-negative. In other words, the feasible set of NCP is $C = \mathbb{R}^n_+$. Assume that the mapping $A : \mathbb{R}^4 \to \mathbb{R}^4$ is given by

$$Ax = \begin{pmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{pmatrix}.$$

The feasible set C is expressed as $C = \left\{x \in \mathbb{R}^4_+ \mid x_1 + x_2 + x_3 + x_4 = 4\right\}$. The parameters of all algorithms are set as follows. In all algorithms, set inertial parameters $\theta_n = 0.2$. Take $\mu = 0.5$ and $\lambda_0 = 0.05$ in the suggested iterative schemes. In LQ Alg. 1 and TSIT Alg. 3.2, choose $\gamma = 0.5$, $\ell = 0.5$, $\mu = 0.4$. Note that the Lipschitz constant of mapping A in this example is unknown, which will affect the implementation of algorithms that need to know the prior information of the Lipschitz constant. Therefore, our suggested self-adaptive step-size methods are more useful than the fixed step-size algorithms in the literature. Moreover, we do not know the exact solution of the problem, so we use $D_n = \|w_n - y_n\|^2$ to study the error of the n-th iteration and use the maximum iteration 500 as a common stopping criterion. It is known that $y_n \in \mathrm{VI}(C, A)$ if and only if $D_n = 0$. The initial values $x_0, x_1 \in C$ are randomly generated and the numerical results are shown in Fig. 1.

Example 4.2. In the second example, we consider the form of linear operator $A : R^m \to R^m$ ($m = 5, 10, 15, 20$) as follows: $A(x) = Gx + g$, where $g \in R^m$ and $G = BB^{\mathsf{T}} + M + E$, matrix $B \in R^{m\times m}$, matrix $M \in R^{m\times m}$ is skew-symmetric, and matrix $E \in R^{m\times m}$ is diagonal matrix whose diagonal terms are non-negative (hence G is positive symmetric definite). We choose the feasible set as

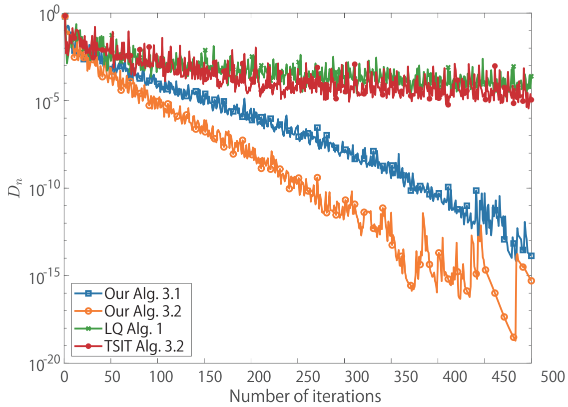
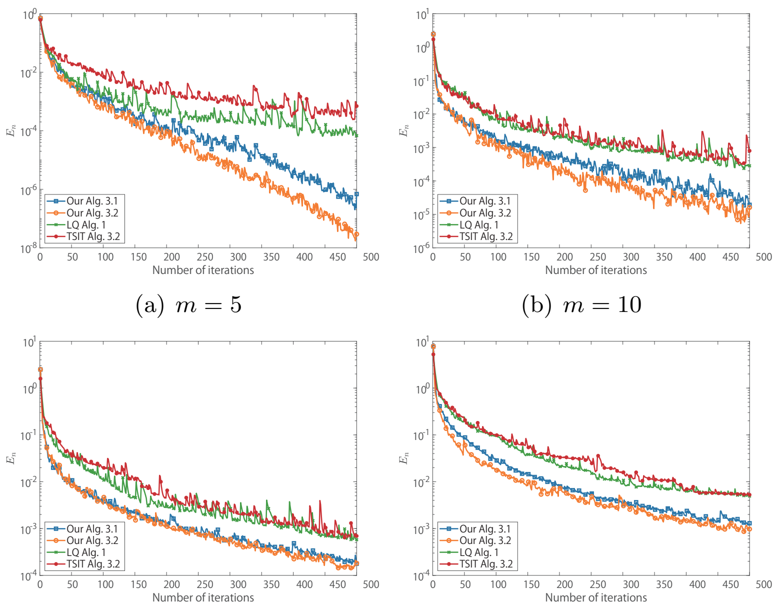


FIGURE 1. Numerical behavior of all algorithms in Example 4.1

$C = \{x \in R^m : -2 \le x_i \le 5,\ i = 1, \ldots, m\}$. It is known that mapping A is monotone and Lipschitz continuous with constant $L = \|G\|$. In this numerical example, both B, M entries are randomly created in $[-2, 2]$, E is generated randomly in $[0, 2]$ and $g = \mathbf{0}$. It can be easily seen that the solution to the problem is $x^* = \{\mathbf{0}\}$. The parameters of all algorithms are set the same as in Example 4.1, and $D_n = \|x_n - x^*\|^2$ is used to measure the calculation error of all algorithms in the n-th step. The maximum iteration 500 as a common stopping criterion and the initial values $x_0 = x_1$ are randomly generated by *rand(m,1)* in MATLAB. The numerical behavior of all algorithms under different dimensions is shown in Fig. 2.



(a) $m = 5$ (b) $m = 10$

(c) $m = 15$ (d) $m = 20$

FIGURE 2. Numerical results for Example 4.2

Remark 4.3. From Figs.1–2, we can see that the proposed iterative schemes converge faster and have better computational performance than the existing algorithms. In addition, these results are independent of the selection of initial values and the size of dimensions. Therefore, our algorithms are robust and efficient.

5. Final remarks

In this paper, we combined the inertial method, the subgradient extragradient method, the Tseng extragradient method and the shrinking projection method to proposed two new algorithms for discovering the solution set of pseudomonotone and Lipschitz continuous (the Lipschitz constant does not need to be known) variational inequality problems in real Hilbert spaces. Under some standard and relaxed assumptions, we have proved the strong convergence of the proposed iterative schemes. Finally, some numerical experiments are given to demonstrate the computational efficiency and competitive advantages of the suggested approaches. The methods obtained in this paper improved and extended some existing results in the literature.

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