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RESEARCH

Extragradient algorithms for solving variational inequalities on Hadamard manifolds

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Abstract

We provide five techniques for solving variational inequality problems on Hadamard manifolds that are based on the adaptive extragradient method. These algorithms operate adaptively, eliminating the need for prior knowledge of the Lipschitz constant associated with the vector field. Furthermore, the iterative sequences produced by the algorithms are shown to converge to the solution of the problem under the conditions that the vector fields are pseudo-monotone and Lipschitz continuous. Additionally, we establish global error bounds and R-linear convergence rates when the vector fields exhibit strong pseudo-monotonicity. Lastly, the theoretical results are illustrated with two numerical instances.

Keywords Variational inequality · Extragradient algorithm · Adaptive stepsize · Hadamard manifolds · Pseudo-monotone vector field

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1 Introduction

This paper aims to present adaptive extragradient algorithms for solving variational inequality problems (VIPs) within Hadamard manifolds. The VIP serves as a versatile framework applicable to a range of fields, including equilibrium programming, economics, transportation regulation, optimal control and compressed sensing (see, e.g., [1–4]). Over the past decades, extensive research has focused on VIPs and related algorithms in linear spaces (see [5–11]). Extending numerical methods from Euclidean spaces to Riemannian manifolds offers advantages, including the transformation of non-convex (resp., non-monotone) problems into convex (resp., monotone) ones through suitable Riemannian metrics (see [12, Section 4]). This underlines the necessity of developing algorithms for VIPs in manifold contexts. In recent years, significant progress has been made in studying optimization problems and solution techniques on manifolds (see, e.g., [13–31]).

1.1 Our contributions

This paper extends the study of VIPs and iterative algorithms on Hadamard manifolds. We address the open question from [14, Section 8] and enhance existing results by developing adaptive algorithms that accommodate pseudo-monotone and Lipschitz continuous vector fields using a non-monotone step size criterion. The generated sequences converge to a solution under mild conditions, assuming the solution's existence. We establish global error bounds and linear convergence results when the vector fields are strongly pseudo-monotone. Additionally, numerical examples are provided to demonstrate the theoretical findings. The proposed algorithms improve and extend several known methods in the literature (see [13–18]).

1.2 Background and organization

Hadamard manifold have several notable properties (see Section 2 for further details), which have drawn considerable attention from researchers. Examples of optimization problems set in Hadamard manifolds can be found in [32, Section 1]. This paper focuses on developing novel extragradient algorithms to solve VIPs. We follow the formulation of the VIP on Hadamard manifolds as introduced by Németh [13]. Let $\mathcal{X}$ be a Hadamard manifold, and let C be a nonempty, closed, and convex subset of $\mathcal{X}$. The tangent bundle of $\mathcal{X}$ is denoted by $T\mathcal{X}$, and $B\colon C \to T\mathcal{X}$ represents a single-valued vector field. The VIP associated with B and C is formulated as follows:

$$\text{Find } u^* \in C \text{ such that } \langle Bu^*, \mathrm{E}_{u^*}^{-1} u\rangle \geq 0 \ \ (\forall u \in C), \tag{1.1}$$

where E^{-1} denotes the inverse of the exponential map. Throughout this paper, we assume that the solution set of VIP (1.1) is nonempty. Notably, the VIP (1.1) generalizes the classical variational inequality problem in linear Euclidean spaces. It is known that there are two popular methods in the literature to solve VIP (1.1): one of which is the proximal point method (see, e.g., [16, 23, 26]) and the other is the extragradient-based method (see, e.g., [14, 15, 18]). Note that the proximal point algorithm is actually an implicit iterative scheme, which means that an optimization subproblem needs to be solved in each iteration. In this case, the proximal point algorithm may be time-consuming. On the other hand, our concern in this paper is mainly on extragradient-based algorithms with explicit forms. Korpelevich [5] invented the projection-based extragradient technique, which has been expanded to solve equilibrium issues in both linear and nonlinear areas and is known to be a useful instrument for addressing VIP. Each iteration of the extragradient method includes two projections on the feasible set, which could reduce the program's computational efficiency if the projections are hard to calculate. To improve the numerical performance of the extragradient algorithm, there are several main variants of the extragradient algorithm; see, e.g., [7–11].

In contrast to linear spaces, there are few papers studying solution algorithms for VIP (1.1) on Hadamard manifolds. In [13], Németh extended some findings on the existence and uniqueness of solutions for VIPs in Euclidean spaces to Hadamard manifolds. Subsequently, an extragradient approach with Armijo line search step sizes was devised by Ferreira et al. [14] to identify singularities of continuous monotone vector fields on Hadamard manifolds. Inspired by the work of Iusem and Svaiter [6], Tang et al. [15, 17] provided two extragradient methods with Armijo line search step sizes to find the solution of VIP (1.1). When the vector field is continuous and pseudo-monotone, they demonstrated that the sequences produced by the proposed algorithms converge to the solution set. Recently, Batista et al. [18] extended the results in [15] from univalued vector fields to multivalued maximal monotone vector fields. Very recently, Sahu et al. [28] proposed an extragradient algorithm with an Armijo-type step size rule to solve both monotone and non-monotone variational inequalities. Comparing the algorithm to [15, 17], they showed its computational efficiency with numerical examples. The drawback of the algorithms described in [15, 17, 18, 28] is that using the Armijo-type rule to update the step size significantly increases the computation time. Therefore, it is interesting and necessary to continuously develop some new results based on extragradient algorithms.

The structure of this paper is outlined as follows: Section 2 introduces key results within the framework of Riemannian geometry. Section 3 details adaptive numerical methods developed to address the VIP (1.1) by utilizing pseudo-monotone vector fields, along with a convergence analysis of these algorithms. We also establish global error bounds and demonstrate R-linear convergence for cases where the vector field exhibits strong pseudo-monotonicity. In Section 4, we present fundamental tests on Hadamard manifolds to validate the convergence of our methods. Finally, Section 5 wraps up the paper and suggests future research directions.

2 Notation and terminology

This section aims to present some useful concepts and results regarding Hadamard manifolds, essential for understanding the content of this paper. These concepts are standard in Riemannian geometry (see, e.g., [15, 23, 33–36]).

Let $\mathcal{X}$ be a connected k-dimensional manifold. The `tangent bundle` of $\mathcal{X}$, represented as $T\mathcal{X}$, is given as $T\mathcal{X} = \cup_{u\in\mathcal{X}} T_u\mathcal{X}$, where $T_u\mathcal{X}$ is the `tangent space` at $u \in \mathcal{X}$ which consists of all tangent vectors at that point. A Riemannian metric on $T_u\mathcal{X}$ is an inner product $\langle \cdot\,,\cdot\rangle_u$ that maps pairs of tangent vectors to nonnegative real numbers, $\langle \cdot\,,\cdot\rangle_u : T_u\mathcal{X} \times T_u\mathcal{X} \to \mathbb{R}_{\geq 0}$, and induces a norm $\|\cdot\|_u$ such that $\|u\|_u = \langle u, u\rangle_u^{1/2}$. When this inner product defines a Riemannian metric for every point $u \in \mathcal{X}$, the collection of inner products $\langle \cdot\,,\cdot\rangle$ constitutes a `Riemannian metric` on $\mathcal{X}$. `Riemannian manifold` is the name given to a differentiable manifold $\mathcal{X}$ that has this metric enabled. To keep things simple, the inner product and norm on $T_u\mathcal{X}$ will be represented by the notations $\langle \cdot\,,\cdot\rangle$ and $\|\cdot\|$, respectively.

The `length` of a piecewise smooth curve $\eta : [a, b] \to \mathcal{X}$ connecting points u and q (where $\eta(a) = u$ and $\eta(b) = q$) is defined as $L(\eta) := \int_a^b \left\|\eta'(t)\right\| \mathrm{d}t$, where $\eta'(t) = \frac{\mathrm{d}\eta(t)}{\mathrm{d}t}$ represents a tangent vector in $T_{\eta(t)}\mathcal{X}$. The minimum length over all such curves linking u and q is known as the `Riemannian distance` between u and q, denoted by $\mathrm{s}(u, q)$. The topology induced by this distance function on $\mathcal{X}$ matches the original topology of the manifold, making $(\mathcal{X}, \mathrm{s})$ a complete metric space, which implies that any closed and bounded subsets of $\mathcal{X}$ are compact (see [33, p. 146, Proposition 2.6 and Theorem 2.8]).

A smooth curve η is called a `geodesic` if its derivative η' remains parallel along η, which implies that $\left\|\eta'\right\|$ remains constant. A geodesic connecting points u and q in $\mathcal{X}$ is termed `minimal` if its length is equal to $\mathrm{s}(u, q)$. The `parallel transport` $\mathrm{PT}_{\eta(b),\eta(a)} : T_{\eta(a)}\mathcal{X} \to T_{\eta(b)}\mathcal{X}$ on the tangent bundle $T\mathcal{X}$ along a minimal geodesic from $\eta(a)$ to $\eta(b)$ is given as $\mathrm{PT}_{\eta(b),\eta(a)}(v) := B(\eta(b))$, where B is the unique vector field such that $\nabla_{\eta'(t)}B = \theta$ (where θ is the zero tangent vector) for all $t \in [a, b]$ and satisfies $B(\eta(a)) = v$. It is important to note that $\mathrm{PT}_{\eta(b),\eta(a)}$ acts as an isometry from $T_{\eta(a)}\mathcal{X}$ to $T_{\eta(b)}\mathcal{X}$.

The `exponential map` $\mathrm{E}_u : T_u\mathcal{X} \to \mathcal{X}$ at a point u is denoted as $\mathrm{E}_u v := \eta_v(1, u)$, where $\eta_v(\cdot, u)$ denotes the geodesic originating from u with initial velocity v. For any $t \in \mathbb{R}$, $\mathrm{E}_u tv = \eta_v(t, u)$, and $\mathrm{E}_u\theta = \eta_v(0, u) = u$. The exponential map has an inverse $\mathrm{E}_u^{-1} : \mathcal{X} \to T_u\mathcal{X}$. Additionally, we have (see, e.g., [33, p. 146, Proposition 2.5] and [35, p. 39, Corollary 2.8]):

$$\left\|\mathrm{E}_u^{-1} g\right\| = \left\|\mathrm{E}_g^{-1} u\right\| = \mathrm{s}(u, g) = \mathrm{s}(g, u) \quad (\forall u, g \in \mathcal{X}).$$

A `Hadamard manifold` refers to a complete, simply connected Riemannian manifold characterized by nonpositive sectional curvature. Throughout this paper, we denote an k-dimensional Hadamard manifold by $\mathcal{X}$ and use C to represent a nonempty, closed, and convex subset within $\mathcal{X}$. According to the well-known Hadamard-Cartan Theorem [35, p. 221, Theorem 4.1], the topology and differential structure of $\mathcal{X}$ are identical to those of the Euclidean space $\mathbb{R}^m$. We now proceed to recall some geometric properties relevant to Hadamard manifolds.

Consider points u_1, u_2, and u_3 on $\mathcal{X}$. Let $\Delta\,(u_1, u_2, u_3)$ denote a geodesic triangle on $\mathcal{X}$ formed by three minimal geodesics η_i that connect u_i to u_{i+1}, where $i = 1, 2, 3 \pmod 3$. Let α_i be the angles of $\Delta\,(u_1, u_2, u_3)$ at the vertices u_i. Based on the comparison theorem for triangles (see [35, p. 223, Prop. 4.5] and [22, Thm. 2.2]), along with properties of the exponential map and distance on $\mathcal{X}$, we have the following results (refer to [23, Eq. (2.3)], [24, Eq. (9)], and [31, p. 280, Prop. 14.16]):

$$\left\langle \mathrm{E}_{u_{i+1}}^{-1} u_i, \mathrm{E}_{u_{i+1}}^{-1} u_{i+2} \right\rangle = \mathrm{s}\left(u_i, u_{i+1}\right) \mathrm{s}\left(u_{i+1}, u_{i+2}\right) \cos \alpha_{i+1}, \tag{2.1}$$

$$\mathrm{s}^2\left(u_i, u_{i+1}\right) + \mathrm{s}^2\left(u_{i+1}, u_{i+2}\right) - 2\left\langle \mathrm{E}_{u_{i+1}}^{-1} u_i, \mathrm{E}_{u_{i+1}}^{-1} u_{i+2} \right\rangle \leq \mathrm{s}^2\left(u_{i+2}, u_i\right), \tag{2.2}$$

$$\mathrm{s}^2\left(u_i, u_{i+2}\right) \leq \left\langle \mathrm{E}_{u_{i+2}}^{-1} u_i, \mathrm{E}_{u_{i+2}}^{-1} u_{i+1} \right\rangle + \left\langle \mathrm{E}_{u_i}^{-1} u_{i+1}, \mathrm{E}_{u_i}^{-1} u_{i+2} \right\rangle.$$

By letting $u_{i+2} = u_i$ in (2.1), one obtains

$$\left\langle \mathrm{E}_{u_{i+1}}^{-1} u_i, \mathrm{E}_{u_{i+1}}^{-1} u_i \right\rangle = \mathrm{s}^2\left(u_{i+1}, u_i\right) = \left\| \mathrm{E}_{u_{i+1}}^{-1} u_i \right\|^2.$$

Let the triangle $\Delta\left(u', q', t'\right)$ represent the comparison triangle for the geodesic triangle $\Delta(u, q, t)$. It is important to note that the comparison triangle is unique up to isometry within $\mathcal{X}$.

Lemma 2.1 ([40, p. 24, Lemma 2.14]) *Consider the geodesic triangle $\Delta(u_1, u_2, u_3)$ in $\mathcal{X}$. There exist points $u_i' \in \mathbb{R}^2$ $(i = 1, 2, 3)$ such that $\mathrm{s}(u_i, u_j) = \|u_i' - u_j'\|$, $\forall i, j \in \{1, 2, 3\}$, $i \neq j$.*

Lemma 2.2 ([30, Lemma 3.5]) *Let $\Delta(u, g, t)$ be a geodesic triangle in $\mathcal{X}$, and let $\Delta\left(u', g', t'\right)$ be its corresponding comparison triangle. Denote the angles at u, g, t of $\Delta(u, g, t)$ by τ, δ, η, and the angles at u', g', t' of $\Delta(u', g', t')$ by τ', δ', η'. Then $\tau' \geq \tau$, $\delta' \geq \delta$, and $\eta' \geq \eta$.*

Proposition 2.1 ([23, p. 671]) *Let $u, q \in \mathcal{X}$ and $v \in T_q\mathcal{X}$. Then*

$$\langle v, -\mathrm{E}_q^{-1} u \rangle = \langle v, \mathrm{PT}_{q,u} \mathrm{E}_u^{-1} q \rangle = \langle \mathrm{PT}_{u,q} v, \mathrm{E}_u^{-1} q \rangle.$$

Lemma 2.3 below is crucial for the convergence analysis in this paper.

Lemma 2.3 *Let $t_m, r_m, g_m, g_{m+1} \in \mathcal{X}$. Consider the geodesic triangle $\Delta(t_m, r_m, g_m)$ and its comparison triangle $\Delta(t_m', r_m', g_m')$. Similarly, consider the geodesic triangle $\Delta(t_m, r_m, g_{m+1})$ and its comparison triangle $\Delta(t_m', r_m', g_{m+1}')$. It follows that*

$$\langle g_{m+1}' - r_m', t_m' - g_m' \rangle \leq \langle \mathrm{PT}_{g_m, r_m} \mathrm{E}_{r_m}^{-1} g_{m+1}, \mathrm{E}_{g_m}^{-1} t_m \rangle.$$

Proof Let $\Psi = \mathrm{E}_{r_m}^{-1} g_{m+1}$ and $a = \mathrm{E}_{g_m} \mathrm{PT}_{g_m, r_m} \mathrm{E}_{r_m}^{-1} g_{m+1}$. The comparison point of a is $a' = g_m' + g_{m+1}' - r_m'$. Let δ (resp., δ') be the angle of $\Delta(a, g_m, t_m)$ (resp., $\Delta(a', g_m', t_m')$) at the vertice g_m (resp., g_m'). According to Lemma 2.2, it follows that $\delta' \geq \delta$ and thus $\cos\delta' \leq \cos\delta$ since $\delta, \delta' \in (0, \pi)$. By using Lemma 2.1 and (2.1), one deduces that

$$\begin{aligned} \langle a' - g_m', t_m' - g_m' \rangle &= \left\| a' - g_m' \right\| \left\| t_m' - g_m' \right\| \cos\delta' \\ &\leq \mathrm{s}\left(a, g_m\right) \mathrm{s}\left(t_m, g_m\right) \cos\delta = \langle \mathrm{PT}_{g_m, r_m} \mathrm{E}_{r_m}^{-1} g_{m+1}, \mathrm{E}_{g_m}^{-1} t_m \rangle. \end{aligned}$$

This completes the proof. □

Proposition 2.2 ([23, Lemma 2.4],[18, Lemma 1.1]) *Let $g_0, r_0 \in \mathcal{X}$, with sequences $\{g_m\} \subset \mathcal{X}$ and $\{r_m\} \subset \mathcal{X}$. If $\lim_{m\to\infty} g_m = g_0$ and $\lim_{m\to\infty} r_m = r_0$, then the following claims are true.*

(i) *For any point $y \in \mathcal{X}$, we have $\lim_{m\to\infty} \mathrm{E}_{g_m}^{-1} y = \mathrm{E}_{g_0}^{-1} y$ and $\lim_{m\to\infty} \mathrm{E}_y^{-1} g_m = \mathrm{E}_y^{-1} g_0$.*
(ii) *If $a_m \in T_{g_m}\mathcal{X}$ and $\lim_{m\to\infty} a_m = a_0$, then $a_0 \in T_{g_0}\mathcal{X}$.*
(iii) *Given $s_m, a_m \in T_{g_m}\mathcal{X}$ and $s_0, a_0 \in T_{g_0}\mathcal{X}$, if $\lim_{m\to\infty} s_m = s_0$ and $\lim_{m\to\infty} a_m = a_0$, then $\lim_{m\to\infty}\langle s_m, a_m\rangle = \langle s_0, a_0\rangle$.*
(iv) *For any $u \in T_{g_0}\mathcal{X}$, the function $B\colon \mathcal{X} \to T\mathcal{X}$, defined by $B(x) = \mathrm{PT}_{x,g_0} u$ for each $x \in \mathcal{X}$ is continuous on $\mathcal{X}$.*
(v) *We have $\lim_{m\to\infty} \mathrm{E}_{g_m}^{-1} r_m = \mathrm{E}_{g_0}^{-1} r_0$.*

Proposition 2.3 ([18, Lemma 1.2]) *Let $\{g_m\} \subset \mathcal{X}$, $\{a_m\} \subset T_{g_m}\mathcal{X}$, and $\{t_m\} \subset (0, 1)$, with $g_0 \in \mathcal{X}$, $a_0 \in T_{g_0}\mathcal{X}$, and $t_0 \in [0, 1)$ such that $\lim_{m\to\infty} g_m = g_0$, $\lim_{m\to\infty} a_m = a_0$, and $\lim_{m\to\infty} t_m = t_0$. Define $\{r_m\}$ by $r_m := \mathrm{E}_{g_m} t_m a_m$. Then, it holds that $\lim_{m\to\infty} \mathrm{PT}_{r_m,g_m} a_m = \mathrm{PT}_{r_0,g_0} a_0$, where $r_0 := \mathrm{E}_{g_0} t_0 a_0 = \lim_{m\to\infty} r_m$.*

Definition 2.1 ([34, p. 59, Definition 1.3]) A subset $C \subset \mathcal{X}$ is called (geodesic) `convex` if for any two points $p, q \in C$, the geodesic connecting p to q lies completely within C. Specifically, if $\eta\colon [a, b] \to \mathcal{X}$ is a geodesic with $p = \eta(a)$ and $q = \eta(b)$, then $\eta((1-t)a + tb) \in C$ for all $t \in [0, 1]$.

Definition 2.2 ([39, Theorem 1], [22, Proposition 3.2]) Let $C \subset \mathcal{X}$ and $t' \in \mathcal{X}$. The point $t \in C$ is called the `projection` of t' onto C, denoted as $\mathbf{Pj}_C\left(t'\right)$, if it satisfies $\mathrm{s}\left(t', t\right) \le \mathrm{s}\left(t', y\right)$ for all $y \in C$.

Let $\mathbf{Pj}_C\colon \mathcal{X} \to C$ denote the projection onto C. According to [39, Theorem 2], the projection $\mathbf{Pj}_C$ is Lipschitz continuous and characterized by Proposition 2.4 below.

Proposition 2.4 ([39, Theorem 2], [22, Corollary 3.1]) *For $C \subset \mathcal{X}$ and $u \in \mathcal{X}$, we have $\mathbf{Pj}_C$ is single-valued and $z = \mathbf{Pj}_C(u)$ if and only if*

$$\langle \mathrm{E}_z^{-1} u, \mathrm{E}_z^{-1} q\rangle \le 0 \quad (\forall q \in C).$$

Remark 2.1 From Proposition 2.4, u^* solves problem (1.1) if and only if $u^* = \mathbf{Pj}_C\left(\mathrm{E}_{u^*}(-\vartheta B u^*)\right)$ for all $\vartheta > 0$. When $\mathcal{X} = \mathbb{R}^m$ in Proposition 2.4, it follows that

$$\left\langle u - \mathbf{Pj}_C(u), q - \mathbf{Pj}_C(u)\right\rangle \le 0 \quad (\forall u \in \mathbb{R}^m)(\forall q \in C).$$

This inequality relates to the fundamental property of projection (see, e.g., [42, p. 53, Theorem 3.16]).

Lemma 2.4 ([15, Lemmas 2.2 and 2.4]) *Let C be a closed and convex subset in a Hadamard manifold $\mathcal{X}$. Let $u \in \mathcal{X}$ and $q \in C$. Then*

(i) $\mathrm{s}^2(q, \mathbf{Pj}_C(u)) \le \mathrm{s}^2(q, u) - \mathrm{s}^2\left(u, \mathbf{Pj}_C(u)\right) \quad (\forall q \in C)$;
(ii) $\mathrm{s}^2(q, \mathbf{Pj}_C(u)) \le \langle \mathrm{E}_q^{-1} u, \mathrm{E}_q^{-1} \mathbf{Pj}_C(u)\rangle \quad (\forall q \in C)$.

Remark 2.2 It is established that a class of firmly quasi-nonexpansive mappings contains firmly nonexpansive mappings and quasi-nonexpansive mappings (refer to [42, p. 69, Definition 4.1] for definitions). From (i) of Lemma 2.4, it follows that the projection operator $\mathbf{Pj}_C$ in Hadamard manifolds also exhibits firmly quasi-nonexpansiveness. This observation is additionally noted in [24, Corollary 1].

Definition 2.3 ([37, Definition 2.2], [23, Definition 3.1], [15, Definition 3.1], [38, p. 702]) Let $C \subset \mathcal{X}$. A vector field B on C is a mapping $B: C \to T\mathcal{X}$ such that $Bu \in T_u\mathcal{X}$ for each $u \in \mathcal{X}$. Then B is said to be:

(i) `monotone` if

$$\langle Bu, \mathrm{E}_u^{-1}q\rangle \leq \langle Bq, -\mathrm{E}_q^{-1}u\rangle, \quad \forall u, q \in C.$$

(ii) μ-`strongly monotone` ($\mu > 0$) if

$$\langle Bu, \mathrm{E}_u^{-1}q\rangle - \langle Bq, -\mathrm{E}_q^{-1}u\rangle \leq -\mu \mathrm{s}^2(u, q), \quad \forall u, q \in C.$$

(iii) μ-`strongly pseudo-monotone` ($\mu > 0$) if

$$\langle Bu, \mathrm{E}_u^{-1}q\rangle \geq 0 \Rightarrow \langle Bq, \mathrm{E}_q^{-1}u\rangle \leq -\mu \mathrm{s}^2(u, q), \quad \forall u, q \in C.$$

(iv) `pseudo-monotone` if

$$\langle Bu, \mathrm{E}_u^{-1}q\rangle \geq 0 \Rightarrow \langle Bq, \mathrm{E}_q^{-1}u\rangle \leq 0, \quad \forall u, q \in C.$$

(v) L-`Lipschitz continuous` ($L > 0$) if

$$\left\|\mathrm{PT}_{q,u}Bu - Bq\right\| \leq L\mathrm{s}(u, q), \quad \forall u, q \in C.$$

Lemma 2.5 ([17, Lemma 2.8]) *If vector field $B: C \to T\mathcal{X}$ is continuous and C is compact and convex, then VIP (1.1) has a solution.*

The notion of R-linear convergence in linear spaces can be generalized to Hadamard manifolds.

Definition 2.4 In a Hadamard manifold $\mathcal{X}$, a sequence $\{g_m\}$ converges R-linearly to u^* with rate $\alpha \in [0, 1)$ if there is a constant $c > 0$ such that $\mathrm{s}(g_m, u^*) \leq c\alpha^m$ for all $m \in \mathbb{N}$.

Definition 2.5 ([22, p. 268, Eq. (25)]) Given a complete metric space X, let C be a nonempty set. With regard to C, a sequence $\{g_m\} \subset X$ is Fejér convergent provided

$$\mathrm{s}\left(g_{m+1}, q\right) \leq \mathrm{s}\left(g_m, q\right) \quad (\forall q \in C)(\forall m \geq 0).$$

Lemma 2.6 ([22, Lemma 6.1], [14, Lemma 7.2]) *Let C be a nonempty subset of a complete metric space X. If the sequence $\{g_m\} \subset X$ is Fejér convergent to C, then $\{g_m\}$ is bounded. Furthermore, if every cluster point of $\{g_m\}$ lies within C, then $\{g_m\}$ converges to a point in C.*

3 Main results

In this section, we give five new iterative algorithms with adaptive step sizes designed to solve variational inequalities governed by pseudo-monotone vector fields in Hadamard manifolds. Our three algorithms utilize an adaptive step size rule (see (3.1)) that does not require a line search. This rule automatically adjusts the step size for the next iteration based on information from previous iterations. The adaptive step size approach proposed in this paper demonstrates improved performance over Armijo-type step size rules used in previous works such as [15, 17, 18, 28], particularly regarding computational efficiency. A key benefit of these algorithms is that they do not require prior knowledge of the Lipschitz constant for the pseudo-monotone vector field. To analyze the convergence of the proposed algorithms, we assume that the following three conditions are satisfied.

(C1) The solution set $\Gamma(B, C)$ of VIP (1.1) is assumed to be nonempty, that is $\Gamma(B, C) \neq \emptyset$.
(C2) The feasible set C is a nonempty, closed, and convex subset of Hadamard manifold $\mathcal{X}$.
(C3) The vector field $B : C \to T\mathcal{X}$ is pseudo-monotone and L-Lipschitz continuous on C. Let $\{\varpi_m\} \subset [1, \infty)$ satisfies $\sum_{m=1}^{\infty}(\varpi_m - 1) < \infty$, and $\{\mu_m\} \subset [0, \infty)$ such that $\sum_{m=1}^{\infty} \mu_m < \infty$.

3.1 Five self-adaptive extragradient algorithms

In this subsection, we provide five modified extragradient-type methods to solve VIP (1.1) in Hadamard manifolds. The five proposed algorithms offer a dual advantage of simplicity and efficiency. The first three algorithms are explicit iterative schemes, while the last two are implicit ones. Building upon the subgradient extragradient algorithm in Euclidean spaces [9], we start by presenting an adaptive modified subgradient extragradient algorithm (see Algorithm 3.1 below) to solve VIP (1.1) with a pseudo-monotone and Lipschitz continuous vector field.

Remark 3.1 If $g_m = r_m$ in *Step 1* of Algorithm 3.1, then by Remark 2.1,

$$\langle Bg_m, \mathrm{E}_{g_m}^{-1} z\rangle \geq 0 \ \ (\forall z \in C).$$

This yields that $g_m \in \Gamma(B, C)$ by the definition of VIP (1.1) and thus the iterative process of Algorithm 3.1 stops.

In all subsequent convergence analyses, we assume that the proposed algorithms do not terminate in a finite number of steps. Before proving the convergence theorem of Algorithm 3.1, we first present two important lemmas.

Lemma 3.1 *Let step size $\{\vartheta_m\}$ be a sequence generated by (3.1). Then it is well defined.*

Proof Since B is L-Lipschitz continuous, in the case of $\left\|\mathrm{PT}_{r_m, g_m} Bg_m - Br_m\right\| \neq 0$, one obtains

$$\frac{\nu \mathrm{s}(g_m, r_m)}{\left\|\mathrm{PT}_{r_m, g_m} Bg_m - Br_m\right\|} \geq \frac{\nu \mathrm{s}(g_m, r_m)}{L\mathrm{s}(g_m, r_m)} = \frac{\nu}{L}.$$

Algorithm 3.1 The first type of modified subgradient extragradient algorithm

Initialization: Take $\vartheta_0 > 0$, $\nu \in (0,1)$, and $\beta \in (0, 2/(1+\nu))$. Let $\{\varpi_m\}$ and $\{\mu_m\}$ satisfy Condition (C3). Let $g_0 \in \mathcal{X}$ and set $m = 0$.

Step 1. Compute

$$r_m = \mathbf{Pj}_C \left(\mathrm{E}_{g_m}(-\vartheta_m B g_m)\right).$$

If $g_m = r_m$, then stop the iterative process and $g_m \in \Gamma(B, C)$; otherwise, go to *Step 2*.

Step 2. Compute

$$g_{m+1} = \mathbf{Pj}_{H_m} \left(\mathrm{E}_{g_m} \left(\mathrm{PT}_{g_m, r_m} (-\beta \vartheta_m B r_m)\right)\right),$$

where

$$H_m := \left\{x \in \mathcal{X} \colon \langle \mathrm{E}_{r_m}^{-1} g_m - \vartheta_m \mathrm{PT}_{r_m, g_m} B g_m, \mathrm{E}_{r_m}^{-1} x\rangle \le 0\right\},$$

and update ϑ_{m+1} by

$$\vartheta_{m+1} = \begin{cases} \min\left\{\dfrac{\nu \mathrm{s}(g_m, r_m)}{\left\|\mathrm{PT}_{r_m, g_m} B g_m - B r_m\right\|}, \varpi_m \vartheta_m + \mu_m\right\}, & \text{if } \left\|\mathrm{PT}_{r_m, g_m} B g_m - B r_m\right\| \neq 0; \\ \varpi_m \vartheta_m + \mu_m, & \text{otherwise.} \end{cases} \tag{3.1}$$

Set $m := m + 1$ and go to *Step 1*.

Here, $\mathrm{PT}_{r_m, g_m} B g_m$ represents the parallel transport of $B g_m$ from $T_{g_m}\mathcal{X}$ to $T_{r_m}\mathcal{X}$ along a minimal geodesic connecting g_m and r_m. Combining the above inequality with (3.1) leads to $\vartheta_{m+1} \ge \min\{\vartheta_m, \nu/L\}$. By induction, it follows that $\vartheta_m \ge \min\{\vartheta_0, \nu/L\}$. Additionally, from (3.1), we observe that $\vartheta_{m+1} \le \varpi_m \vartheta_m + \mu_m$ for any $m \ge 0$. Given condition (C3) and [41, Lemma 1], we can conclude that $\lim_{m\to\infty} \vartheta_m$ exists. Since the sequence $\{\vartheta_m\}$ is bounded below by $\min\{\vartheta_0, \nu/L\}$, it follows that $\lim_{m\to\infty} \vartheta_m := \vartheta > 0$. □

Lemma 3.2 *Let $\{r_m\}$ and $\{g_m\}$ be created by Algorithm 3.1. Fix $p \in \Gamma(B, C)$. It follows that*

$$\mathrm{s}^2(g_{m+1}, p) \le \mathrm{s}^2(g_m, p) - \beta^* \left(\mathrm{s}^2(g_m, r_m) + \mathrm{s}^2(r_m, g_{m+1})\right), \tag{3.2}$$

where

$$\beta^* := \begin{cases} 2 - \beta - \beta\nu\vartheta_m/\vartheta_{m+1}, & \text{if } \beta \in [1, 2/(1+\nu)), \\ \beta(1 - \nu\vartheta_m/\vartheta_{m+1}), & \text{if } \beta \in (0, 1). \end{cases}$$

Moreover, $\{g_m\}$ is Fejér monotone with respect to $\Gamma(B, C)$. Analogously, both $\{r_m\}$ and $\{g_m\}$ are bounded.

Proof Let $t_m := \mathrm{E}_{g_m}\left(\mathrm{PT}_{g_m, r_m}(-\beta\vartheta_m B r_m)\right)$. Consider $\Delta(t_m, r_m, g_m)$ and its comparison triangle $\Delta\left(t'_m, r'_m, g'_m\right)$. The comparison point of t_m is $t'_m = g'_m - \beta\vartheta_m B r'_m$. Similarly, consider the pair of $\Delta(t_m, g_m, p)$ and $\Delta\left(t'_m, g'_m, p'\right)$. By using Lemma 2.1, one has

$$\mathrm{s}(t_m, p) = \left\|t'_m - p'\right\|, \quad \mathrm{s}(g_m, p) = \left\|g'_m - p'\right\|, \quad \mathrm{s}(t_m, g_m) = \left\|t'_m - g'_m\right\|.$$

Also, consider the geodesic triangle $\Delta\left(t_m, g_{m+1}, g_m\right)$ and its comparison triangle $\Delta\left(t_m', g_{m+1}', g_m'\right)$. By Lemma 2.1, one obtains

$$\mathrm{s}\left(g_{m+1}, g_m\right) = \left\|g_{m+1}' - g_m'\right\|, \quad \mathrm{s}\left(t_m, g_{m+1}\right) = \left\|t_m' - g_{m+1}'\right\|.$$

From the definitions of g_{m+1} and t_m, and Lemma 2.4(i) (noting that $p \in \Gamma(B, C) \subset C$), we have

$$\begin{aligned}
&\mathrm{s}^2\left(g_{m+1}, p\right) = \mathrm{s}^2\left(\mathbf{Pj}_{H_m}\left(\mathrm{E}_{g_m}\left(\mathrm{PT}_{g_m,r_m}\left(-\beta\vartheta_m B r_m\right)\right)\right), p\right)\\
&\le \mathrm{s}^2\left(\mathrm{E}_{g_m}\left(\mathrm{PT}_{g_m,r_m}\left(-\beta\vartheta_m B r_m\right)\right), p\right) - \mathrm{s}^2\left(\mathrm{E}_{g_m}\left(\mathrm{PT}_{g_m,r_m}\left(-\beta\vartheta_m B r_m\right)\right), g_{m+1}\right)\\
&= \mathrm{s}^2\left(t_m, p\right) - \mathrm{s}^2\left(t_m, g_{m+1}\right).
\end{aligned}$$

Note that

$$\begin{aligned}
&\mathrm{s}^2\left(t_m, p\right) - \mathrm{s}^2\left(t_m, g_{m+1}\right)\\
&= \left\|t_m' - p'\right\|^2 - \left\|t_m' - g_{m+1}'\right\|^2\\
&= \left\|g_m' - \beta\vartheta_m B r_m' - p'\right\|^2 - \left\|g_m' - \beta\vartheta_m B r_m' - g_{m+1}'\right\|^2\\
&= \left\|g_m' - p'\right\|^2 + \left(\beta\vartheta_m\right)^2\left\|B r_m'\right\|^2 - 2\langle g_m' - p', \beta\vartheta_m B r_m'\rangle\\
&\quad - \left\|g_m' - g_{m+1}'\right\|^2 - \left(\beta\vartheta_m\right)^2\left\|B r_m'\right\|^2 + 2\langle g_m' - g_{m+1}', \beta\vartheta_m B r_m'\rangle\\
&= \mathrm{s}^2\left(g_m, p\right) + 2\langle g_m' - p', t_m' - g_m'\rangle - \mathrm{s}^2\left(g_m, g_{m+1}\right) + 2\langle g_{m+1}' - g_m', t_m' - g_m'\rangle.
\end{aligned}$$

Therefore, it follows that

$$\mathrm{s}^2\left(g_{m+1}, p\right) \le \mathrm{s}^2\left(g_m, p\right) + 2\langle g_{m+1}' - p', t_m' - g_m'\rangle - \mathrm{s}^2\left(g_m, g_{m+1}\right). \tag{3.3}$$

According to $p \in \Gamma(B, C)$ and $r_m \in C$, one arrives at $\langle Bp, \mathrm{E}_p^{-1} r_m\rangle \ge 0$, which together with the pseudomonotonicity of B implies that

$$\langle B r_m, \mathrm{E}_{r_m}^{-1} p\rangle \le 0. \tag{3.4}$$

By using (3.4), Lemma 2.3, and Proposition 2.1, one has

$$\begin{aligned}
&\langle g_{m+1}' - p', t_m' - g_m'\rangle\\
&= \langle g_{m+1}' - r_m', t_m' - g_m'\rangle + \langle r_m' - p', t_m' - g_m'\rangle\\
&\le \langle \mathrm{PT}_{g_m,r_m}\mathrm{E}_{r_m}^{-1} g_{m+1}, \mathrm{E}_{g_m}^{-1} t_m\rangle + \langle \mathrm{PT}_{g_m,p}\mathrm{E}_p^{-1} r_m, \mathrm{E}_{g_m}^{-1} t_m\rangle\\
&= \langle \mathrm{PT}_{g_m,r_m}\mathrm{E}_{r_m}^{-1} g_{m+1}, \mathrm{E}_{g_m}^{-1} t_m\rangle + \langle -\mathrm{E}_{g_m}^{-1} t_m, \mathrm{PT}_{g_m,r_m}\mathrm{E}_{r_m}^{-1} p\rangle\\
&= -\beta\langle \vartheta_m B r_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle + \beta\vartheta_m\langle B r_m, \mathrm{E}_{r_m}^{-1} p\rangle\\
&\le -\beta\langle \vartheta_m B r_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle.
\end{aligned} \tag{3.5}$$

Combining (3.3) and (3.5), one obtains

$$\mathrm{s}^2\left(g_{m+1}, p\right) \leq \mathrm{s}^2\left(g_m, p\right) - \mathrm{s}^2\left(g_m, g_{m+1}\right) - 2\beta\langle\vartheta_m B r_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle. \tag{3.6}$$

Consider the geodesic triangle $\Delta(g_m, r_m, g_{m+1})$. It follows from (2.2) that

$$\mathrm{s}^2(r_m, g_{m+1}) + \mathrm{s}^2(r_m, g_m) - 2\langle\mathrm{E}_{r_m}^{-1} g_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle \leq \mathrm{s}^2(g_m, g_{m+1}). \tag{3.7}$$

Note that

$$\begin{aligned}
&\langle\mathrm{E}_{r_m}^{-1} g_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle \\
&= \langle\mathrm{E}_{r_m}^{-1} g_m - \vartheta_m \mathrm{PT}_{r_m, g_m} B g_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle + \langle\vartheta_m \mathrm{PT}_{r_m, g_m} B g_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle \\
&= \langle\mathrm{E}_{r_m}^{-1} g_m - \vartheta_m \mathrm{PT}_{r_m, g_m} B g_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle + \langle\vartheta_m B r_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle \\
&\quad + \langle\vartheta_m \mathrm{PT}_{r_m, g_m} B g_m - \vartheta_m B r_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle.
\end{aligned} \tag{3.8}$$

From the definition of H_m and $g_{m+1} \in H_m$, one sees that

$$\langle\mathrm{E}_{r_m}^{-1} g_m - \vartheta_m \mathrm{PT}_{r_m, g_m} B g_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle \leq 0. \tag{3.9}$$

According to the definition of ϑ_{m+1}, one has

$$\begin{aligned}
\langle\vartheta_m \mathrm{PT}_{r_m, g_m} B g_m - \vartheta_m B r_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle &\leq \vartheta_m \left\|\mathrm{PT}_{r_m, g_m} B g_m - B r_m\right\| \cdot \left\|\mathrm{E}_{r_m}^{-1} g_{m+1}\right\| \\
&\leq \frac{\nu\vartheta_m}{\vartheta_{m+1}} \mathrm{s}\left(g_m, r_m\right) \mathrm{s}\left(g_{m+1}, r_m\right) \\
&\leq \frac{1}{2} \frac{\nu\vartheta_m}{\vartheta_{m+1}} \left(\mathrm{s}^2\left(g_m, r_m\right) + \mathrm{s}^2\left(g_{m+1}, r_m\right)\right).
\end{aligned} \tag{3.10}$$

Combining (3.8), (3.9), and (3.10), we have

$$\langle\mathrm{E}_{r_m}^{-1} g_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle \leq \frac{1}{2} \frac{\nu\vartheta_m}{\vartheta_{m+1}} \left(\mathrm{s}^2\left(g_m, r_m\right) + \mathrm{s}^2\left(g_{m+1}, r_m\right)\right) + \langle\vartheta_m B r_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle.$$

This together with (3.7) yields that

$$\begin{aligned}
-\mathrm{s}^2(g_m, g_{m+1}) &\leq -\mathrm{s}^2(r_m, g_{m+1}) - \mathrm{s}^2(r_m, g_m) + 2\langle\vartheta_m B r_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle \\
&\quad + \nu\vartheta_m/\vartheta_{m+1} \left(\mathrm{s}^2\left(g_m, r_m\right) + \mathrm{s}^2\left(g_{m+1}, r_m\right)\right) \\
&= -\left(1 - \nu\vartheta_m/\vartheta_{m+1}\right) \left(\mathrm{s}^2\left(g_m, r_m\right) + \mathrm{s}^2\left(g_{m+1}, r_m\right)\right) \\
&\quad + 2\langle\vartheta_m B r_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle,
\end{aligned}$$

which is equivalent to

$$-2\beta\langle\vartheta_m Br_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle \le -\beta\,(1-\nu\vartheta_m/\vartheta_{m+1})\left(\mathrm{s}^2\,(g_m, r_m) + \mathrm{s}^2\,(g_{m+1}, r_m)\right) + \beta \mathrm{s}^2(g_m, g_{m+1}) \quad (\forall\beta > 0). \tag{3.11}$$

From (3.6) and (3.11), one has

$$\mathrm{s}^2\,(g_{m+1}, p) \le \mathrm{s}^2\,(g_m, p) - \beta\,(1-\nu\vartheta_m/\vartheta_{m+1})\left(\mathrm{s}^2\,(g_m, r_m) + \mathrm{s}^2\,(g_{m+1}, r_m)\right) - (1-\beta)\mathrm{s}^2(g_m, g_{m+1}) \quad (\forall\beta > 0). \tag{3.12}$$

Consider the geodesic triangle $\Delta\,(g_m, r_m, g_{m+1})$ and its comparison triangle $\Delta\left(g_m', r_m', g_{m+1}'\right)$. It follows from Lemma 2.1 that

$$\mathrm{s}\,(g_m, r_m) = \|g_m' - r_m'\|, \quad \mathrm{s}\left(r_m, g_{m+1}\right) = \|r_m' - g_{m+1}'\|, \quad \mathrm{s}\left(g_m, g_{m+1}\right) = \|g_m' - g_{m+1}'\|.$$

According to the Cauchy-Schwarz inequality,

$$\begin{aligned}\|g_m' - g_{m+1}'\|^2 &= \|g_m' - r_m' + r_m' - g_{m+1}'\|^2 \\ &\le \|g_m' - r_m'\|^2 + \|g_{m+1}' - r_m'\|^2 + 2\|g_m' - r_m'\| \cdot \|r_m' - g_{m+1}'\| \\ &\le 2\left(\|g_m' - r_m'\|^2 + \|g_{m+1}' - r_m'\|^2\right).\end{aligned}$$

That is

$$\mathrm{s}^2\,(g_m, g_{m+1}) \le 2\left(\mathrm{s}^2\,(g_m, r_m) + \mathrm{s}^2\,(r_m, g_{m+1})\right). \tag{3.13}$$

The conclusion required in (3.2) can be directly derived from (3.12) and (3.13). On the other hand, by virtue of Lemma 3.1, one has $\lim_{m\to\infty}\beta^* > 0$ for any $\beta \in (0, 2/(1+\nu))$. That is, there exists a positive integer N such that $\beta^* > 0$ for all $m \ge N$. Combining this with (3.2), we have

$$\mathrm{s}^2\,(g_{m+1}, p) \le \mathrm{s}^2\,(g_m, p) \quad (\forall m \ge N).$$

This implies that $\{g_m\}$ is Fejér monotone with respect to $\Gamma(B, C)$ and $\{g_m\}$ is bounded. By letting $m \to \infty$ in (3.2), one arrives at $\lim_{m\to\infty}\mathrm{s}\,(g_m, r_m) = 0$ and $\lim_{m\to\infty}\mathrm{s}\,(r_m, g_{m+1}) = 0$. Thus $\{r_m\}$ is also bounded. □

Now we can prove the convergence of the proposed Algorithm 3.1.

Theorem 3.1 *Let $\{g_m\}$ be generated by Algorithm 3.1 and let Conditions* (C1)–(C3) *hold. Then $\{g_m\}$ converges to a solution of VIP* (1.1).

Proof From Lemma 3.2, we know that $\{g_m\}$ is Fejér convergent to the solution set $\Gamma(B, C)$ of the VIP (1.1). According to Lemma 2.6, it is necessary to demonstrate that the weak cluster points of $\{g_m\}$ belong to $\Gamma(B, C)$. Let u^* be a cluster point of $\{g_m\}$. Due to the boundedness of $\{g_m\}$, there exists a subsequence $\{g_{m_j}\}$ of $\{g_m\}$ such that $g_{m_j} \to u^*$. From the condition $\lim_{m\to\infty} \mathrm{s}(g_m, r_m) = 0$, it follows that $r_{m_j} \to u^*$ and thus $u^* \in C$. From $r_{m_j} = P_C\big(\mathrm{E}_{g_{m_j}}(-\vartheta_{m_j} B g_{m_j})\big)$ and considering $j \to \infty$, by using Proposition 2.3, the Lipschitz continuous of B, and $\lim_{j\to\infty} \vartheta_{m_j} = \vartheta > 0$, one obtains

$$u^* = P_C\big(\mathrm{E}_{u^*}\left(-\vartheta B u^*\right)\big).$$

This together with Remark 2.1 yields that $u^* \in \Gamma(B, C)$, as desired. □

Next, we provide a modified version of the suggested Algorithm 3.1, which differs from Algorithm 3.1 in the computation of r_m and g_{m+1}. This approach is shown in Algorithm 3.2.

Algorithm 3.2 The second type of modified subgradient extragradient algorithm

Initialization: Take $\vartheta_0 > 0$, $\nu \in (0, 1)$, and $\beta \in (1/(2-\nu), 1/\nu)$. Let $\{\varpi_m\}$ and $\{\mu_m\}$ satisfy Condition (C3). Let $g_0 \in \mathcal{X}$ and set $m = 0$.

Iterative Steps: Assume that $g_m \in \mathcal{X}$ is known, calculate g_{m+1} as follows.

Step 1. Compute

$$r_m = \mathbf{Pj}_C\left(\mathrm{E}_{g_m}\left(-\beta\vartheta_m B g_m\right)\right).$$

If $g_m = r_m$, then stop the iterative process and $g_m \in \Gamma(B, C)$; otherwise, go to *Step 2*.

Step 2. Compute

$$g_{m+1} = \mathbf{Pj}_{H_m}\left(\mathrm{E}_{g_m}\left(\mathrm{PT}_{g_m, r_m}\left(-\vartheta_m B r_m\right)\right)\right),$$

where

$$H_m := \left\{x \in \mathcal{X} \colon \langle \mathrm{E}_{r_m}^{-1} g_m - \beta\vartheta_m \mathrm{PT}_{r_m, g_m} B g_m, \mathrm{E}_{r_m}^{-1} x\rangle \le 0\right\},$$

and update ϑ_{m+1} by (3.1).

Set $m := m + 1$ and go to *Step 1*.

Theorem 3.2 *Let* $\{g_m\}$ *be created by Algorithm 3.2 and Conditions* (C1)–(C3) *hold. Then* $\{g_m\}$ *converges to a solution of VIP* (1.1).

Proof In the light of (3.3)-(3.6), one sees that

$$\mathrm{s}^2\left(g_{m+1}, p\right) \le \mathrm{s}^2\left(g_m, p\right) - \mathrm{s}^2\left(g_m, g_{m+1}\right) - 2\langle \vartheta_m B r_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle. \tag{3.14}$$

Consider the geodesic triangle $\Delta(g_m, r_m, g_{m+1})$. By using (2.2), one arrives at

$$\mathrm{s}^2(r_m, g_{m+1}) + \mathrm{s}^2(r_m, g_m) - 2\langle \mathrm{E}_{r_m}^{-1} g_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle \le \mathrm{s}^2(g_m, g_{m+1}). \tag{3.15}$$

Note that

$$\begin{aligned} &\langle \mathrm{E}_{r_m}^{-1} g_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle \\ &= \langle \mathrm{E}_{r_m}^{-1} g_m - \beta\vartheta_m \mathrm{PT}_{r_m, g_m} B g_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle + \langle \beta\vartheta_m B r_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle \\ &\quad + \beta\vartheta_m \langle \mathrm{PT}_{r_m, g_m} B g_m - B r_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle. \end{aligned} \tag{3.16}$$

From $g_{m+1} \in H_m$, one obtains

$$\langle \mathrm{E}_{r_m}^{-1} g_m - \beta\vartheta_m \mathrm{PT}_{r_m, g_m} B g_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle \le 0. \tag{3.17}$$

Combining (3.10), (3.16), and (3.17), we have

$$\langle \mathrm{E}_{r_m}^{-1} g_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle \leq \beta\nu\vartheta_m/(2\vartheta_{m+1}) \left(\mathrm{s}^2\left(g_m, r_m\right) + \mathrm{s}^2\left(g_{m+1}, r_m\right)\right) \\ + \beta\langle\vartheta_m B r_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle.$$

This combining with (3.15) infers that

$$-\mathrm{s}^2(g_m, g_{m+1}) \leq -\left(1 - \beta\nu\vartheta_m/\vartheta_{m+1}\right) \left(\mathrm{s}^2\left(g_m, r_m\right) + \mathrm{s}^2\left(g_{m+1}, r_m\right)\right) \\ + 2\beta\langle\vartheta_m B r_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle.$$

This implies that

$$-2\langle\vartheta_m B r_m, \mathrm{E}_{r_m}^{-1} g_{m+1}\rangle \leq -\left(1/\beta - \nu\vartheta_m/\vartheta_{m+1}\right) \left(\mathrm{s}^2\left(g_m, r_m\right) + \mathrm{s}^2\left(g_{m+1}, r_m\right)\right) \\ + \beta^{-1}\mathrm{s}^2(g_m, g_{m+1}) \quad (\forall\beta > 0). \tag{3.18}$$

Combining (3.14) and (3.18), one has

$$\mathrm{s}^2\left(g_{m+1}, p\right) \leq \mathrm{s}^2\left(g_m, p\right) - \left(1/\beta - \nu\vartheta_m/\vartheta_{m+1}\right) \left(\mathrm{s}^2\left(g_m, r_m\right) + \mathrm{s}^2\left(g_{m+1}, r_m\right)\right) \\ - \left(1 - \beta^{-1}\right)\mathrm{s}^2(g_m, g_{m+1}) \quad (\forall\beta > 0). \tag{3.19}$$

By using (3.13) and (3.19), we conclude that

$$\mathrm{s}^2\left(g_{m+1}, p\right) \leq \mathrm{s}^2\left(g_m, p\right) - \beta^{\dagger}\left(\mathrm{s}^2\left(g_m, r_m\right) + \mathrm{s}^2\left(r_m, g_{m+1}\right)\right), \tag{3.20}$$

where

$$\beta^{\dagger} := \begin{cases} 2 - 1/\beta - \nu\vartheta_m/\vartheta_{m+1}, & \text{if } \beta \in (1/(2-\nu), 1); \\ 1/\beta - \nu\vartheta_m/\vartheta_{m+1}, & \text{if } \beta \in [1, 1/\nu). \end{cases}$$

Then $\lim_{m\to\infty}\beta^{\dagger} > 0$ for any $\beta \in (1/(2-\nu), 1/\nu)$ by means of Lemma 3.1. In other words, there exists a positive integer N_1 such that $\beta^{\dagger} > 0$ for all $m \geq N_1$ and $\beta^{\dagger}$ has a positive bound from below. This together with (3.20) yields that

$$\mathrm{s}\left(g_{m+1}, p\right) \leq \mathrm{s}\left(g_m, p\right) \quad (\forall m \geq N_1),$$

which implies that $\{g_m\}$ is Fejér monotone with respect to $\Gamma(B, C)$ and thus $\{g_m\}$ is bounded. By letting $m \to \infty$ in (3.20), one obtains $\lim_{m\to\infty}\mathrm{s}\left(g_m, r_m\right) = 0$ and $\lim_{m\to\infty}\mathrm{s}\left(r_m, g_{m+1}\right) = 0$. Thus $\{r_m\}$ is also bounded. The rest of the argument proceeds identically to Theorem 3.1. □

The projection and contraction algorithm [7] is an effective method for solving variational inequalities in Euclidean spaces. It is important to note that this algorithm only requires a single projection onto the feasible set at each iteration, which results in a computational complexity similar to that of the subgradient extragradient algorithm. Numerical experiments by Cai et al. [43] showed that the projection and contraction algorithm runs twice as fast as the extragradient algorithm [5]. In the following, we introduce two adaptive modified projection and contraction algorithms to solve VIP (1.1) with a pseudo-monotone and Lipschitz continuous vector field on Hadamard manifolds. The first approach is outlined in Algorithm 3.3.

Algorithm 3.3 The first type of modified projection and contraction algorithm

Initialization: Take $\vartheta_0 > 0$, $\nu \in (0, 1)$, $\sigma \in (0, 2/\nu)$, and $\beta \in (\sigma/2, 1/\nu)$. Let $\{\varpi_m\}$ and $\{\mu_m\}$ satisfy Condition (C3). Let $g_0 \in \mathcal{X}$ and set $m = 0$.

Step 1. Compute

$$r_m = \mathbf{Pj}_C\left(\mathrm{E}_{g_m}\left(-\beta\vartheta_m B g_m\right)\right).$$

If $g_m = r_m$, then stop the iterative process and $g_m \in \Gamma(B, C)$; otherwise, go to *Step 2*.

Step 2. Compute

$$g_{m+1} = \mathbf{Pj}_{H_m}\left(\mathrm{E}_{g_m}\left(\mathrm{PT}_{g_m,r_m}\left(-\sigma\zeta_m\vartheta_m B r_m\right)\right)\right),$$

where

$$\zeta_m := \frac{\langle \mathrm{E}_{r_m}^{-1} g_m, \alpha_m\rangle}{\|\alpha_m\|^2}, \quad \alpha_m := \mathrm{E}_{r_m}^{-1} g_m + \beta\vartheta_m\left(B r_m - \mathrm{PT}_{r_m,g_m} B g_m\right), \tag{3.21}$$

and

$$H_m := \left\{x \in \mathcal{X} : \langle \mathrm{E}_{r_m}^{-1} g_m - \beta\vartheta_m \mathrm{PT}_{r_m,g_m} B g_m, \mathrm{E}_{r_m}^{-1} x\rangle \le 0\right\},$$

and update the next step size ϑ_{m+1} by (3.1).

Set $m := m + 1$ and go to *Step 1*.

Before starting the analysis of the convergence of Algorithm 3.3, we show that $\{\zeta_m\}$ in (3.21) is well defined.

Lemma 3.3 *Let $\{\zeta_m\}$ be generated by (3.21). Then $\alpha_m = 0$ if and only if $g_m = r_m$.*

Proof From the definition of α_m and (3.1), one has

$$\begin{aligned} \|\alpha_m\| &= \left\|\mathrm{E}_{r_m}^{-1} g_m + \beta\vartheta_m\left(B r_m - \mathrm{PT}_{r_m,g_m} B g_m\right)\right\| \\ &\ge \left\|\mathrm{E}_{r_m}^{-1} g_m\right\| - \beta\vartheta_m\left\|B r_m - \mathrm{PT}_{r_m,g_m} B g_m\right\| \\ &\ge \left(1 - \beta\nu\vartheta_m/\vartheta_{m+1}\right)\mathrm{s}\left(g_m, r_m\right). \end{aligned} \tag{3.22}$$

Similarly, we have

$$\left\|\alpha_m\right\| \le \left\|\mathrm{E}_{r_m}^{-1} g_m\right\| + \beta\vartheta_m\left\|B r_m - \mathrm{PT}_{r_m,g_m} B g_m\right\| \le \left(1 + \beta\nu\vartheta_m/\vartheta_{m+1}\right)\mathrm{s}\left(g_m, r_m\right). \tag{3.23}$$

Combining (3.22) and (3.23), we obtain

$$\left(1 - \beta\nu\vartheta_m/\vartheta_{m+1}\right)\mathrm{s}\left(g_m, r_m\right) \le \|\alpha_m\| \le \left(1 + \beta\nu\vartheta_m/\vartheta_{m+1}\right)\mathrm{s}\left(g_m, r_m\right).$$

By using Lemma 3.1, one obtains that $\lim_{m\to\infty} \vartheta_m$ exists. Due to $\beta < 1/\nu$, one sees that

$$\lim_{m\to\infty} (1 - \beta\nu\vartheta_m/\vartheta_{m+1}) > 0.$$

Therefore, we conclude that $\alpha_m = 0$ if and only if $g_m = r_m$. According to Remark 3.1, we know that the iteration of Algorithm 3.3 stops when $g_m = r_m$. That is, the *Step 2* in Algorithm 3.3 is not performed if $g_m = r_m$, so ζ_m is well defined. □

Theorem 3.3 *Let* $\{g_m\}$ *be formed by Algorithm 3.3 and let Conditions* (C1)–(C3) *hold. Then* $\{g_m\}$ *converges to a solution of VIP* (1.1).

Proof By means of Lemma 3.1 and $\beta < 1/\nu$, one has $1 - \beta\nu\vartheta_m/\vartheta_{m+1} > 0$ for all $m \geq m_0$. It should be noted that $\zeta_m > 0$ for all $m \geq m_0$. Indeed, in view of the definition of α_m and (3.1), we deduce that

$$\begin{aligned}
\zeta_m \|\alpha_m\|^2 &= \langle \mathrm{E}_{r_m}^{-1} g_m, \alpha_m\rangle \\
&= \langle \mathrm{E}_{r_m}^{-1} g_m, \mathrm{E}_{r_m}^{-1} g_m\rangle - \langle \mathrm{E}_{r_m}^{-1} g_m, \beta\vartheta_m \left(\mathrm{PT}_{r_m,g_m} B g_m - B r_m\right)\rangle \\
&= \mathrm{s}^2 (g_m, r_m) - \beta\vartheta_m \langle \mathrm{PT}_{r_m,g_m} B g_m - B r_m, \mathrm{E}_{r_m}^{-1} g_m\rangle \\
&\geq \mathrm{s}^2 (g_m, r_m) - \beta\vartheta_m \left\|\mathrm{PT}_{r_m,g_m} B g_m - B r_m\right\| \cdot \left\|\mathrm{E}_{r_m}^{-1} g_m\right\| \\
&\geq \mathrm{s}^2 (g_m, r_m) - \beta\nu\vartheta_m/\vartheta_{m+1} \mathrm{s}^2 (g_m, r_m) \\
&= (1 - \beta\nu\vartheta_m/\vartheta_{m+1})\, \mathrm{s}^2 (g_m, r_m).
\end{aligned} \tag{3.24}$$

By using the definitions of ζ_m and α_m, (3.23), and (3.24), one has

$$\begin{aligned}
\zeta_m = \frac{\langle \mathrm{E}_{r_m}^{-1} g_m, \alpha_m\rangle}{\|\alpha_m\|^2} &\geq \frac{(1 - \beta\nu\vartheta_m/\vartheta_{m+1})\, \mathrm{s}^2 (g_m, r_m)}{\|\alpha_m\|^2} \\
&\geq \frac{(1 - \beta\nu\vartheta_m/\vartheta_{m+1})}{(1 + \beta\nu\vartheta_m/\vartheta_{m+1})^2} > 0, \quad (\forall m \geq m_0).
\end{aligned}$$

Let $t_m := \mathrm{E}_{g_m}\left(\mathrm{PT}_{g_m,r_m}(-\sigma\zeta_m\vartheta_m B r_m)\right)$. Consider $\Delta(t_m, r_m, g_m)$ and its comparison triangle $\Delta\left(t'_m, r'_m, g'_m\right)$. The comparison point of t_m is $t'_m = g'_m - \sigma\zeta_m\vartheta_m B r'_m$. Similarly to the derivation (3.3), one gives

$$\mathrm{s}^2 (g_{m+1}, p) \leq \mathrm{s}^2 (g_m, p) - \mathrm{s}^2 (g_m, g_{m+1}) + 2\langle g'_{m+1} - p', t'_m - g'_m\rangle. \tag{3.25}$$

Let $a = \mathrm{E}_{g_m}\mathrm{PT}_{g_m,r_m}\alpha_m$. Consider $\Delta(r_m, g_m, a)$ and its comparison triangle $\Delta\left(r'_m, g'_m, a'\right)$. The comparison point of a is $a' = 2g'_m - r'_m + \beta\vartheta_m\left(B r'_m - B g'_m\right)$. Let $b = \mathrm{E}_{g_{m+1}}\mathrm{PT}_{g_{m+1},r_m}\alpha_m$. Consider a pair of $\Delta(r_m, g_m, b)$ and $\Delta\left(r'_m, g'_m, b'\right)$. The comparison point of b is $b' = g'_{m+1} + g'_m - r'_m + \beta\vartheta_m\left(B r'_m - B g'_m\right)$. Note that

$$\langle g'_{m+1} - p', t'_m - g'_m\rangle = \langle g'_{m+1} - r'_m, t'_m - g'_m\rangle + \langle r'_m - p', t'_m - g'_m\rangle, \tag{3.26}$$

and

$$
\begin{aligned}
&\langle g'_{m+1} - r'_m, t'_m - g'_m \rangle = \langle g'_{m+1} - r'_m, -\sigma \zeta_m \vartheta_m B r'_m \rangle \\
&= \sigma \zeta_m \beta^{-1} \big\langle g'_{m+1} - r'_m, -\big(g'_m - r'_m + \beta \vartheta_m \left(B r'_m - B g'_m\right)\big) \\
&\qquad\qquad + \left(g'_m - r'_m - \beta \vartheta_m B g'_m\right) \big\rangle \\
&= \sigma \zeta_m \beta^{-1} \langle r'_m - g'_{m+1}, a' - g'_m \rangle + \sigma \zeta_m \beta^{-1} \langle g'_{m+1} - r'_m, g'_m - r'_m - \beta \vartheta_m B g'_m \rangle \\
&= \sigma \zeta_m \beta^{-1} \left(\langle r'_m - g'_m, a' - g'_m \rangle + \langle g'_m - g'_{m+1}, b' - g'_{m+1} \rangle \right) \\
&\quad + \sigma \zeta_m \beta^{-1} \langle g'_{m+1} - r'_m, g'_m - r'_m - \beta \vartheta_m B g'_m \rangle.
\end{aligned}
$$

Therefore, by Lemma 2.3 we have

$$
\langle a' - g'_m, r'_m - g'_m \rangle \leq \langle \mathrm{PT}_{g_m, r_m} \alpha_m, \mathrm{E}^{-1}_{g_m} r_m \rangle.
$$

Similarly, we obtain

$$
\langle g'_m - g'_{m+1}, b' - g'_{m+1} \rangle \leq \langle \mathrm{PT}_{g_{m+1}, r_m} \alpha_m, \mathrm{E}^{-1}_{g_{m+1}} g_m \rangle,
$$

$$
\langle g'_{m+1} - r'_m, g'_m - r'_m - \beta \vartheta_m B g'_m \rangle \leq \langle \mathrm{E}^{-1}_{r_m} g_m - \beta \vartheta_m \mathrm{PT}_{r_m, g_m} B g_m, \mathrm{E}^{-1}_{r_m} g_{m+1} \rangle,
$$

and

$$
\langle r'_m - p', t'_m - g'_m \rangle \leq \langle \mathrm{PT}_{g_m, p} \mathrm{E}^{-1}_p r_m, \mathrm{E}^{-1}_{g_m} t_m \rangle.
$$

It follows from $g_{m+1} \in H_m$ that

$$
\langle \mathrm{E}^{-1}_{r_m} g_m - \beta \vartheta_m \mathrm{PT}_{r_m, g_m} B g_m, \mathrm{E}^{-1}_{r_m} g_{m+1} \rangle \leq 0. \tag{3.27}
$$

From (3.4), (3.26), and (3.27), by using Proposition 2.1, and the definition of ζ_m, we deduce that

$$
\begin{aligned}
\langle g'_{m+1} - p', t'_m - g'_m \rangle &\leq \sigma \zeta_m \beta^{-1} \left(\langle \mathrm{PT}_{g_m, r_m} \alpha_m, \mathrm{E}^{-1}_{g_m} r_m \rangle + \langle \mathrm{PT}_{g_{m+1}, r_m} \alpha_m, \mathrm{E}^{-1}_{g_{m+1}} g_m \rangle \right) \\
&\quad + \sigma \zeta_m \beta^{-1} \langle \mathrm{E}^{-1}_{r_m} g_m - \beta \vartheta_m \mathrm{PT}_{r_m, g_m} B g_m, \mathrm{E}^{-1}_{r_m} g_{m+1} \rangle \\
&\quad + \langle \mathrm{PT}_{g_m, p} \mathrm{E}^{-1}_p r_m, \mathrm{E}^{-1}_{g_m} t_m \rangle \\
&\leq \sigma \zeta_m \beta^{-1} \left(\langle \alpha_m, \mathrm{PT}_{r_m, g_m} \mathrm{E}^{-1}_{g_m} r_m \rangle + \langle \alpha_m, \mathrm{PT}_{r_m, g_{m+1}} \mathrm{E}^{-1}_{g_{m+1}} g_m \rangle \right) \\
&\quad + \langle \mathrm{PT}_{r_m, p} \mathrm{E}^{-1}_p r_m, \mathrm{PT}_{r_m, g_m} \mathrm{E}^{-1}_{g_m} t_m \rangle \\
&\leq \sigma \zeta_m \beta^{-1} \left(\langle \alpha_m, \mathrm{PT}_{r_m, g_m} \mathrm{E}^{-1}_{g_m} r_m \rangle + \langle \alpha_m, \mathrm{PT}_{r_m, g_{m+1}} \mathrm{E}^{-1}_{g_{m+1}} g_m \rangle \right) \\
&\quad + \sigma \zeta_m \vartheta_m \langle \mathrm{E}^{-1}_{r_m} p, B r_m \rangle \\
&\leq \sigma \zeta_m \beta^{-1} \left(\langle \alpha_m, -\mathrm{E}^{-1}_{r_m} g_m \rangle + \|\alpha_m\| \cdot \|\mathrm{E}^{-1}_{g_{m+1}} g_m\| \right) \\
&\leq -\sigma \zeta_m^2 \beta^{-1} \|\alpha_m\|^2 + 2^{-1} \sigma^2 \zeta_m^2 \beta^{-2} \|\alpha_m\|^2 + 2^{-1} \mathrm{s}^2 \left(g_m, g_{m+1}\right).
\end{aligned} \tag{3.28}
$$

Combining (3.25) and (3.28), one obtains

$$
\mathrm{s}^2 \left(g_{m+1}, p\right) \leq \mathrm{s}^2 \left(g_m, p\right) - \sigma \zeta_m^2 (2\beta - \sigma) \beta^{-2} \|\alpha_m\|^2. \tag{3.29}
$$

By using (3.23) and (3.24), one arrives at

$$\zeta_m^2\|\alpha_m\|^2=\frac{\zeta_m^2\|\alpha_m\|^4}{\|\alpha_m\|^2}\geq\frac{(1-\beta\nu\vartheta_m/\vartheta_{m+1})^2}{(1+\beta\nu\vartheta_m/\vartheta_{m+1})^2}\mathrm{s}^2\left(g_m,r_m\right). \tag{3.30}$$

Combining (3.25), (3.29), and (3.30), we have

$$\mathrm{s}^2\left(g_{m+1},p\right)\leq\mathrm{s}^2\left(g_m,p\right)-\frac{\sigma}{\beta^2}(2\beta-\sigma)\frac{(1-\beta\nu\vartheta_m/\vartheta_{m+1})^2}{(1+\beta\nu\vartheta_m/\vartheta_{m+1})^2}\mathrm{s}^2\left(g_m,r_m\right)\quad(\forall p\in\Gamma(B,C)). \tag{3.31}$$

Since $\sigma\in(0,2/\nu)$ and $\beta\in(\sigma/2,1/\nu)$, one sees that $\sigma\beta^{-2}(2\beta-\sigma)>0$, which combining with (3.31) implies that

$$\mathrm{s}^2\left(g_{m+1},p\right)\leq\mathrm{s}^2\left(g_m,p\right)\quad(\forall m\geq m_0).$$

This means that $\{g_m\}$ is Fejér monotone with respect to $\Gamma(B,C)$ and thus $\{g_m\}$ is bounded. By letting $m\to\infty$ in (3.31), one concludes that $\lim_{m\to\infty}\mathrm{s}\left(g_m,r_m\right)=0$. Hence $\{r_m\}$ is also bounded. The continuation of the proof is analogous to that of Theorem 3.1. □

It is important to note that the proposed Algorithms 3.1–3.3 involve computing projections onto the set C and the half-space H_m at each iteration. Next, we present two implicit iterative algorithms to address VIP (1.1). Now, we propose another type of projection and contraction algorithm, which requires solving a convex optimization problem at each iteration and does not involve projection onto a half-space. This approach is displayed in Algorithm 3.4 below.

Algorithm 3.4 The second type of modified projection and contraction algorithm

Initialization: Take $\vartheta_0>0$, $\nu\in(0,1)$, $\sigma\in(0,2)$, and $\beta\in(0,1/\nu)$. Let $\{\varpi_m\}$ and $\{\mu_m\}$ satisfy Condition (C3). Let $g_0\in\mathcal{X}$ and set $m=0$.

Step 1. Compute $r_m\in C$ such that

$$\langle\mathrm{E}_{r_m}^{-1}g_m-\beta\vartheta_m\mathrm{PT}_{r_m,g_m}Bg_m,\ \mathrm{E}_{r_m}^{-1}y\rangle\leq0,\quad\forall y\in C.$$

If $g_m=r_m$, then stop the iterative process and $g_m\in\Gamma(B,C)$; otherwise, go to *Step 2*.

Step 2. Compute

$$g_{m+1}=\mathrm{E}_{g_m}\left(\mathrm{PT}_{g_m,r_m}\left(-\sigma\zeta_m\alpha_m\right)\right),$$

where ζ_m and α_m are defined in (3.21). Update ϑ_{m+1} by (3.1).

Set $m:=m+1$ and go to *Step 1*.

Theorem 3.4 *Let $\{g_m\}$ be created by Algorithm 3.4 and Conditions* (C1)–(C3) *hold. Then $\{g_m\}$ converges to a solution of VIP* (1.1).

Proof Fix $p \in \Gamma(B, C)$. Consider $\Delta(p, g_{m+1}, g_m)$ and its comparison triangle $\Delta\left(p', g'_{m+1}, g'_m\right)$. By using Lemma 2.1, one arrives at

$$\mathrm{s}(p, g_m) = \left\|p' - g'_m\right\|, \quad \mathrm{s}(g_{m+1}, g_m) = \left\|g'_{m+1} - g'_m\right\|, \quad \mathrm{s}(p, g_{m+1}) = \left\|p' - g'_{m+1}\right\|.$$

The comparison point for g_{m+1} is $g'_{m+1} = g'_m - \sigma\zeta_m\left(g'_m - r'_m + \beta\vartheta_m(Br'_m - Bg'_m)\right)$. For convenience, we let $\chi'_m = g'_m - r'_m + \beta\vartheta_m(Br'_m - Bg'_m)$. From the definitions of g_{m+1} and g'_{m+1}, we have

$$\sigma\zeta_m\left\|\alpha_m\right\| = \mathrm{s}(g_{m+1}, g_m) = \left\|g'_{m+1} - g'_m\right\| = \sigma\zeta_m\left\|\chi'_m\right\|.$$

By using the definition of g_{m+1}, one sees that

$$\begin{aligned} \mathrm{s}^2(g_{m+1}, p) &= \|g'_{m+1} - p'\|^2 = \|g'_m - \sigma\zeta_m\chi'_m - p'\|^2 \\ &= \|g'_m - p'\|^2 - 2\sigma\zeta_m\langle g'_m - p', \chi'_m\rangle + \sigma^2\zeta_m^2\|\chi'_m\|^2 \\ &= \mathrm{s}^2(g_m, p) + 2\sigma\zeta_m\langle p' - g'_m, \chi'_m\rangle + \sigma^2\zeta_m^2\|\alpha_m\|^2. \end{aligned} \tag{3.32}$$

Let $a := \mathrm{E}_{r_m}\alpha_m$ and $b := \mathrm{E}_{g_m}\mathrm{PT}_{g_m,r_m}\alpha_m$. The comparison points of a and b are $a' = g'_m + \beta\vartheta_m\left(Br'_m - Bg'_m\right)$ and $b' = 2g'_m - r'_m + \beta\vartheta_m\left(Br'_m - Bg'_m\right)$, respectively. It follows from the definition of χ'_m and Lemma 2.3 that

$$\begin{aligned} \langle p' - g'_m, \chi'_m\rangle &= \langle r'_m - g'_m, \chi'_m\rangle + \langle p' - r'_m, \chi'_m\rangle \\ &= \langle r'_m - g'_m, b' - g'_m\rangle + \langle p' - r'_m, a' - r'_m\rangle \\ &\le \langle \mathrm{E}_{g_m}^{-1}r_m, \mathrm{E}_{g_m}^{-1}b\rangle + \langle \mathrm{E}_{r_m}^{-1}p, \mathrm{E}_{r_m}^{-1}a\rangle \\ &= \langle \mathrm{E}_{g_m}^{-1}r_m, \mathrm{PT}_{g_m,r_m}\alpha_m\rangle + \langle \mathrm{E}_{r_m}^{-1}p, \alpha_m\rangle. \end{aligned} \tag{3.33}$$

From the definition of α_m and Proposition 2.1, one obtains

$$\langle \mathrm{E}_{g_m}^{-1}r_m, \mathrm{PT}_{g_m,r_m}\alpha_m\rangle = -\langle \mathrm{E}_{r_m}^{-1}g_m, \alpha_m\rangle = -\zeta_m\left\|\alpha_m\right\|^2. \tag{3.34}$$

By using the definition of r_m and $p \in C$, one sees that

$$\langle \mathrm{E}_{r_m}^{-1}g_m - \beta\vartheta_m\mathrm{PT}_{r_m,g_m}Bg_m, \mathrm{E}_{r_m}^{-1}p\rangle \le 0. \tag{3.35}$$

Owing to $p \in \Gamma(B, C)$ and $r_m \in C$, one gives that $\langle Bp, \mathrm{E}_p^{-1}r_m\rangle \ge 0$. It follows from the pseudomonotonicity of B that $\langle Br_m, \mathrm{E}_{r_m}^{-1}p\rangle \le 0$. This combining with (3.35) yields that

$$\langle \mathrm{E}_{r_m}^{-1}g_m + \beta\vartheta_m\left(Br_m - \mathrm{PT}_{r_m,g_m}Bg_m\right), \mathrm{E}_{r_m}^{-1}p\rangle \le 0. \tag{3.36}$$

Combining (3.33), (3.34), and (3.36), we have

$$\langle p' - g'_m, \chi'_m\rangle \le -\zeta_m\left\|\alpha_m\right\|^2.$$

This together with (3.30) and (3.32) yields

$$
\begin{aligned}
\mathrm{s}^2(g_{m+1}, p) &\le \mathrm{s}^2(g_m, p) - 2\sigma\zeta_m^2\|\alpha_m\|^2 + \sigma^2\zeta_m^2\|\alpha_m\|^2 \\
&\le \mathrm{s}^2(g_m, p) - \sigma(2-\sigma)\frac{(1-\beta\nu\vartheta_m/\vartheta_{m+1})^2}{(1+\beta\nu\vartheta_m/\vartheta_{m+1})^2}\mathrm{s}^2(g_m, r_m) \quad (\forall m \ge m_0).
\end{aligned}
\tag{3.37}
$$

By using $\sigma \in (0, 2)$ and (3.37), one has

$$
\mathrm{s}(g_{m+1}, p) \le \mathrm{s}(g_m, p) \quad (\forall m \ge m_0),
$$

which implies that $\{g_m\}$ is Fejér monotone with respect to $\Gamma(B, C)$. Thus $\{g_m\}$ is bounded by means of Lemma 2.6. By setting $m \to \infty$ in (3.37), we can show that $\lim_{m\to\infty}\mathrm{s}(g_m, r_m) = 0$. Thus $\{r_m\}$ is bounded. By means of Lemma 2.6, it is left to prove that any cluster point of $\{g_m\}$ belongs to $\Gamma(B, C)$. Let u^* be a cluster point of $\{g_m\}$. From the fact that $\{g_m\}$ is bounded, there exists a subsequence $\left\{g_{m_j}\right\}$ of $\{g_m\}$ satisfies $\lim_{j\to\infty} g_{m_j} = u^*$. We also have $\lim_{j\to\infty} r_{m_j} = u^*$ and $u^* \in C$ due to $\lim_{m\to\infty}\mathrm{s}(g_m, r_m) = 0$. By using the definition of r_m, one has

$$
\left\langle \mathrm{E}_{r_{m_j}}^{-1} g_{m_j}, \mathrm{E}_{r_{m_j}}^{-1} x\right\rangle \le \beta\vartheta_{m_j}\left\langle \mathrm{PT}_{r_{m_j}, g_{m_j}} B g_{m_j}, \mathrm{E}_{r_{m_j}}^{-1} x\right\rangle \ (\forall x \in C).
$$

Note that $\lim_{j\to\infty}\vartheta_{m_j} = \vartheta > 0$ and $\beta > 0$. Letting $j \to \infty$ in the above inequality and applying Propositions 2.2 and 2.3, we obtain

$$
\langle Bu^*, \mathrm{E}_{u^*}^{-1} x\rangle \ge 0 \ \ (\forall x \in C).
$$

This implies that $u^* \in \Gamma(B, C)$, as required. □

Finally, we introduce an implicit modified Tseng's extragradient method, as shown in Algorithm 3.5.

Algorithm 3.5 The modified Tseng's extragradient algorithm

Initialization: Take $\vartheta_0 > 0$ and $\nu \in (0, 1)$. Let $\{\varpi_m\}$ and $\{\mu_m\}$ satisfy Condition (C3). Let $g_0 \in \mathcal{X}$ and set $m = 0$.

Step 1. Compute $r_m \in C$ such that

$$
\left\langle \mathrm{E}_{r_m}^{-1} g_m - \vartheta_m \mathrm{PT}_{r_m, g_m} B g_m, \mathrm{E}_{r_m}^{-1} y\right\rangle \le 0, \quad \forall y \in C.
$$

If $g_m = r_m$, then stop the iterative process and $g_m \in \Gamma(B, C)$; otherwise, go to *Step 2*.

Step 2. Compute

$$
g_{m+1} = \mathrm{E}_{r_m}\left(\vartheta_m\left(\mathrm{PT}_{r_m, g_m} B g_m - B r_m\right)\right),
$$

and update ϑ_{m+1} by (3.1).

Set $m := m + 1$ and go to *Step 1*.

Theorem 3.5 *Let $\{g_m\}$ be formed by Algorithm 3.5 and Conditions* (C1)–(C3) *hold. Then $\{g_m\}$ converges to a solution of VIP* (1.1).

Proof Fix $p \in \Gamma(B, C)$. Consider $\Delta(g_m, r_m, p)$ and its comparison triangle $\Delta\left(g'_m, r'_m, p'\right)$. By 2.1, one has

$$\mathrm{s}(g_m, p) = \left\|g'_m - p'\right\|, \quad \mathrm{s}(r_m, p) = \left\|r'_m - p'\right\|, \quad \mathrm{s}(g_m, r_m) = \left\|g'_m - r'_m\right\|.$$

Similarly, consider a pair of $\Delta(g_{m+1}, r_m, p)$ and $\Delta\left(g'_{m+1}, r'_m, p'\right)$. By using Lemma 2.1, one has

$$\mathrm{s}(g_{m+1}, p) = \left\|g'_{m+1} - p'\right\|, \quad \mathrm{s}(r_m, p) = \left\|r'_m - p'\right\|, \quad \mathrm{s}(g_{m+1}, r_m) = \left\|g'_{m+1} - r'_m\right\|.$$

From the definition of g_{m+1} in Algorithm 3.5, one obtains that the comparison point of g_{m+1} is $g'_{m+1} = r'_m + \vartheta_m\left(Bg'_m - Br'_m\right)$. Moreover,

$$\left\|\mathrm{E}_{r_m}^{-1} g_{m+1}\right\| = \left\|\mathrm{E}_{r_m}^{-1}\mathrm{E}_{r_m}\left(\vartheta_m\left(\mathrm{PT}_{r_m,g_m} Bg_m - Br_m\right)\right)\right\|.$$

Thus $\vartheta_m\left\|\mathrm{PT}_{r_m,g_m} Bg_m - Br_m\right\| = \mathrm{s}(g_{m+1}, r_m) = \|g'_{m+1} - r'_m\| = \vartheta_m\left\|Br'_m - Bg'_m\right\|$. This together with Lemma 2.3 yields

$$\begin{aligned}
\mathrm{s}^2(g_{m+1}, p) = \|g'_{m+1} - p'\|^2 &= \left\|r'_m + \vartheta_m\left(Bg'_m - Br'_m\right) - p'\right\|^2 \\
&= \left\|r'_m - p'\right\|^2 + \vartheta_m^2\left\|Bg'_m - Br'_m\right\|^2 \\
&\quad + 2\vartheta_m\langle Br'_m - Bg'_m, p' - r'_m\rangle \\
&\leq \left\|r'_m - p'\right\|^2 + \vartheta_m^2\left\|\mathrm{PT}_{r_m,g_m} Bg_m - Br_m\right\|^2 \\
&\quad + 2\vartheta_m\big\langle Br_m - \mathrm{PT}_{r_m,g_m} Bg_m, \mathrm{E}_{r_m}^{-1} p\big\rangle.
\end{aligned} \tag{3.38}$$

By using Lemma 2.3 again, one has

$$\begin{aligned}
\left\|r'_m - p'\right\|^2 &= \left\|r'_m - g'_m\right\|^2 + \left\|g'_m - p'\right\|^2 + 2\langle r'_m - g'_m, g'_m - p'\rangle \\
&= \left\|g'_m - p'\right\|^2 + \left\|r'_m - g'_m\right\|^2 - 2\langle r'_m - g'_m, r'_m - g'_m\rangle \\
&\quad + 2\langle r'_m - g'_m, r'_m - p'\rangle \\
&= \left\|g'_m - p'\right\|^2 - \left\|r'_m - g'_m\right\|^2 + 2\langle g'_m - r'_m, p' - r'_m\rangle \\
&\leq \mathrm{s}^2(g_m, p) - \mathrm{s}^2(g_m, r_m) + 2\langle \mathrm{E}_{r_m}^{-1} g_m, \mathrm{E}_{r_m}^{-1} p\rangle.
\end{aligned} \tag{3.39}$$

By using the definition of r_m and $p \in C$, one sees that

$$\langle \mathrm{E}_{r_m}^{-1} g_m - \vartheta_m \mathrm{PT}_{r_m,g_m} Bg_m, \mathrm{E}_{r_m}^{-1} p\rangle \leq 0. \tag{3.40}$$

From $p \in \Gamma(B, C)$ and $r_m \in C$, one arrives at $\left\langle Bp, \mathrm{E}_p^{-1} r_m \right\rangle \geq 0$. It follows from the pseudomonotonicity of B that

$$\langle Br_m, \mathrm{E}_{r_m}^{-1} p \rangle \leq 0. \tag{3.41}$$

Combining (3.1), (3.38), (3.39), (3.40), and (3.41), we have

$$\begin{aligned} \mathrm{s}^2\left(g_{m+1}, p\right) &\leq \mathrm{s}^2\left(g_m, p\right) - \mathrm{s}^2\left(g_m, r_m\right) + \vartheta_m^2 \left\| \mathrm{PT}_{r_m, g_m} B g_m - B r_m \right\|^2 \\ &\leq \mathrm{s}^2\left(g_m, p\right) - \left(1 - \nu^2 \vartheta_m^2 / \vartheta_{m+1}^2\right) \mathrm{s}^2\left(g_m, r_m\right). \end{aligned} \tag{3.42}$$

From Lemma 3.1, it follows that $1 - \nu^2 \vartheta_m^2 / \vartheta_{m+1}^2 > 0$ for all $m \geq m_0$. The rest of the proof is the same as Theorem 3.4. $\square$

Remark 3.2 We have the following observations for the proposed algorithms.

(i) If Condition (C2) in Section 3 adds the restriction that C is a bounded set, one knows by Lemma 2.5 that VIP (1.1) always has a unique solution. In this case, Condition (C1) is no longer required for the convergence analysis of our algorithms.

(ii) The Algorithms 3.1-3.5 proposed in this paper employ a new non-monotonic step size criterion. The advantage of our algorithms is that they can perform simple calculations to update the step size using some previously known information without involving the computation of projections, which greatly improves their convergence speed. Notice that our step size criterion (3.1) generates a non-monotonic sequence of step sizes, which is better in practice than the Armijo-type step size applied in [15, 17, 18, 28].

(iii) It should be noted that the Algorithms 3.1–3.5 are all explicit, which makes them easier to implement than the implicit proximal point methods in [16, 23, 26]. On the other hand, the convergence analysis of our algorithms only requires that the vector fields involved are pseudo-monotone, which is weaker than the monotonicity imposed by the approaches in [14, 18] and the strongly pseudo-monotonicity required by the method in [15].

(iv) The proposed Algorithms 3.1–3.5 insert a new parameter β making them use different step sizes when computing r_m and g_{m+1} in each iteration. This new technique improves the range of parameters of the corresponding original algorithms and these algorithms have a better performance when the appropriate parameter β is chosen (see the numerical results in Section 4). Note that if $\beta = 1$ in Algorithms 3.1 and 3.2, then Algorithms 3.1 and 3.2 are equivalent.

In the Euclidean space $\mathbb{R}^n$, the Hadamard manifold $\mathcal{X}$ reduces to $\mathbb{R}^n$, the exponential map $\mathrm{E}_u(v)$ becomes $\mathrm{E}_u(v) = u + v$, and the parallel transport $\mathrm{PT}_{u,v}$ reduces to the identity mapping. Under these simplifications, Algorithm 3.1 can be rewritten in the Euclidean setting as follows.

Algorithm 3.6 The Algorithm 3.1 in Euclidean space

Initialization: Take $\vartheta_0 > 0$, $\nu \in (0,1)$, and $\beta \in (0, 2/(1+\nu))$. Let $g_0 \in \mathbb{R}^n$ and set $m = 0$.
Step 1. Compute

$$r_m = \mathbf{Pj}_C\left(g_m - \vartheta_m B g_m\right).$$

If $g_m = r_m$, then stop the iterative process; otherwise, go to *Step 2*.
Step 2. Compute

$$g_{m+1} = \mathbf{Pj}_{H_m}\left(g_m - \beta\vartheta_m B r_m\right),$$

where

$$H_m := \left\{x \in \mathbb{R}^n : \langle g_m - r_m - \vartheta_m B g_m, x - r_m\rangle \le 0\right\},$$

and update ϑ_{m+1} by

$$\vartheta_{m+1} = \begin{cases} \min\left\{\dfrac{\nu\|g_m - r_m\|}{\|Bg_m - Br_m\|}, \varpi_m\vartheta_m + \mu_m\right\}, & \text{if } \|Bg_m - Br_m\| \neq 0; \\ \varpi_m\vartheta_m + \mu_m, & \text{otherwise.} \end{cases}$$

Set $m := m + 1$ and go to *Step 1*.

Theorem 3.6 *Let $\{g_m\}$ be the sequence generated by Algorithm 3.6 in the Euclidean space $\mathbb{R}^n$. Suppose that the following conditions hold:*

1. *The solution set of the variational inequality problem*

$$\text{Find } u^* \in C \text{ such that } \langle Bu^*, u - u^*\rangle \ge 0, \quad \forall u \in C$$

   is nonempty.
2. *The feasible set C is a nonempty, closed, and convex subset of $\mathbb{R}^n$.*
3. *The mapping $B : C \to \mathbb{R}^n$ is pseudo-monotone and L-Lipschitz continuous on C. The sequences $\{\varpi_m\} \subset [1,\infty)$ and $\{\mu_m\} \subset [0,\infty)$ satisfy*

$$\sum_{m=1}^{\infty}(\varpi_m - 1) < \infty, \quad \sum_{m=1}^{\infty}\mu_m < \infty.$$

Then the sequence $\{g_m\}$ converges to a solution of the variational inequality problem.

3.2 Error bound and linear convergence

In this section, we analyze the global error bounds and establish the R-linear convergence of the proposed algorithms for solving variational inequalities with strongly pseudo-monotone vector fields on Hadamard manifolds. Error bounds are essential for developing stopping criteria and assessing the convergence rate of algorithms. For further theoretical insights on error bounds for variational inequalities in Euclidean spaces and Hilbert spaces, see [1, Chapter 6], [45, Section 4], and [46, Section 3]. Recently, Nguyen et al. [44, Theorem 3.2] established the existence and uniqueness of solutions for the VIP (1.1) arising from strongly pseudo-monotone vector fields. To advance our analysis, we replace Condition (C3) from Section 3 with the following slightly stronger Condition (C3′).

(C3′) The vector field $B\colon C \to T\mathcal{X}$ is strongly pseudo-monotone with a modulus μ and L-Lipschitz continuous on C. Let $\{\varpi_m\} \subset [1, \infty)$ satisfies $\sum_{m=1}^{\infty}(\varpi_m - 1) < \infty$, and $\{\mu_m\} \subset [0, \infty)$ such that $\sum_{m=1}^{\infty} \mu_m < \infty$.

Next, we establish global error bounds for our algorithms embedding the adaptive step size criterion (3.1).

Theorem 3.7 *Let $\{g_m\}$ be generated by Algorithm 3.1 (or Algorithm 3.5). If Conditions* (C1)*,* (C2)*, and* (C3′) *hold, then*

$$\mathrm{s}(g_m, p) \le \left(1 + \frac{1 + \nu\vartheta_m/\vartheta_{m+1}}{\mu\vartheta_m}\right)\mathrm{s}(g_m, r_m),$$

where p is the unique solution of VIP (1.1)*. Moreover, $\{g_m\}$ converges R-linearly to p.*

Proof From [44, Theorem 3.2], it yields that VIP (1.1) has a unique solution. Set $\Gamma(B, C) := \{p\}$. From the definition of H_m in Algorithm 3.1 (or by the definition of r_m in Algorithm 3.5), one has

$$\langle \mathrm{E}_{r_m}^{-1} g_m, \mathrm{E}_{r_m}^{-1} p\rangle \le \vartheta_m \langle \mathrm{PT}_{r_m, g_m} Bg_m, \mathrm{E}_{r_m}^{-1} p\rangle. \tag{3.43}$$

Combining $p \in \Gamma(B, C)$ with $r_m \in C$, one yields $\langle Bp, \mathrm{E}_p^{-1} r_m\rangle \ge 0$, which together with the μ-strongly pseudomonotonicity of B implies that

$$\langle Br_m, \mathrm{E}_{r_m}^{-1} p\rangle \le -\mu \mathrm{s}^2(r_m, p). \tag{3.44}$$

From (3.1), (3.43), and (3.44), it follows that

$$\begin{aligned}
\langle \mathrm{E}_{r_m}^{-1} g_m, \mathrm{E}_{r_m}^{-1} p\rangle &\le \vartheta_m \langle \mathrm{PT}_{r_m, g_m} Bg_m - Br_m, \mathrm{E}_{r_m}^{-1} p\rangle + \vartheta_m \langle Br_m, \mathrm{E}_{r_m}^{-1} p\rangle \\
&\le \vartheta_m \left\| \mathrm{PT}_{r_m, g_m} Bg_m - Br_m \right\| \mathrm{s}(r_m, p) - \mu\vartheta_m \mathrm{s}^2(r_m, p) \\
&\le \nu\vartheta_m/\vartheta_{m+1} \mathrm{s}(g_m, r_m)\mathrm{s}(r_m, p) - \mu\vartheta_m \mathrm{s}^2(r_m, p),
\end{aligned}$$

which implies that

$$\begin{aligned}
\mu\vartheta_m \mathrm{s}^2(r_m, p) &\le \nu\vartheta_m/\vartheta_{m+1} \mathrm{s}(g_m, r_m)\mathrm{s}(r_m, p) + \langle \mathrm{E}_{r_m}^{-1} g_m, -\mathrm{E}_{r_m}^{-1} p\rangle \\
&\le \nu\vartheta_m/\vartheta_{m+1} \mathrm{s}(g_m, r_m)\mathrm{s}(r_m, p) + \mathrm{s}(g_m, r_m)\mathrm{s}(r_m, p).
\end{aligned}$$

Thus

$$\mathrm{s}(r_m, p) \le \frac{1 + \nu\vartheta_m/\vartheta_{m+1}}{\mu\vartheta_m} \mathrm{s}(g_m, r_m).$$

Let $\psi_m := (1 + \nu\vartheta_m/\vartheta_{m+1})/(\mu\vartheta_m)$. Then

$$\mathrm{s}(g_m, p) \le \mathrm{s}(g_m, r_m) + \mathrm{s}(r_m, p) \le (1 + \psi_m)\mathrm{s}(g_m, r_m). \tag{3.45}$$

This provides the required error bound for $\mathrm{s}(g_m, p)$. By using (3.45), one has

$$\mathrm{s}(g_m, r_m) \geq (1 + \psi_m)^{-1}\mathrm{s}(g_m, p). \tag{3.46}$$

Let us show the linear convergence of the sequences generated by Algorithm 3.1 and Algorithm 3.5, respectively.

Case 1: Consider $\{g_m\}$ generated by Algorithm 3.1. From Lemma 3.2 and (3.46), one obtains

$$\mathrm{s}^2(g_{m+1}, p) \leq \mathrm{s}^2(g_m, p) - \beta^*\mathrm{s}^2(g_m, r_m) \leq \left(1 - \beta^*(1+\psi_m)^{-2}\right)\mathrm{s}^2(g_m, p). \tag{3.47}$$

By using the definition of β^* in (3.2) and Lemma 3.1, one sees that $0 < \lim_{m\to\infty}\beta^* < 1$. Moreover, we have $0 < \lim_{m\to\infty}(1 + \psi_m)^{-2} < 1$. Thus

$$0 < \phi_0 := \lim_{m\to\infty}\left(\beta^*(1 + \psi_m)^{-2}\right) < 1.$$

Therefore, there exists a positive constant $N_1 \in \mathbb{N}$ such that $\beta^*(1 + \psi_m)^{-2} \geq \phi_0$ for all $m \geq N_1$. By letting $\phi_1 := (1 - \phi_0)^{1/2}$ and using (3.47), we have

$$\begin{aligned}\mathrm{s}(g_{m+1}, p) &\leq \phi_1\mathrm{s}(g_m, p) \leq \phi_1^2\mathrm{s}(g_{n-1}, p)\\ &\leq \cdots \leq \phi_1^{k+1-N_1}\mathrm{s}\left(g_{N_1}, p\right) = \frac{\mathrm{s}\left(g_{N_1}, p\right)}{\phi_1^{N_1-1}}\phi_1^n \quad (\forall m \geq N_1).\end{aligned}$$

Thus we deduce that $\{g_m\}$ converges R-linearly to p. □

Case 2: Consider $\{g_m\}$ generated by Algorithm 3.5. According to the inequalities (3.42) and (3.46), one has

$$\mathrm{s}^2(g_{m+1}, p) \leq \left(1 - (1 - \nu^2\vartheta_m^2/\vartheta_{m+1}^2)(1 + \psi_m)^{-2}\right)\mathrm{s}^2(g_m, p).$$

Let $\phi_2 := \lim_{m\to\infty}(1 - \nu^2\vartheta_m^2/\vartheta_{m+1}^2)(1 + \psi_m)^{-2}$. Note that $\phi_2 \in (0, 1)$. By using Lemma 3.1, there exists a positive constant $N_2 \in \mathbb{N}$ such that $\left(1 - \nu^2\vartheta_m^2/\vartheta_{m+1}^2\right)(1 + \psi_m)^{-2} \geq \phi_2$ for all $m \geq N_2$. Let $\phi_3 := (1 - \phi_2)^{1/2}$. Consequently,

$$\mathrm{s}(g_{m+1}, p) \leq \phi_3\mathrm{s}(g_m, p) \leq \cdots \leq \phi_3^{k+1-N_2}\mathrm{s}\left(g_{N_2}, p\right) = \frac{\mathrm{s}\left(g_{N_2}, p\right)}{\phi_3^{N_2-1}}\phi_3^n \quad (\forall m \geq N_2).$$

This implies that $\{g_m\}$ converges R-linearly to p. □

Similar to the proof of Theorem 3.7, we can easily arrive at the R-linear convergence of the proposed Algorithms 3.2–3.4.

Theorem 3.8 *Let $\{g_m\}$ be generated by Algorithm 3.2 (or Algorithms 3.3 and 3.4). If Conditions* (C1), (C2), *and* (C3′) *hold, then*

$$\mathrm{s}(g_m, p) \leq \left(1 + \frac{1 + \beta\nu\vartheta_m/\vartheta_{m+1}}{\beta\mu\vartheta_m}\right)\mathrm{s}(g_m, r_m),$$

where p is the unique solution of VIP (1.1). Furthermore, $\{g_m\}$ converges R-linearly to p.

Proof By means of [44, Theorem 3.2], one has that $\Gamma(B, C)$ is a singleton. Set $\Gamma(B, C) := \{p\}$. By using the definition of H_m in Algorithm 3.2 (or by the definition of H_m in Algorithm 3.3, or by the definition of r_m in Algorithm 3.4), one arrives at

$$\langle \mathrm{E}_{r_m}^{-1} g_m, \mathrm{E}_{r_m}^{-1} p\rangle \leq \beta\vartheta_m \langle \mathrm{PT}_{r_m, g_m} B g_m, \mathrm{E}_{r_m}^{-1} p\rangle.$$

According to a proof similar to Theorem 3.7, we obtain $\mathrm{s}(g_m, p) \leq (1+\psi_m)\mathrm{s}(g_m, r_m)$, where $\psi_m := (1 + \beta\nu\vartheta_m/\vartheta_{m+1})/(\beta\mu\vartheta_m)$. Hence,

$$\mathrm{s}(g_m, r_m) \geq (1 + \psi_m)^{-1}\mathrm{s}(g_m, p). \tag{3.48}$$

Next, we show the linear convergence of the sequences formed by Algorithms 3.2, 3.3, and 3.4, respectively.

Case 1: Consider $\{g_m\}$ generated by Algorithm 3.2. The linear convergence of $\{g_m\}$ can be obtained through a statement similar to *Case 1* of Theorem 3.7.

Case 2: Consider $\{g_m\}$ generated by Algorithm 3.3. From (3.31) and (3.48), one has

$$\mathrm{s}^2(g_{m+1}, p) \leq \mathrm{s}^2(g_m, p) - \frac{\sigma}{\beta^2}(2\beta - \sigma)\frac{(1 - \beta\nu\vartheta_m/\vartheta_{m+1})^2}{(1 + \beta\nu\vartheta_m/\vartheta_{m+1})^2}(1 + \psi_m)^{-2}\mathrm{s}^2(g_m, p). \tag{3.49}$$

Note that $0 < f(\sigma) := \frac{\sigma}{\beta^2}(2\beta - \sigma) \leq 1$ ($f(\sigma) = 1$ when $\sigma = \beta$). Let

$$\phi_4 := \lim_{m\to\infty} \frac{\sigma}{\beta^2}(2\beta - \sigma)\frac{(1 - \beta\nu\vartheta_m/\vartheta_{m+1})^2}{(1 + \beta\nu\vartheta_m/\vartheta_{m+1})^2}(1 + \psi_m)^{-2}.$$

Combining Lemma 3.1 with the definition of ψ_m, we have

$$\frac{\sigma}{\beta^2}(2\beta - \sigma)\frac{(1 - \beta\nu\vartheta_m/\vartheta_{m+1})^2}{(1 + \beta\nu\vartheta_m/\vartheta_{m+1})^2}(1 + \psi_m)^{-2} \geq \phi_4 \quad (\forall m \geq N_3).$$

Let $\phi_5 := (1 - \phi_4)^{1/2}$. This together with (3.49) gives

$$\mathrm{s}\left(g_{m+1}, p\right) \leq \frac{\mathrm{s}\left(g_{N_3}, p\right)}{\phi_5^{N_3-1}}\phi_5^n \quad (\forall m \geq N_3),$$

which implies that $\{g_m\}$ converges R-linearly to u^*.

Case 3: Consider $\{g_m\}$ generated by Algorithm 3.4. Notice that $f(\sigma) := 0 < \sigma(2 - \sigma) \leq 1$ ($f(\sigma) = 1$ when $\sigma = 1$). The linear convergence of $\{g_m\}$ can be easily obtained by using (3.37) and following a proof process similar to the one in *Case 2* above. □

Remark 3.3 Theorems 3.7 and 3.8 demonstrate that the distance $\mathrm{s}(g_m, p)$ between the m-th iteration point g_m of the proposed algorithms and the unique solution p of VIP (1.1) is bounded by certain known parameters and values from the iteration. Importantly, the right-hand side of the resulting inequalities is known at each iteration. This enables us to establish a stopping criterion that meets any desired accuracy level when the VIP (1.1) is driven by strongly pseudo-monotone vector fields. Furthermore, our results on the error bounds of the algorithms generalize the findings of [45, Theorem 4.2] and [46, Theorem 3.2], extending them from linear Hilbert spaces to Hadamard manifolds.

4 Numerical experiments

In this section, we provide two fundamental numerical examples (see, e.g., [25–27]) to illustrate the computational performance of the proposed algorithms. All our code was implemented in MATLAB R2023b and executed on a MacBook with 8 GB of memory.

Example 4.1 Let $\mathbb{R}_{++} = \{z \in \mathbb{R} : z > 0\}$, and define the Riemannian metric $\langle \cdot, \cdot \rangle$ as follows:

$$\langle q, w \rangle := \frac{qw}{z^2} \quad (\forall q, w \in T_z\mathcal{X},\ \forall z \in \mathcal{X}).$$

Consequently, $\mathcal{X} = (\mathbb{R}_{++}, \langle \cdot, \cdot \rangle)$ forms a Riemannian manifold. The Riemannian distance $\mathrm{s}\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{++}$ for points $z, q \in \mathcal{X}$ is defined as

$$\mathrm{s}(z, q) = \left| \ln\left(\frac{z}{q}\right) \right| \quad (\forall z, q \in \mathcal{X}),$$

as referenced in [25, Example 1]. Hence, $\mathcal{X}$ qualifies as a Hadamard manifold. We have $\mathrm{E}_z(tv) = ze^{\left(\frac{v}{z}\right)t}$ for all $z \in \mathcal{X}$, $t \in \mathbb{R}$, and $v \in T_z\mathcal{X}$ and the inverse of the exponential map is

$$\mathrm{E}_z^{-1}q = z \ln\left(\frac{q}{z}\right) \quad (\forall z, q \in \mathcal{X}).$$

Consider the set $C = [1, 100]$, which is a bounded, closed, and convex subset of $\mathcal{X}$, thereby making C compact within $\mathcal{X}$. Define the single-valued vector field $B\colon C \to T\mathcal{X}$ as:

$$Bz = z(5 - \ln z) \quad (\forall z \in C).$$

We first demonstrate that B is pseudo-monotone but not monotone on C. For any $z, q \in C$, we have:

$$\langle Bz, \mathrm{E}_z^{-1}q \rangle = (5 - \ln z) \ln\left(\frac{q}{z}\right) \geq 0.$$

Since $5 - \ln z > 0$ for all $z \in C$, it follows that $\ln\left(\frac{q}{z}\right) \geq 0$. Therefore,

$$\langle Bq, \mathrm{E}_q^{-1} z\rangle = (5 - \ln q) \ln\left(\frac{z}{q}\right) \leq 0.$$

This confirms that B is pseudo-monotone on C. In contrast, for any $z, q \in C$,

$$\langle Bz, \mathrm{E}_z^{-1} q\rangle + \langle Bq, \mathrm{E}_q^{-1} z\rangle = \left(\ln\left(\frac{z}{q}\right)\right)^2 = \mathrm{s}^2(z, q) \geq 0.$$

This shows that B is not monotone on C.

Next, we demonstrate that B is Lipschitz continuous on C. According to [25, Example 1], for a function $f : \mathbb{R}_{++} \to \mathbb{R}$ that is twice differentiable, the gradient and Hessian are given by

$$\operatorname{grad} f(z) = z^2 f'(z), \quad \operatorname{Hess} f(z) = f''(z) + z^{-1} f'(z),$$

where f' and f'' denote the first and second derivatives of f in the Euclidean context. We define $f : C \to \mathbb{R}$ as $f(z) = 5 \ln z - \frac{(\ln z)^2}{2}$ for all $z \in C$. This function is twice continuously differentiable on $\mathcal{X}$ in the Euclidean sense. From the previous formulas, we find:

$$\operatorname{grad} f(z) = z(5 - \ln z) = Bz, \quad \operatorname{Hess} f(z) = -\frac{1}{z^2}. \tag{4.1}$$

It follows that B is L-Lipschitz continuous on C with Lipschitz constant $L = 1$, since $\|\operatorname{Hess} f(z)\| \leq 1$ for all $z \in C$ (see [47, Lemma 2.3] for further details). To verify that B is 1-Lipschitz continuous, we can also apply Definition 2.3. For any $z, q \in C$, we have:

$$\begin{aligned} \left\|\mathrm{PT}_{q,z} Bz - Bq\right\| &= \|q(5 - \ln z) - q(5 - \ln q)\| \\ &= \|q \ln(q/z)\| = \mathrm{s}(z, q). \end{aligned}$$

Let u^* denote the solution to the VIP (1.1). This requires us to find u^* such that:

$$\langle Bu^*, \mathrm{E}_{u^*}^{-1} z\rangle \geq 0 \quad (\forall z \in C) \Leftrightarrow (5 - \ln u^*) \ln\left(\frac{z}{u^*}\right) \geq 0 \quad (\forall z \in [1, 100]).$$

This leads to the conclusion that $u^* = 1$, which is the unique solution to the VIP (1.1) as established by Lemma 2.5.

Example 4.2 Let $\langle \cdot\,, \cdot\rangle$, s, and $\mathcal{X}$ be the same as in Example 4.1. Let $C = [1, 100]$ be a subset of $\mathbb{R}_{++}$. Let the single-valued vector field $B : C \to T\mathcal{X}$ be given by

$$Bz = z \ln z \quad (\forall z \in C).$$

One obtains that B is pseudo-monotone and monotone on C. Let $f : C \to \mathbb{R}$ be given by $f(z) = \frac{(\ln z)^2}{2}$ for all $z \in C$. It follows from (4.1) that $\operatorname{grad} f(z) = z \ln z = Bz$ and $\operatorname{Hess} f(z) = \frac{1}{z^2}$. Consequently, B is 1-Lipschitz continuous. It is obvious that the VIP (1.1) with B and C given above has a unique solution $u^* = 1$.

Next, we use the proposed algorithms to solve Examples 4.1 and 4.2 and compare them with the Algorithm 2 of Sahu et al. [28] (shortly, SFS Alg. 2), the Algorithm 3.1 of Tang et al. [17] (shortly, TWL Alg. 3.1), the Algorithm 4.1 of Tang and Huang [15] (shortly, TH Alg. 4.1). The parameters of the algorithms are set as follows. Take $\nu = 0.5$, $\varpi_m = 1 + 0.1/(m+1)^2$, $\vartheta_0 = 0.5$, and $\mu_m = 0.1/(m+1)^2$ for the proposed Algorithms 3.1–3.5. Set $\beta \in \{0.8, 0.9, 1.0, 1.2, 1.3\}$ for Algorithms 3.1–3.4. Pick $\sigma = 1.5$ for Algorithms 3.3 and 3.4. Set $\vartheta = 1$, $\sigma = 0.8$, and $\eta = 0.5$ for SFS Alg. 2. Choose $\vartheta_m = 0.5$ and $\alpha = 0.9$ for TH Alg. 4.1. Select $\sigma = 0.8$ and $\nu = 0.5$ for TWL Alg. 3.1. In Examples 4.1 and 4.2, we denote the iteration error at step m of algorithms by $\mathrm{s}(g_m, u^*)$ (where $u^* = 1$ for both examples) and use $\mathrm{s}(g_m, u^*) < 10^{-5}$ or the maximum number of iterations 100 as their common stopping condition. Table 1 and Fig. 1 present the numerical performance of Algorithms 3.1–3.4 with different parameters β in Example 4.2. Finally, we choose two initial points to test the convergence performance of the proposed algorithms with $\beta = 1.3$ and the compared ones in [15, 17, 28] for Examples 4.1 and 4.2, as shown in Table 2 and Fig. 2.

Remark 4.1 Based on the numerical results from Examples 4.1 and 4.2, we can draw the following conclusions:

Table 1 The results of Algorithms 3.1–3.4 in Example 4.2 for different parameters β and initial values

Algorithms	β	g_0				
		20	40	60	80	100
Our Alg. 3.1	0.8	32	30	31	29	31
	0.9	27	29	28	33	29
	1.0	29	28	30	26	31
	1.2	25	30	27	27	28
	1.3	25	27	25	25	25
Our Alg. 3.2	0.8	31	30	32	32	31
	0.9	30	28	30	30	33
	1.0	29	28	30	26	31
	1.2	24	24	34	20	27
	1.3	22	26	23	19	24
Our Alg. 3.3	0.8	13	13	12	15	15
	0.9	14	13	12	15	13
	1.0	13	13	14	14	13
	1.2	12	12	12	11	12
	1.3	11	10	11	11	11
Our Alg. 3.4	0.8	14	14	15	15	15
	0.9	12	12	12	12	12
	1.0	10	10	10	10	10
	1.2	6	6	6	6	6
	1.3	4	4	4	4	4

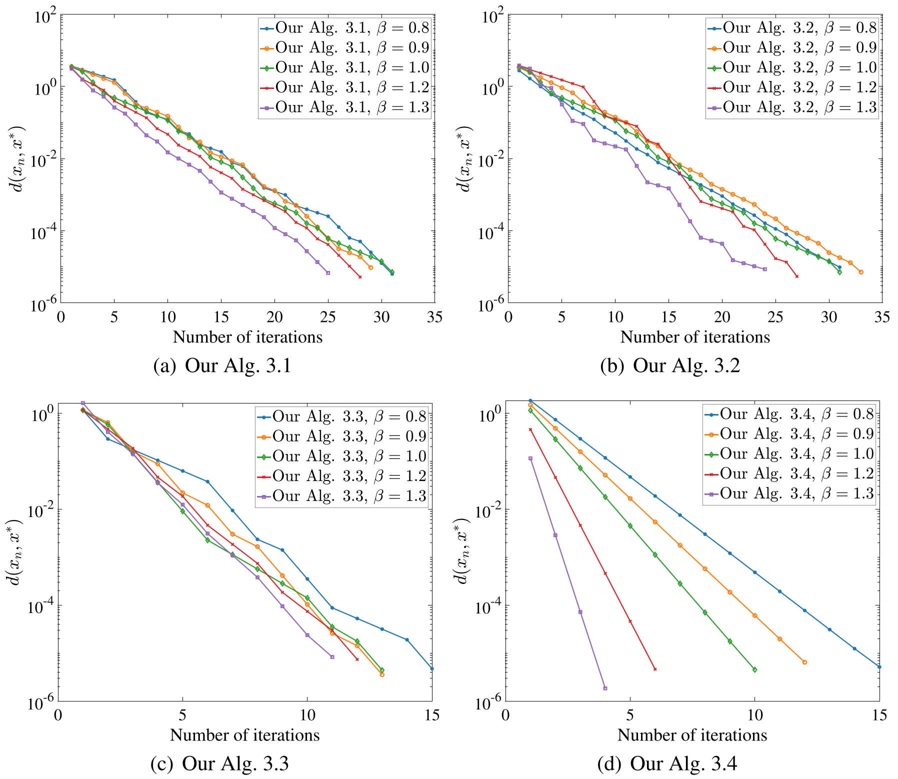

Fig. 1 Numberical behavior of our algorithms with different β in Example 4.2 ($g_0 = 100$)

(i) The numerical results presented in Examples 4.1 and 4.2 demonstrate that the algorithms proposed in this paper are effective for solving variational inequality problems with Lipschitz continuous vector fields on Hadamard manifolds.

Table 2 The number of iterations and execution time for all algorithms in Examples 4.1 and 4.2

Algorithms	Example 4.1				Example 4.2			
	$g_0 = 50$		$g_0 = 90$		$g_0 = 50$		$g_0 = 90$	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
Our Alg. 3.1	3	0.0057	6	0.0049	26	0.0079	27	0.0089
Our Alg. 3.2	3	0.0006	4	0.0009	27	0.0074	22	0.0044
Our Alg. 3.3	3	0.0007	5	0.0011	10	0.0014	12	0.0025
Our Alg. 3.4	19	0.5477	20	0.4862	4	0.3220	4	0.3114
Our Alg. 3.5	20	0.2257	19	0.2311	45	0.5294	46	0.5143
SFS Alg. 2	3	0.9390	4	1.4161	100	48.6086	100	79.0538
TH Alg. 4.1	23	0.5316	26	0.4916	100	0.8488	100	0.7630
TWL Alg. 3.1	20	0.3986	22	0.6514	93	0.9285	97	0.9847

(ii) Our algorithms converge faster than the compared one in [15, 17, 28] based on observations of iteration count and execution time. These results are independent of the initial value selection, demonstrating the efficiency and robustness of the proposed algorithms.

(iii) From the numerical results in Table 2, we can see the following:

(1) The explicit subgradient extragradient type algorithms (Algorithms 3.1–3.3) require less execution time compared to the implicit proximal point type algorithms (Algorithms 3.4 and 3.5).

(2) The projection and contraction type algorithms (Algorithms 3.3 and 3.4) converge faster than the subgradient extragradient algorithms (Algorithms 3.1 and 3.2) and the Tseng's extragradient algorithm (Algorithm 3.5).

(3) The proposed algorithms (Algorithms 3.1–3.5) using the adaptive step size rule (3.1) require less execution time compared to those using the Armijo-type rule (the algorithms in [15, 17, 28]). This is because the latter requires multiple evaluations of projections in each iteration to find a suitable step size (notably, SFS Alg. 2 requires many projection evaluations when updating g_{m+1} as the number of iterations increases, significantly increasing the execution time).

(iv) On the other hand, Figures 1 and 2 verify the theoretical results obtained in Section 3 about the distance $\mathrm{s}(g_m, u^*)$ being Fejér monotone. Indeed, the iteration errors in the y-axis of Figs. 1 and 2 indicate that the iterative sequences generated by our proposed algorithms are Fejér monotone with respect to the solution set $\Gamma(B, C)$ of VIP (1.1). That is, $\mathrm{s}\,(g_{m+1}, u^*) \leq \mathrm{s}\,(g_m, u^*)$ for all $u^* \in \Gamma(B, C)$.

(v) The performance of our algorithms improves when the parameter β is appropriately selected, as shown in Table 1 and Fig. 1.

(vi) It should be pointed out that the vector fields in Example 4.1 is pseudo-monotone rather than monotone, which means that algorithms proposed in the literature

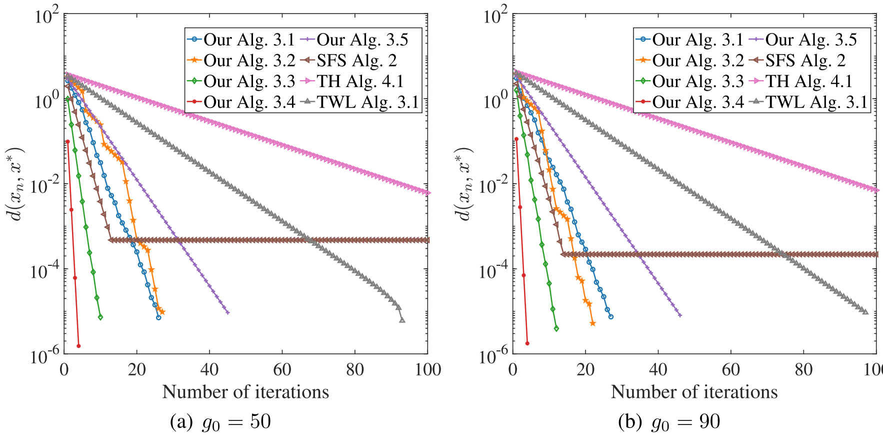

Fig. 2 Convergence behavior of all algorithms in Example 4.2

(see, e.g., [14, 18, 23]) for solving variational inequality problems induced by monotone vector fields on Hadamard manifolds will not be available in this case.

5 Conclusions

This paper introduces five adaptive numerical algorithms for finding solutions to variational inequality problems (VIPs) on Hadamard manifolds. The proposed algorithms draw inspiration from the extragradient method, subgradient extragradient algorithm, and the projection and contraction approach. They incorporate adaptive step size strategies, allowing for dynamic adjustment throughout the process. Under the conditions of a pseudo-monotone and Lipschitz continuous vector field, we demonstrate that the generated sequences converge to the solution of the VIP, provided that a solution exists. Additionally, we establish global error bounds and prove R-linear convergence for the algorithms when the vector fields governing the VIP are strongly pseudo-monotone. Some computational experiments indicate the efficiency of these algorithms. Our results extend and improve upon existing algorithms for solving VIPs on Hadamard manifolds. Given that the VIP is a specific case of equilibrium programming, future work will aim to extend the proposed algorithms to tackle equilibrium problems on Hadamard manifolds. Another promising direction is to investigate practical applications of these algorithms within Hadamard manifolds. Furthermore, exploring the application of inertial techniques to accelerate the convergence rate of the extragradient-type algorithms could be a promising direction.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Ethical Approval Not applicable.

Competing interests The authors declare no competing interests.

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