Pure and Applied Functional Analysis
Volume 8, Number 3, 2023, 965–986

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

THREE APPROXIMATION METHODS FOR SOLVING CONSTRAINT VARIATIONAL INEQUALITIES AND RELATED PROBLEMS

BING TAN, AVIV GIBALI*, AND XIAOLONG QIN

Abstract. In this paper, we present three new self-adaptive one-projection algorithms to find common solutions for the pseudo-monotone variational inequality problem and the fixed point problem of a demi-contractive mapping. Note that the suggested approaches use a non-monotonic self-adaptive step size so that they can work well without knowing the prior knowledge of the Lipschitz constant of the mapping. Strong convergence theorems of the proposed iterative schemes are established in real Hilbert spaces. Several mathematical experiments are reported to demonstrate the numerical behavior of the suggested algorithms and compare them with the existing ones. Finally, the suggested methods are used to solve optimal control problems.

1. Introduction

Throughout the paper, assume that C is a nonempty closed and convex subset of a real Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let us first review the classical variational inequality problem of Fichera [12, 13] and Stampacchia [25]. We denote the problem as (VI) and it is formulated as follows.

$$\text{(VI)} \qquad \text{Find } x^* \in C \text{ such that } \langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C\,,$$

where $A : C \to \mathcal{H}$ is an operator. It is known that variational inequalities play a significant role in applied science and optimization theory. They provide a general and useful framework for solving engineering problems, data sciences and other fields; for example, see [1, 4, 5, 30]. Therefore, the numerical methods for studying variational inequalities have attracted numerous interest from researchers. In the past few decades, many scholars proposed various iterative schemes to solve (VI), see, e.g., [6, 14, 15, 26, 27] and the references therein.

One of the earliest and simple methods for solving VIs is an extension of the projected gradient method for constrained optimization. Given the current iterate x_n, calculate the next iterate x_{n+1} as follows.

$$\text{(PGM)} \qquad x_{n+1} = P_C(x_n - \lambda_n A x_n)\,,$$

2020 *Mathematics Subject Classification.* 47J20, 47J25, 47J30, 68W10, 65K15.

Key words and phrases. Variational inequality, fixed point, extragradient method, hybrid steepest descent method, pseudomonotone mapping.

*Corresponding author.

where $\{\lambda_n\}$ is a sequence of positive real numbers and $P_C : \mathcal{H} \to C$ denotes the metric (nearest point) projection from $\mathcal{H}$ onto C, characterized by $P_C(x) := \arg\min\{\|x - y\|, y \in C\}$ and $P_C(x) \in C$ for all $x \in \mathcal{H}$. Under some assumptions on $\{\lambda_n\}$ and A, it is known that a solution of (VI) is a fixed point of the mapping $P_C(I - \lambda_n A)$ (I is the identity mapping), that is

$$x^* \text{ solves (VI) } \Leftrightarrow x^* = P_C(x^* - \lambda_n A x^*) .$$

While our study here does not enable to translate VIs to fixed point problem, we still wish to focus on them independently since many problems in optimization theory can be transformed in such a way. Recall that the fixed point problem (FPP) is expressed as follows.

$$\text{(FPP)} \qquad \text{Find } x^* \in \mathcal{H} \text{ such that } Ux^* = x^* ,$$

where $U : \mathcal{H} \to \mathcal{H}$ is a mapping. The fixed point problem has a lot of research results in the optimization community.

In this paper, we focus on studying a feasibility problem that consists of fixed point and variational inequality and also can be seen as a constraint variational inequality problem. The problem is phrased as follows.

$$\text{(VIFPP)} \qquad \text{Find } x^* \in \mathcal{H} \text{ such that } x^* \in \mathrm{F}(U) \cap \mathrm{VI}(C, A) ,$$

where $\mathrm{VI}(C, A)$ and $\mathrm{F}(U)$ represent the solution set of (VI) and (FPP), respectively. Motivation for studying this topic, common solution problem with constraints, is mainly due to its potential applications in mathematical modeling of specific complex issues. Indeed, problems in practical applications may have certain constraints, which can be given by fixed-point problems, variational inequalities, or other types of problems.

In order to obtain a specific solution of (VIFPP), we focus on the following bi-level variational inequality problem: find $x^* \in \mathrm{F}(U) \cap \mathrm{VI}(C, A)$ such that

$$\text{(VI-VIFPP)} \qquad \langle Gx^*, x - x^* \rangle \geq 0, \ \forall x \in \mathrm{F}(U) \cap \mathrm{VI}(C, A) ,$$

where the mapping G is strong monotone and Lipschitz continuous.

Next we review related results in the literature both theatrical and practical.

1.1. **Relation with previous work.** In this section, we wish to give some of the approaches for solving (VI) and (VIFPP), which inspired us to propose our new iterative schemes for solving (VI-VIFPP).

The earliest numerical approach for solving (VI) is the projected gradient method (PGM) presented in (PGM). The convergence assumptions of the PGM are quite restrictive, that is, operator A is strongly monotone and Lipschitz continuous, which limits the implementation of such methods in practical applications.

In a way to weakening the assumptions, Korpelevich [18] proposed a two-step iterative scheme (known as the extragradient method, EGM) in which the A is assumed to be pseudo-monotone only.

$$\text{(EGM)} \qquad \begin{cases} y_n = P_C\left(x_n - \lambda_n A x_n\right) , \\ x_{n+1} = P_C\left(x_n - \lambda_n A y_n\right) . \end{cases}$$

The main drawback of the method is the need to compute two projections onto the VI's feasible set per each iteration. So, in case that the set C has a complex structure, the extragradient method would require an intensive computational resources.

Further extensions of the extragradient method focus on the elimination of the extra projection. Next we present three improvements. The first is Tseng's extragradient method (TEGM) [31], that replaces the second step of the EGM by an explicit calculation step. Another idea is known as the subgradient extragradient method [8–10], where an additional easy and constructible set is introduced and the second projection is calculated with respect to it.

The third development is He [17] projection and contraction method (PCM) and it uses only one projection onto C per each iteration and the second step is updated via some previous information (see (3.25)).

Note that all the above method need to calculate the projection onto the feasible set at least once. So a natural extension is a method that doesn't require any projections. Yamada in [38] consider the variational inequality (VI-VIFPP) with $A = 0$ and this translates to the following problem.

$$\text{(VI-FPP)} \qquad \text{Find } x^* \in \mathrm{F}(U) \text{ such that } \langle Gx^*, x - x^* \rangle \geq 0, \ \forall x \in \mathrm{F}(U)\,,$$

where G is an η-strongly monotone and k-Lipschitz continuous mapping, U is a nonexpansive mapping having a fixed point $\mathrm{F}(U)$. Yamada's hybrid steepest descent method is formulated as follows.

$$x_{n+1} = (I - \sigma\theta_{n+1}G)\,Ux_n\,,$$

where $\sigma \in (0, 2\eta/k^2)$ and $\{\theta_n\}$ is some sequence. This method attracts much interest and many extended it to various optimization problems, such as split feasibility problems and variational inequalities, see, e.g., [11, 20].

The above mentioned methods, besides Yamada's, converge only weakly in infinite-dimensional spaces, see also [10, 32, 33]. Besides theoretical advantages of strong convergence method, in many applications strongly convergent is essential, for example in machine learning and quantum computation, see, e.g., [2].

Many recent developments in this directions combines various techniques such as Halpern, Mann, viscosity and hybrid steepest descent method, see [7, 15, 16, 19, 22, 29, 34, 36]. Maingé [22] proposed the so-called hybrid viscosity-like extragradient method that combines the extragradient method, the Mann-type method and the hybrid steepest descent method for solving (VI-VIFPP). The iterative step is formulated as follows.

$$\text{(HVEGM)} \qquad \begin{cases} y_n = P_C\left(x_n - \lambda_n A x_n\right)\,, \\ z_n = P_C\left(x_n - \lambda_n A y_n\right)\,, \\ q_n = z_n - \theta_n G\left(z_n\right)\,, \\ x_{n+1} = [(1-\gamma_n)I + \gamma_n U]q_n\,, \end{cases}$$

where A is a monotone and L-Lipschitz continuous mapping, U is a ρ-demi-contractive and demiclosed mapping, G is an η-strongly monotone and k-Lipschitz continuous mapping, and the parameters fulfil some conditions.

Another interesting result is Gibali and Shehu [15] projection and contraction extragradient method (PCEGM). The algorithm has the following form.

$$\text{(PCEGM)} \qquad \begin{cases} y_n = P_C\left(x_n - \lambda_n A x_n\right) , \\ z_n = x_n - \phi \delta_n d_n , \\ q_n = z_n - \theta_n G\left(z_n\right) , \\ x_{n+1} = [(1-\gamma_n) I + \gamma_n U] q_n , \end{cases}$$

where the mappings A, U, G and other parameters are the same as in (HVEGM).

Recently, Tong and Tian [36] combined Tseng's extragradient method with the hybrid steepest descent method and the Mann-like method, and offered a new self-adaptive iterative scheme to solve (VI-VIFPP).

$$\text{(STEGM)} \qquad \begin{cases} y_n = P_C\left(x_n - \lambda_n A x_n\right) , \\ z_n = y_n - \lambda_n\left(A y_n - A x_n\right) , \\ q_n = \left(1-\gamma_n\right) z_n + \gamma_n U z_n , \\ x_{n+1} = \left(I - \mu \theta_n G\right) q_n , \end{cases}$$

where U is quasi-nonexpansive with a demiclosedness property, A is monotone and L-Lipschitz continuous, and G is strongly monotone and Lipschitz continuous, and the other parameters fulfil some conditions. While the method converges strongly under some assumptions, the line search update rule increases the computational burden since many evaluations of A are needed per each iteration.

Motivated and inspired by all the above work, that is [10, 15, 22, 31, 36], in Section 3 we propose and analyse three self-adaptive one-projection methods for solving pseudo-monotone variational inequalities and other related optimization problems. Moreover, we present a new step size selection scheme that does not include any line search process, which uses previously known information to generate a non-monotonic step size sequence in each iteration. Under some suitable conditions imposed on the mappings and parameters, we established the strong convergence theorems of the suggested iterative algorithms in a real Hilbert space. In Section 4 we provide some numerical experiments including an optimal control problems, show the efficiency and applicability of our algorithms.

2. Preliminaries

Throughout this paper, the weak convergence and strong convergence of $\{x_n\}$ to x are represented by $x_n \rightharpoonup x$ and $x_n \rightarrow x$, respectively. It is known that the following inequalities hold.

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle,\ \forall x, y \in \mathcal{H} . \tag{2.1}$$

$$\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x-y\rangle ,\ \forall x, y \in \mathcal{H} . \tag{2.2}$$

$$\langle x - P_C(x), y - P_C(x)\rangle \leq 0,\ \forall x \in \mathcal{H}, y \in C . \tag{2.3}$$

Definition 2.1. Recall that an operator $M : \mathcal{H} \rightarrow \mathcal{H}$ with its fixed point set $\mathrm{F}(M)$ is said to be:

(1) *L-Lipschitz continuous* with $L > 0$ if $\|Mx - My\| \leq L\|x - y\|, \ \forall x, y \in \mathcal{H}$. If $L = 1$ then M is said to be *nonexpansive.*
(2) *η-strongly monotone* with $\eta > 0$ if $\langle Mx - My, x - y\rangle \geq \eta\|x - y\|^2, \ \forall x, y \in \mathcal{H}$.
(3) *Monotone* if $\langle Mx - My, x - y\rangle \geq 0, \ \forall x, y \in \mathcal{H}$.
(4) *Pseudo-monotone* if $\langle Mx, y - x\rangle \geq 0 \Longrightarrow \langle My, y - x\rangle \geq 0, \ \forall x, y \in \mathcal{H}$.
(5) *Quasi-nonexpansive* if $\|Mx - z\| \leq \|x - z\|, \ \forall z \in \mathrm{F}(M), x \in \mathcal{H}$.
(6) *ρ-demi-contractive* with $0 \leq \rho < 1$ if

$$\|Mx - z\|^2 \leq \|x - z\|^2 + \rho\|(I - M)x\|^2, \quad \forall z \in \mathrm{F}(M), x \in \mathcal{H}\,, \tag{2.4}$$

or equivalently

$$\langle Mx - z, x - z\rangle \leq \|x - z\|^2 + \frac{\rho - 1}{2}\|x - Mx\|^2, \quad \forall z \in \mathrm{F}(M), x \in \mathcal{H}\,. \tag{2.5}$$

From the above definitions, it is easy to see that the class of demi-contractive mappings includes the class of quasi-nonexpansive mappings and the class of pseudo-monotone mappings contains the class of monotone mappings, in other words, (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6).

To prove the convergence of the proposed algorithms, we need the following lemmas.

Lemma 2.2 ([39])**.** *Assume that $U : \mathcal{H} \to \mathcal{H}$ is a nonlinear operator with $\mathrm{F}(U) \neq \emptyset$. Then, $I - U$ is said to be demiclosed at zero if for any $\{x_n\}$ in $\mathcal{H}$, the following implication holds: $x_n \rightharpoonup x$ and $(I - U)x_n \to 0 \Rightarrow x \in \mathrm{F}(U)$.*

Lemma 2.3 ([38])**.** *Let $\mu > 0$ and $\theta \in (0, 1]$. Let $G : \mathcal{H} \to \mathcal{H}$ be a k-Lipschitz continuous and η-strongly monotone mapping with $0 < \eta \leq k$. Associating with a nonexpansive mapping $U : \mathcal{H} \to \mathcal{H}$, define the mapping $U^\mu : \mathcal{H} \to \mathcal{H}$ by $U^\mu x = (I - \mu\theta G)(Ux), \forall x \in \mathcal{H}$. Then, U^μ is a contraction provided $\mu < \frac{2\eta}{k^2}$, that is*

$$\|U^\mu x - U^\mu y\| \leq (1 - \theta\omega)\|x - y\|, \quad \forall x, y \in \mathcal{H}\,,$$

where $\omega = 1 - \sqrt{1 - \mu\left(2\eta - \mu k^2\right)} \in (0, 1)$.

Lemma 2.4 ([22])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}\,.$$

In fact, m_k is the largest number n in the set $\{1, 2, \ldots, k\}$ such that $a_n < a_{n+1}$.

3. Main results

In this section, we introduce our new iterative methods for solving variational inequalities and fixed point problems and analyze their convergence behavior.

For the convergence analysis we assume following conditions.

(C1) The mapping $A : \mathcal{H} \to \mathcal{H}$ is pseudo-monotone and L-Lipschitz continuous on $\mathcal{H}$, and sequentially weakly continuous on C.
(C2) The mapping $U : \mathcal{H} \to \mathcal{H}$ is ρ-demi-contractive such that $(I - U)$ is demiclosed at zero.

(C3) The solution set $\mathrm{F}(U) \cap \mathrm{VI}(C, A) \neq \emptyset$.
(C4) The mapping $G : \mathcal{H} \to \mathcal{H}$ is η-strongly monotone and k-Lipschitz continuous, where $\eta > 0$ and $k > 0$.
(C5) Let $\{\xi_n\}$ be a nonnegative sequence satisfies $\sum_{n=1}^{\infty} \xi_n < \infty$. Assume that $\{\theta_n\} \subset (0,1)$ and $\{\gamma_n\}$ are two real sequences such that $\lim_{n\to\infty} \theta_n = 0$, $\sum_{n=1}^{\infty} \theta_n = \infty$ and $\gamma_n \subset (a, (1-\rho)/2)$ for some $a > 0$.

3.1. **First algorithm.** Inspired by the subgradient extragradient method, the hybrid steepest descent method and the Mann-like method, we introduce the first iterative method that uses non-monotonic adaptive step sizes.

Algorithm 1

Initialization: Take $\lambda_1 > 0$, $\sigma \in (0,1)$ and $\mu \in \left(0, \frac{2\eta}{k^2}\right)$. Let $x_1 \in \mathcal{H}$ be arbitrary.
Iterative Steps: Calculate x_{n+1} as follows:

$$\begin{cases} y_n = P_C\left(x_n - \lambda_n A x_n\right), \\ z_n = P_{T_n}\left(x_n - \lambda_n A y_n\right), \\ T_n := \{x \in \mathcal{H} \mid \langle x_n - \lambda_n A x_n - y_n, x - y_n \rangle \leq 0\}, \\ q_n = \left(I - \mu\theta_n G\right) z_n, \\ x_{n+1} = \left(1-\gamma_n\right) q_n + \gamma_n U q_n, \end{cases}$$

where the step size λ_{n+1} is updated by the following

$$\lambda_{n+1} = \begin{cases} \min\left\{ \dfrac{\sigma \|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n + \xi_n \right\}, & \text{if } Ax_n - Ay_n \neq 0; \\ \lambda_n + \xi_n, & \text{otherwise.} \end{cases} \tag{3.1}$$

We start with the validation of the sequence $\{\lambda_n\}$.

Lemma 3.1 ([21, Lemma 3.1])**.** . *Suppose that the condition (C1) holds. Then the sequence $\{\lambda_n\}$ generated by* (3.1) *is well defined and* $\lim_{n\to\infty} \lambda_n = \lambda$ *and* $\lambda \in \left[\min\{\frac{\sigma}{L}, \lambda_1\}, \lambda_1 + \Xi\right]$, *where* $\Xi = \sum_{n=1}^{\infty} \xi_n$.

Remark 3.2. The idea of the step size λ_n defined in (3.1) is derived from [21]. It is worth noting that the step size λ_n generated in Algorithm 1 is allowed to increase when the iteration increases. Therefore, the use of this type of step size reduces the dependence on the initial step size λ_1. On the other hand, because of $\sum_{n=1}^{\infty} \xi_n < +\infty$, which implies that $\lim_{n\to\infty} \xi_n = 0$. Thus, the step size λ_n may not increase when n is large enough. If $\xi_n = 0$, then the step size λ_n in Algorithm 1 is similar to the approaches in [27, 35].

Lemma 3.3. *Assume that Conditions (C1) and (C3) hold. Let $\{y_n\}$ and $\{z_n\}$ be two sequences generated by Algorithm 1. Then, for all $p \in \mathrm{VI}(C, A)$,*

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \left(1 - \frac{\sigma\lambda_n}{\lambda_{n+1}}\right)\left(\|y_n - x_n\|^2 + \|z_n - y_n\|^2 \right).$$

Proof. Using the definition of λ_n, one obtains

$$\|Ax_n - Ay_n\| \leq \frac{\sigma}{\lambda_{n+1}} \|x_n - y_n\|, \quad \forall n. \tag{3.2}$$

Indeed, if $Ax_n = Ay_n$, then the inequality (3.2) holds. Otherwise, it implies from (3.1) that

$$\lambda_{n+1} = \min\left\{\frac{\sigma\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n + \xi_n\right\} \leq \frac{\sigma\|x_n - y_n\|}{\|Ax_n - Ay_n\|}.$$

Consequently,

$$\|Ax_n - Ay_n\| \leq \frac{\sigma}{\lambda_{n+1}}\|x_n - y_n\|.$$

Therefore, the inequality (3.2) holds when $Ax_n = Ay_n$ and $Ax_n \neq Ay_n$. By the definition of z_n and (2.2), one sees that

$$\begin{aligned}
2\|z_n - p\|^2 &= 2\|P_{T_n}(x_n - \lambda_n Ay_n) - P_{T_n}(p)\|^2 \leq 2\langle z_n - p, x_n - \lambda_n Ay_n - p\rangle \\
&= \|z_n - p\|^2 + \|x_n - \lambda_n Ay_n - p\|^2 - \|z_n - x_n + \lambda_n Ay_n\|^2 \\
&= \|z_n - p\|^2 + \|x_n - p\|^2 + \lambda_n^2\|Ay_n\|^2 - 2\langle x_n - p, \lambda_n Ay_n\rangle \\
&\quad - \|z_n - x_n\|^2 - \lambda_n^2\|Ay_n\|^2 - 2\langle z_n - x_n, \lambda_n Ay_n\rangle \\
&= \|z_n - p\|^2 + \|x_n - p\|^2 - \|z_n - x_n\|^2 - 2\langle z_n - p, \lambda_n Ay_n\rangle.
\end{aligned}$$

This implies that

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \|z_n - x_n\|^2 - 2\langle z_n - p, \lambda_n Ay_n\rangle. \tag{3.3}$$

Since $p \in \mathrm{VI}(C, A)$, one has $\langle Ap, x - p\rangle \geq 0$ for all $x \in C$. By the pseudomontonicity of A on $\mathcal{H}$, we get $\langle Ax, x - p\rangle \geq 0$ for all $x \in C$. Taking $x = y_n \in C$, one infers that $\langle Ay_n, p - y_n\rangle \leq 0$. Consequently,

$$\langle Ay_n, p - z_n\rangle = \langle Ay_n, p - y_n\rangle + \langle Ay_n, y_n - z_n\rangle \leq \langle Ay_n, y_n - z_n\rangle. \tag{3.4}$$

From (3.3) and (3.4), one obtains

$$\begin{aligned}
\|z_n - p\|^2 &\leq \|x_n - p\|^2 - \|z_n - x_n\|^2 + 2\lambda_n\langle Ay_n, y_n - z_n\rangle \\
&= \|x_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - x_n\|^2 - 2\langle z_n - y_n, y_n - x_n\rangle \\
&\quad + 2\lambda_n\langle Ay_n, y_n - z_n\rangle \\
&= \|x_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - x_n\|^2 \\
&\quad + 2\langle z_n - y_n, x_n - \lambda_n Ay_n - y_n\rangle.
\end{aligned} \tag{3.5}$$

Since $z_n \in T_n$ and (3.2), one has

$$\begin{aligned}
&2\langle x_n - \lambda_n Ay_n - y_n, z_n - y_n\rangle \\
&= 2\langle x_n - \lambda_n Ax_n - y_n, z_n - y_n\rangle + 2\lambda_n\langle Ax_n - Ay_n, z_n - y_n\rangle \\
&\leq 2\lambda_n\|Ay_n - Ax_n\|\|y_n - z_n\| \leq 2\frac{\sigma\lambda_n}{\lambda_{n+1}}\|x_n - y_n\|\|y_n - z_n\| \\
&\leq \frac{\sigma\lambda_n}{\lambda_{n+1}}\left(\|x_n - y_n\|^2 + \|y_n - z_n\|^2\right).
\end{aligned}$$

This combining with (3.5) concludes that

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \left(1 - \frac{\sigma\lambda_n}{\lambda_{n+1}}\right)\left(\|y_n - x_n\|^2 + \|z_n - y_n\|^2\right).$$

This completes the proof of the lemma. □

Lemma 3.4 ([28, Lemma 5.8]). *Assume that the condition (C1) holds. Let $\{x_n\}$ and $\{y_n\}$ be two sequences created by Algorithm 1. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to $z \in \mathcal{H}$ and $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$, then $z \in \mathrm{VI}(C, A)$.*

We are now in position to prove the strong convergence result of Algorithm 1.

Theorem 3.5. *Assume that Conditions (C1)–(C5) hold. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges to an element p in norm, where $p = P_{\mathrm{F}(U)\cap\mathrm{VI}(C,A)}(I - \mu G)p$.*

Proof. According to Lemma 2.3, we know that $(I-\mu G)$ is a contractive mapping and hence $P_{\mathrm{F}(U)\cap\mathrm{VI}(C,A)}(I - \mu G)$ is also a contraction mapping. By Banach contraction principle, there exists a unique point $p \in \mathcal{H}$ such that $p = P_{\mathrm{F}(U)\cap\mathrm{VI}(C,A)}(I - \mu G)p$, and thus $p \in \mathrm{F}(U) \cap \mathrm{VI}(C, A)$.

To better organize the proof process, we divide the proof into four parts.

Claim 1. The sequence $\{x_n\}$ is bounded. By Lemma 3.1, we have $\lim_{n\to\infty}(1 - \frac{\sigma\lambda_n}{\lambda_{n+1}}) = 1 - \sigma > 0$. Thus, there exists $n_0 \in \mathbb{N}$ such that

$$1 - \frac{\sigma\lambda_n}{\lambda_{n+1}} > 0, \quad \forall n \geq n_0 .$$

This together with Lemma 3.3 yields that

$$\|z_n - p\| \leq \|x_n - p\| , \quad \forall n \geq n_0 . \tag{3.6}$$

From the definition of x_{n+1}, (2.4) and (2.5), we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1-\gamma_n)(q_n - p) + \gamma_n (Uq_n - p)\|^2 \\ &= (1-\gamma_n)^2 \|q_n - p\|^2 + \gamma_n^2 \|Uq_n - p\|^2 \\ &\quad + 2(1-\gamma_n)\gamma_n \langle Uq_n - p, q_n - p\rangle \\ &\leq (1-\gamma_n)^2 \|q_n - p\|^2 + \gamma_n^2 \|q_n - p\|^2 + \gamma_n^2\rho \|Uq_n - q_n\|^2 \\ &\quad + 2(1-\gamma_n)\gamma_n \Big[\|q_n - p\|^2 - \frac{1-\rho}{2} \|Uq_n - q_n\|^2 \Big] \\ &= \|q_n - p\|^2 + \gamma_n [\gamma_n - (1-\rho)] \|Uq_n - q_n\|^2 . \end{aligned} \tag{3.7}$$

It follows from the assumption on $\{\gamma_n\}$ that

$$\|x_{n+1} - p\| \leq \|q_n - p\| . \tag{3.8}$$

From (3.6) and (3.8), one has

$$\|z_{n+1} - p\| \leq \|x_{n+1} - p\| \leq \|q_n - p\|, \quad \forall n \geq n_0 . \tag{3.9}$$

By $q_{n+1} = (I - \mu\theta_{n+1}G)\, z_{n+1}$ and Lemma 2.3, we obtain

$$\begin{aligned} \|q_{n+1} - p\| &= \|(I - \mu\theta_{n+1}G)\, z_{n+1} - (I - \mu\theta_{n+1}G)\, p - \mu\theta_{n+1}Gp\| \\ &\leq \|(I - \mu\theta_{n+1}G)\, z_{n+1} - (I - \mu\theta_{n+1}G)\, p\| + \mu\theta_{n+1}\|Gp\| \\ &\leq (1 - \omega\theta_{n+1})\, \|z_{n+1} - p\| + \mu\theta_{n+1}\|Gp\| \\ &\leq (1 - \omega\theta_{n+1})\, \|q_n - p\| + \omega\theta_{n+1} \cdot (\mu\|Gp\|/\omega) \\ &\leq \max\{\|q_n - p\| , \mu\|Gp\|/\omega\} \\ &\leq \cdots \leq \max\{\|q_0 - p\| , \mu\|Gp\|/\omega\} , \end{aligned}$$

where $\omega = 1-\sqrt{1-\mu\left(2\eta-\mu k^2\right)} \in (0,1)$. Therefore, the sequence $\{q_n\}$ is bounded. So, through expression (3.9), the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are also bounded.

Claim 2.

$$\begin{aligned}&\Big(1-\frac{\sigma\lambda_n}{\lambda_{n+1}}\Big)\big(\,\|y_n-x_n\|^2+\|z_n-y_n\|^2\,\big)+\|x_{n+1}-z_n\|^2\\&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\theta_n M_1,\quad \forall n\ge n_0\end{aligned}$$

for some $M_1>0$. Since $Uq_n-q_n=\frac{1}{\gamma_n}\left(x_{n+1}-q_n\right)$, we obtain from (3.7) that

$$\|x_{n+1}-p\|^2\le\|q_n-p\|^2-\iota\,\|x_{n+1}-q_n\|^2\ , \tag{3.10}$$

where $\iota=\frac{1-\rho-\gamma_n}{\gamma_n}$. Since $0<\gamma_n<(1-\rho)/2$, one obtains $\iota>1$. By the boundedness of $\{x_n\}$ and $\{z_n\}$, we can assume that $M_1:=\sup_{n\in\mathbb{N}}|2\mu\left\langle p-x_{n+1},Gz_n\right\rangle|$. Thus, combining (2.1), (3.10), the definition of q_n and Lemma 3.3, we have

$$\begin{aligned}\|x_{n+1}-p\|^2&\le\|q_n-p\|^2-\|x_{n+1}-q_n\|^2\\&\le\|z_n-\mu\theta_nGz_n-p\|^2-\|x_{n+1}-(z_n-\mu\theta_nGz_n)\|^2\\&\le\|z_n-p\|^2-2\mu\theta_n\left\langle x_{n+1}-p,Gz_n\right\rangle-\|x_{n+1}-z_n\|^2\\&\le\|x_n-p\|^2-2\mu\theta_n\left\langle x_{n+1}-p,Gz_n\right\rangle-\|x_{n+1}-z_n\|^2\\&\quad-\Big(1-\frac{\sigma\lambda_n}{\lambda_{n+1}}\Big)\big(\,\|y_n-x_n\|^2+\|z_n-y_n\|^2\,\big)\\&\le\|x_n-p\|^2-\Big(1-\frac{\sigma\lambda_n}{\lambda_{n+1}}\Big)\big(\,\|y_n-x_n\|^2+\|z_n-y_n\|^2\,\big)\\&\quad-\|x_{n+1}-z_n\|^2+\theta_nM_1,\quad\forall n\ge n_0\,.\end{aligned}\tag{3.11}$$

Claim 3. The sequence $\{\|x_n-p\|^2\}$ converges to zero by considering two possible cases on the sequence $\{\|x_n-p\|^2\}$. Let $a_n=\|x_n-p\|^2$.

Case 1. There exists m_0 such that $\{a_n\}$ is decreasing for $n\ge m_0$. In that case, there exists the limit of $\{a_n\}$, i.e., $\lim_{n\to\infty}\|x_n-p\|^2=a$ and $\lim_{n\to\infty}(a_{n+1}-a_n)=0$. From Claim 2, Lemma 3.1 and Condition (C5), we get

$$\lim_{n\to\infty}\|x_n-y_n\|=0,\ \text{and}\ \lim_{n\to\infty}\|z_n-x_{n+1}\|=0,\ \text{and}\ \lim_{n\to\infty}\|z_n-y_n\|=0\,, \tag{3.12}$$

which implies that $\lim_{n\to\infty}\|z_n-x_n\|=0$ and hence $\|z_n-p\|^2\to a$. On the other hand, since $\{x_n\}\subset C$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to some $z\in C$. This together with (3.12) and Lemma 3.4 gives that $z\in\mathrm{VI}(C,A)$. From $q_{n_k}=z_{n_k}-\mu\theta_{n_k}Gz_{n_k}$, G is Lipschitz continuous, $\{z_{n_k}\}$ is bounded and $\lim_{k\to\infty}\theta_{n_k}=0$, we infer that $\|q_{n_k}-z_{n_k}\|\to0$. This together with $\|z_{n_k}-x_{n_k+1}\|\to0$ yields that $\|x_{n+1}-q_{n_k}\|\to0$. In addition, it follows from the definition of x_{n+1} and the assumption on $\{\gamma_n\}$ that

$$\|Uq_{n_k}-q_{n_k}\|=\frac{1}{\gamma_{n_k}}\|x_{n+1}-q_{n_k}\|\to0,\ \text{as}\ k\to\infty\,.$$

From (3.12), one obtains $z_{n_k}\rightharpoonup z$, which, together with $\|Uq_{n_k}-q_{n_k}\|\to0$, $(I-U)$ is demiclosed at zero and Lemma 2.2, indicates that $z\in\mathrm{F}(U)$. Thus, we deduce that $z \in \mathrm{F}(U) \cap \mathrm{VI}(C, A)$. It follows from the definition of p and (2.3) that

$$\lim_{n\to\infty} \langle z_n - p, Gp\rangle = \lim_{k\to\infty} \langle z_{n_k} - p, Gp\rangle = \langle z - p, Gp\rangle \geq 0\,. \tag{3.13}$$

Using G is η-strongly monotone, one has

$$\begin{aligned} \langle x_{n+1} - p, Gz_n\rangle &= \langle x_{n+1} - z_n, Gz_n\rangle + \langle z_n - p, Gp\rangle + \langle z_n - p, Gz_n - Gp\rangle \\ &\geq \langle x_{n+1} - z_n, Gz_n\rangle + \langle z_n - p, Gp\rangle + \eta \left\|z_n - p\right\|^2\,. \end{aligned} \tag{3.14}$$

Combining (3.12), (3.13) and (3.14), it follows that

$$\liminf_{n\to\infty} \langle x_{n+1} - p, Gz_n\rangle \geq \eta a\,. \tag{3.15}$$

Without loss of generality, we assume that $a > 0$. Choosing $\epsilon = \frac{1}{2}\eta a$. It follows from (3.15) that there exists $m_1 > 0$ such that

$$\langle x_{n+1} - p, Gz_n\rangle \geq \eta a - \epsilon = \frac{1}{2}\eta a, \quad \forall n \geq m_1\,,$$

which together with (3.11) implies that

$$a_{n+1} - a_n \leq -\mu\eta a\theta_n, \quad \forall n \geq m_1\,.$$

Therefore, by summarizing the above formula, we can show

$$a_{n+1} - a_{m_1} \leq -\mu\eta a \sum_{k=m_1}^{n} \theta_k\,. \tag{3.16}$$

Since $\sum_{k=1}^{\infty} \theta_k = \infty$, we can conclude from (3.16) that $\liminf_{n\to\infty} a_n = -\infty$, which is a contradiction. Consequently, we get that $a = 0$, i.e., $x_n \to p$ as $n \to \infty$.

Case 2. There exists a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that $a_{n_i} \leq a_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, it follows from Lemma 2.4 that there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_n \leq a_{m_k+1}\,. \tag{3.17}$$

Combining Claim 2, Lemma 3.1 and $\lim_{n\to\infty} \theta_n = 0$, one obtains

$$\|x_{m_k+1} - z_{m_k}\| \to 0,\ \|x_{m_k} - y_{m_k}\| \to 0,\ \|y_{m_k} - z_{m_k}\| \to 0\,, \tag{3.18}$$

which indicates that $\lim_{k\to\infty} \|z_{m_k} - x_{m_k}\| = 0$. Due to the sequence $\{x_{m_k}\}$ is bounded, there exists a subsequence $\{x_{m_{k_j}}\}$ of $\{z_{m_k}\}$ such that $x_{m_{k_j}} \rightharpoonup z$. Thus, by using (3.18) and Lemma 3.4, we have $z \in \mathrm{VI}(C, A)$. Using an analysis similar to Case 1, we can get $z \in \mathrm{F}(U) \cap \mathrm{VI}(C, A)$. According to (3.11), one has

$$2\mu\theta_{m_{k_j}} \langle x_{m_{k_j}+1} - p, Gz_{m_{k_j}}\rangle \leq a_{m_{k_j}} - a_{m_{k_j}+1}\,,$$

and hence

$$\langle x_{m_{k_j}+1} - p, Gz_{m_{k_j}}\rangle \leq 0\,. \tag{3.19}$$

Using G is η-strongly monotone and (3.19), one gets

$$\begin{aligned}\eta\|z_{m_{k_j}}-p\|^2 &\le \langle z_{m_{k_j}}-p, Gz_{m_{k_j}}-Gp\rangle \\ &= \langle x_{m_{k_j}+1}-p, Gz_{m_{k_j}}\rangle + \langle z_{m_{k_j}}-x_{m_{k_j}+1}, Gz_{m_{k_j}}\rangle \\ &\quad - \langle z_{m_{k_j}}-p, Gp\rangle \\ &\le \langle z_{m_{k_j}}-x_{m_{k_j}+1}, Gz_{m_{k_j}}\rangle - \langle z_{m_{k_j}}-p, Gp\rangle\,. \end{aligned} \tag{3.20}$$

Taking the limit in (3.20) as $j \to \infty$, which, together with (3.18) and (3.19) yields

$$\limsup_{j\to\infty} \|z_{m_{k_j}}-p\|^2 \le -\frac{1}{\eta}\langle z-p, Gp\rangle \le 0\,.$$

Hence, $\lim_{j\to\infty}\|z_{m_{k_j}}-p\| = 0$ and thus $\lim_{k\to\infty}\|z_{m_k}-p\| = 0$. This combining with (3.18) implies that $\|x_{m_k+1}-p\| \to 0$, that is, $\lim_{k\to\infty} a_{m_k+1} = 0$. In view of the fact $a_n \le a_{m_k+1}$ in (3.17), we can infer that $\lim_{n\to\infty} a_n = 0$. The proof is completed. ☐

3.2. **Second algorithm.** Here we present our second algorithm.

Algorithm 2

Initialization: Take $\lambda_1 > 0$, $\sigma \in (0,1)$ and $\mu \in \left(0, \frac{2\eta}{k^2}\right)$. Let $x_1 \in \mathcal{H}$ be arbitrary.
Iterative Steps: Calculate x_{n+1} as follows:

$$\begin{cases} y_n = P_C\left(x_n - \lambda_n A x_n\right)\,, \\ z_n = y_n - \lambda_n\left(Ay_n - Ax_n\right)\,, \\ q_n = \left(I - \mu\theta_n G\right) z_n\,, \\ x_{n+1} = \left(1-\gamma_n\right) q_n + \gamma_n U q_n\,, \end{cases}$$

where the step size λ_n is defined in (3.1).

The following lemma is very helpful for analyzing the convergence of Algorithm 2.

Lemma 3.6. *Assume that Conditions (C1) and (C3) hold. Let $\{y_n\}$ and $\{z_n\}$ be two sequences created by Algorithm 2. Then,*

$$\|z_n-p\|^2 \le \|x_n-p\|^2 - \left(1-\sigma^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|x_n-y_n\|^2\,, \quad \forall p \in \mathrm{VI}(C,A)\,,$$

and

$$\|z_n-y_n\| \le \frac{\sigma\lambda_n}{\lambda_{n+1}}\|x_n-y_n\|\,.$$

Proof. From the definition of λ_n, one obtains

$$\|Ax_n - Ay_n\| \le \frac{\sigma}{\lambda_{n+1}}\|x_n-y_n\|\,, \quad \forall n\,. \tag{3.21}$$

By the definition of z_n, one sees that

$$\begin{aligned}
\|z_n - p\|^2 &= \|y_n - p\|^2 + \lambda_n^2 \|Ay_n - Ax_n\|^2 - 2\lambda_n \langle y_n - p, Ay_n - Ax_n\rangle \\
&= \|x_n - p\|^2 + \|y_n - x_n\|^2 - 2\langle y_n - x_n, y_n - x_n\rangle \\
&\quad + 2\langle y_n - x_n, y_n - p\rangle + \lambda_n^2 \|Ay_n - Ax_n\|^2 \\
&\quad - 2\lambda_n \langle y_n - p, Ay_n - Ax_n\rangle \\
&= \|x_n - p\|^2 - \|y_n - x_n\|^2 + 2\langle y_n - x_n, y_n - p\rangle \\
&\quad + \lambda_n^2 \|Ay_n - Ax_n\|^2 - 2\lambda_n \langle y_n - p, Ay_n - Ax_n\rangle \,.
\end{aligned} \tag{3.22}$$

Using $y_n = P_C(x_n - \lambda_n Ax_n)$ and (2.3), we infer that

$$\langle y_n - x_n + \lambda_n Ax_n, y_n - p\rangle \le 0\,,$$

or equivalently

$$\langle y_n - x_n, y_n - p\rangle \le -\lambda_n \langle Ax_n, y_n - p\rangle \,. \tag{3.23}$$

From (3.21), (3.22) and (3.23), we have

$$\begin{aligned}
\|z_n - p\|^2 &\le \|x_n - p\|^2 - \|y_n - x_n\|^2 - 2\lambda_n \langle Ax_n, y_n - p\rangle \\
&\quad + \sigma^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} \|x_n - y_n\|^2 - 2\lambda_n \langle y_n - p, Ay_n - Ax_n\rangle \\
&\le \|x_n - p\|^2 - \left(1 - \sigma^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|x_n - y_n\|^2 - 2\lambda_n \langle y_n - p, Ay_n\rangle \,.
\end{aligned} \tag{3.24}$$

Since $p \in \mathrm{VI}(C, A)$, one gets $\langle Ap, y_n - p\rangle \ge 0$, which together with the pseudomonotonicity of A, implies that $\langle Ay_n, y_n - p\rangle \ge 0$. In view of (3.24), one obtains

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \left(1 - \sigma^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|x_n - y_n\|^2 \,.$$

From the definition of z_n and (3.21), we have

$$\|z_n - y_n\| \le \frac{\sigma\lambda_n}{\lambda_{n+1}} \|x_n - y_n\| \,.$$

This completes the proof of Lemma 3.6. □

Theorem 3.7. *Assume that Conditions (C1)–(C5) hold. Then the sequence $\{x_n\}$ formed by Algorithm 2 converges to an element p in norm, where $p = P_{\mathrm{F}(U)\cap \mathrm{VI}(C,A)}(I - \mu G)p$.*

Proof. By Lemma 3.1, there exists $n_1 \in \mathbb{N}$ such that $1 - \sigma^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} > 0$, $\forall n \ge n_1$. From Lemma 3.6, one gets $\|z_n - p\| \le \|x_n - p\|$, $\forall n \ge n_1$. Using the same arguments as Claim 1 in Theorem 3.5, we get that the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{q_n\}$ are bounded. From (3.11), the definition of q_n and Lemma 3.8, we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|z_n - p\|^2 - 2\mu\theta_n \langle x_{n+1} - p, Gz_n\rangle - \|x_{n+1} - z_n\|^2 \\
&\le \|x_n - p\|^2 - \left(1 - \sigma^2 \frac{\lambda_{n^2}}{\lambda_{n+1}^2}\right) \|y_n - x_n\|^2 \\
&\quad - \|x_{n+1} - z_n\|^2 + \theta_n M_1,\ \forall n \ge n_1 \,,
\end{aligned}$$

where M_1 is defined in Claim 2 of Theorem 3.5. Finally, combining Lemma 3.6 and Claim 3 in Theorem 3.5, we can easily obtain the desired conclusion. □

3.3. **Third algorithm.** Finally, our third algorithm is presented.

Algorithm 3

Initialization: Take $\lambda_1 > 0$, $\sigma \in (0,1)$, $\phi \in (0,2)$ and $\mu \in \left(0, \frac{2\eta}{k^2}\right)$. Let $x_1 \in \mathcal{H}$ be arbitrary.

Iterative Steps: Calculate x_{n+1} as follows:

$$\begin{cases} y_n = P_C\left(x_n - \lambda_n A x_n\right) , \\ z_n = x_n - \phi\delta_n d_n , \\ q_n = \left(I - \mu\theta_n G\right) z_n , \\ x_{n+1} = \left(1-\gamma_n\right) q_n + \gamma_n U q_n , \end{cases}$$

where the step size λ_n is defined in (3.1), d_n and δ_n are generated by the following

$$d_n := x_n - y_n - \lambda_n\left(Ax_n - Ay_n\right) , \quad \delta_n := \begin{cases} \frac{\langle x_n - y_n, d_n\rangle}{\|d_n\|^2}, & \text{if } d_n \neq 0; \\ 0, & \text{otherwize.} \end{cases} \tag{3.25}$$

Lemma 3.8. *Assume that Conditions (C1) and (C3) hold. Let $\{y_n\}$ and $\{z_n\}$ be two sequences formed by Algorithm 3. Then,*

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \frac{2-\phi}{\phi}\|x_n - z_n\|^2, \quad \forall p \in \mathrm{VI}(C, A) ,$$

and

$$\|x_n - y_n\|^2 \le \frac{\left(1 + \frac{\sigma\lambda_n}{\lambda_{n+1}}\right)^2}{\left[(1 - \frac{\sigma\lambda_n}{\lambda_{n+1}})\phi\right]^2}\|x_n - z_n\|^2 .$$

Proof. If there exists $n_2 \ge 1$ such that $d_{n_2} = 0$, then $z_{n_2} = x_{n_2}$ and the first inequality holds. Now, we consider $d_n \neq 0$ for each $n \ge 1$. By using the definition of z_n, one obtains

$$\begin{aligned} \|z_n - p\|^2 &= \|x_n - \phi\delta_n d_n - p\|^2 \\ &= \|x_n - p\|^2 - 2\phi\delta_n\langle x_n - p, d_n\rangle + \phi^2\delta_n^2\|d_n\|^2 . \end{aligned} \tag{3.26}$$

According to the definition of d_n, one sees that

$$\begin{aligned} \langle x_n - p, d_n\rangle &= \langle x_n - y_n, d_n\rangle + \langle y_n - p, d_n\rangle \\ &= \langle x_n - y_n, d_n\rangle + \langle y_n - p, x_n - y_n - \lambda_n(Ax_n - Ay_n)\rangle . \end{aligned} \tag{3.27}$$

From $y_n = P_C(x_n - \lambda_n A x_n)$ and the property of projection (2.3), we have

$$\langle x_n - y_n - \lambda_n A x_n, y_n - p\rangle \ge 0 . \tag{3.28}$$

Using $p \in \mathrm{VI}(C, A)$ and $y_n \in C$, we get that $\langle Ap, y_n - p\rangle \ge 0$, which combining with the pseudomonotonicity of A yields that

$$\lambda_n\langle Ay_n, y_n - p\rangle \ge 0 . \tag{3.29}$$

By using (3.27), (3.28) and (3.29), we obtain

$$\langle x_n - p, d_n\rangle \geq \langle x_n - y_n, d_n\rangle . \tag{3.30}$$

It follows from the definition of z_n that $z_n - x_n = \phi\delta_n d_n$. Combining (3.26), (3.30) and the definition of δ_n, we conclude that

$$\begin{aligned}\|z_n - p\|^2 &\leq \|x_n - p\|^2 - 2\phi\delta_n\langle x_n - y_n, d_n\rangle + \phi^2\delta_n^2\|d_n\|^2 \\ &= \|x_n - p\|^2 - 2\phi\delta_n^2\|d_n\|^2 + \phi^2\delta_n^2\|d_n\|^2 \\ &= \|x_n - p\|^2 - \frac{2-\phi}{\phi}\|\phi\delta_n d_n\|^2 \\ &= \|x_n - p\|^2 - \frac{2-\phi}{\phi}\|x_n - z_n\|^2 .\end{aligned}$$

According to the definition of (3.1), one has $\|Ax_n - Ay_n\| \leq (\sigma/\lambda_{n+1})\|x_n - y_n\|$, $\forall n$, which combining with the definition of δ_n yields that

$$\begin{aligned}\delta_n\|d_n\|^2 = \langle d_n, x_n - y_n\rangle &\geq \|x_n - y_n\|^2 - \lambda_n\|Ax_n - Ay_n\|\|x_n - y_n\| \\ &\geq \left(1 - \frac{\sigma\lambda_n}{\lambda_{n+1}}\right)\|x_n - y_n\|^2 .\end{aligned} \tag{3.31}$$

Using the definition of d_n and (3.1), we get

$$\begin{aligned}\|d_n\| &\leq \|x_n - y_n\| + \lambda_n\|Ax_n - Ay_n\| \leq \|x_n - y_n\| + \frac{\sigma\lambda_n}{\lambda_{n+1}}\|x_n - y_n\| \\ &= \left(1 + \frac{\sigma\lambda_n}{\lambda_{n+1}}\right)\|x_n - y_n\| .\end{aligned} \tag{3.32}$$

Combining (3.31) and (3.32), one obtains

$$\delta_n^2\|d_n\|^2 \geq \left(1 - \frac{\sigma\lambda_n}{\lambda_{n+1}}\right)^2\frac{\|x_n - y_n\|^4}{\|d_n\|^2} \geq \frac{\left(1 - \frac{\sigma\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1 + \frac{\sigma\lambda_n}{\lambda_{n+1}}\right)^2}\|x_n - y_n\|^2 . \tag{3.33}$$

By the definition of z_n and (3.33), we have

$$\|z_n - x_n\|^2 = \phi^2\delta_n^2\|d_n\|^2 \geq \phi^2\frac{\left(1 - \frac{\sigma\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1 + \frac{\sigma\lambda_n}{\lambda_{n+1}}\right)^2}\|x_n - y_n\|^2 .$$

Thus, we get

$$\|x_n - y_n\|^2 \leq \frac{\left(1 + \frac{\sigma\lambda_n}{\lambda_{n+1}}\right)^2}{\left[\left(1 - \frac{\sigma\lambda_n}{\lambda_{n+1}}\right)\phi\right]^2}\|x_n - z_n\|^2 .$$

The proof of the lemma is now complete. □

Theorem 3.9. *Assume that Conditions (C1)–(C5) hold. Then the sequence $\{x_n\}$ created by Algorithm 3 converges to an element p in norm, where $p = P_{\mathrm{F}(U)\cap\mathrm{VI}(C,A)}(I-\mu G)p$.*

Proof. From Lemma 3.8 and $\phi \in (0,2)$, one has $\|z_n - p\| \leq \|x_n - p\|$, $\forall n \geq 1$. Using the same arguments as Claim 1 in Theorem 3.5, we get that the sequences

$\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{q_n\}$ are bounded. Combining (3.11), the definition of q_n and Lemma 3.8, we get

$$\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2 - \frac{2-\phi}{\phi}\|x_n-z_n\|^2 - \|x_{n+1}-z_n\|^2 + \theta_n M_1, \quad \forall n \geq 1\,,$$

where M_1 is defined in Claim 2 of Theorem 3.5. Finally, we show that the sequence $\{\|x_n-p\|\}$ converges to zero. The proof is similar to Claim 3 in Theorem 3.5, so we omit it here. □

Remark 3.10. Observe that if we choose $G(x) = x - f(x)$, where $f : \mathcal{H} \to \mathcal{H}$ is a contraction, all the proposed methods can be easily modified and the generated sequence converge in norm to a point $p \in \mathrm{F}(U) \cap \mathrm{VI}(C, A)$, where $p = P_{\mathrm{F}(U)\cap\mathrm{VI}(C,A)}(f(p))$.

4. Numerical examples

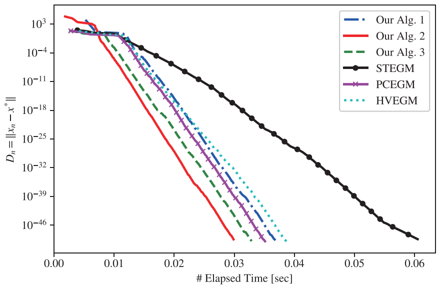
In this section, we present several mathematical examples in finite and infinite-dimensional spaces, demonstrating with comparison to related works, the behavior of our Algorithms 1–3.

In all algorithms, set $\theta_n = 1/n+1$ and $\gamma_n = n/(2n+1)$. For the suggested Algorithms 1–3, we choose $\lambda_1 = 0.5$, $\sigma = 0.5$, $\mu = 1$ and $\xi_n = 1/(n+1)^{1.1}$. Take $r = 0.5$, $l = 0.5$, $\sigma = 0.4$ and $\mu = 0.5$ in (STEGM). For (HVEGM) and (PCEGM), we select step size $\lambda = 0.5/L$ and step size $\lambda = 0.4/L$, respectively. Pick $\phi = 1$ in our Algorithm 3 and (PCEGM). All the programs were implemented in Matlab 2018a on a Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz computer with RAM 8.00 GB.

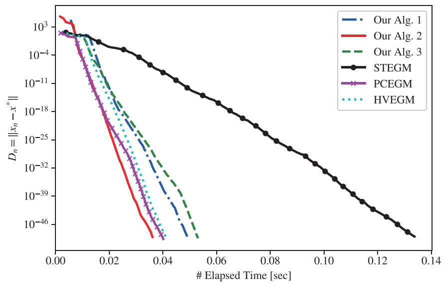
4.1. Theoretical examples.

Example 4.1. In first example, we consider the linear operator $A : \mathbb{R}^m \to \mathbb{R}^m$ ($m = 50, 100, 150, 200$) in the form $A(x) = Mx+q$, where $q \in \mathbb{R}^m$ and $M = NN^{\mathsf{T}}+Q+D$, N is a $m\times m$ matrix, Q is a $m\times m$ skew-symmetric matrix, and D is a $m\times m$ diagonal matrix with its diagonal entries being nonnegative (hence M is positive symmetric definite). The feasible set C is given by $C = \{x \in \mathbb{R}^m : -2 \leq x_i \leq 5,\, i = 1, \ldots, m\}$. It is clear that A is monotone and Lipschitz continuous with constant $L = \|M\|$. In this experiment, all entries of N, D are generated randomly in $[0, 2]$, Q is generated randomly in $[-2, 2]$ and $q = 0$. Let $U : H \to H$ and $G : H \to H$ be given by $Ux := \frac{1}{2}x$ and $Gx := \frac{1}{2}x$, respectively. It is easy to see that the solution of the problem in this case is $x^* = \{\mathbf{0}\}$. We use $D_n = \|x_n - x^*\|$ to measure the iteration error of all algorithms at the n-th step. The maximum iteration 400 as a common stopping criterion and the initial value x_1 is randomly generated by *20rand(m,1)* in Matlab. The numerical results of all algorithms in different dimensions are shown in Figure 1.

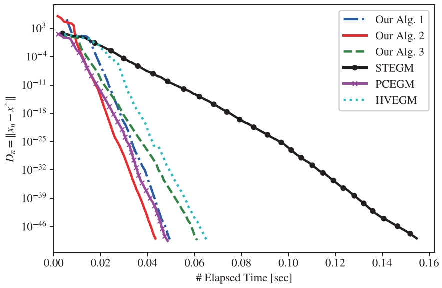
Example 4.2. In this example, we consider our problem in the infinite-dimensional Hilbert space $\mathcal{H} = L^2([0, 1])$ with inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)\,\mathrm{d}t, \forall x, y \in \mathcal{H}$ and norm $\|x\| := (\int_0^1 |x(t)|^2\,\mathrm{d}t)^{1/2}, \forall x \in \mathcal{H}$. Let the feasible set be the unit ball



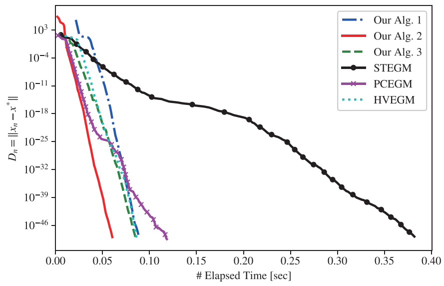
(a) $m = 50$



(b) $m = 100$



(c) $m = 150$



(d) $m = 200$

FIGURE 1. Numerical results of all the algorithms for Example 4.1

$C := \{x \in \mathcal{H} : \|x\| \le 1\}$. It should be noted that the projection on C is inherently explicit, that is,

$$P_C(x) = \begin{cases} \frac{x}{\|x\|}, & \text{if } \|x\| > 1; \\ x, & \text{if } \|x\| \le 1. \end{cases}$$

Let the mapping $U : L^2([0,1]) \to L^2([0,1])$ be created by

$$(Ux)(t) = \int_0^1 tx(s)\,\mathrm{d}s, \quad t \in [0,1]\,.$$

A straightforward computation indicates that U is 0-demi-contractive. Assume that the mapping $A : C \to \mathcal{H}$ is generated by the following

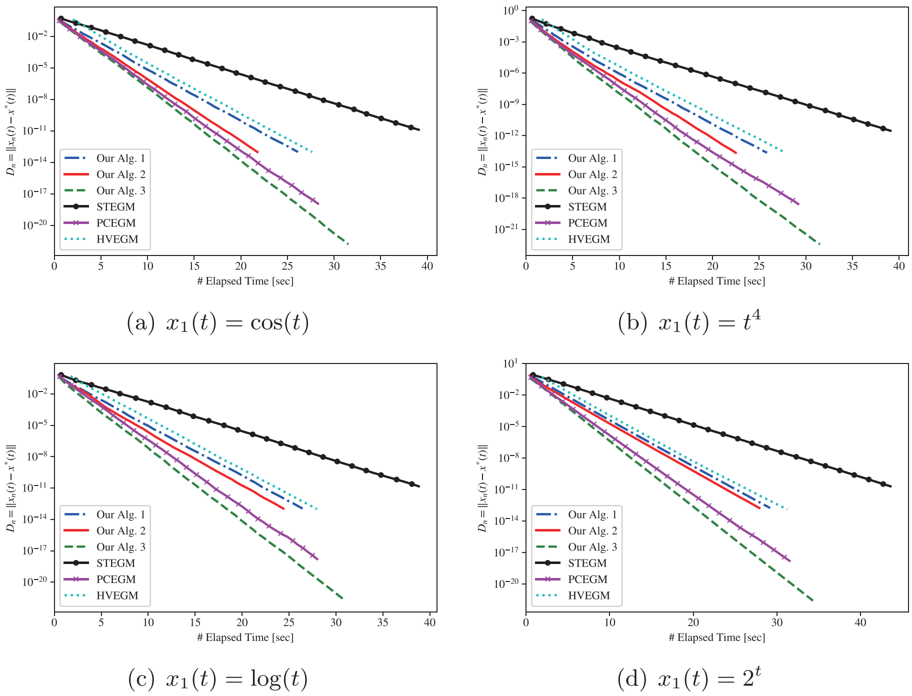
$$(Ax)(t) = \int_0^1 (x(t) - Q(t,s)g(x(s)))\,\mathrm{d}s + h(t), \quad t \in [0,1],\, x \in C,$$

where

$$Q(t,s) = \frac{2tse^{t+s}}{e\sqrt{\mathrm{e}^2 - 1}}\,, \quad g(x) = \cos x\,, \quad h(t) = \frac{2te^{t}}{e\sqrt{\mathrm{e}^2 - 1}}\,.$$

It is known that A is monotone and L-Lipschitz continuous with $L = 2$. Let $G : \mathcal{H} \to \mathcal{H}$ be an operator defined by $(Gx)(t) = \frac{1}{2}x(t), t \in [0,1]$. It is easy to see that operator G is $\frac{1}{2}$-strongly monotone and $\frac{1}{2}$-Lipschitz continuous. The solution of the problem is $x^*(t) = 0$. The maximum iteration 50 is used as the common stopping criterion. Figure 2 describes the numerical behavior $D_n = \|x_n(t) - x^*(t)\|$ of all the algorithms with four different initial values.

Remark 4.3. From Examples 4.1–4.2, we have the following observations:



(a) $x_1(t) = \cos(t)$

(b) $x_1(t) = t^4$

(c) $x_1(t) = \log(t)$

(d) $x_1(t) = 2^t$

FIGURE 2. Numerical results of all the algorithms for Example 4.2

(1) The proposed iterative schemes outperform the known ones and that these results are independent of the size of the dimensions and the choice of initial values. Thus, the algorithms obtained in this paper are efficient and robust.

(2) From Figs. 1, 2, it is easy to see that Algorithm (STEGM) demands more execution time than other algorithms to achieve the same accuracy because the Armijo-type criterion takes a lot of time to find a suitable step size in each iteration. Moreover, it should be pointed out that Algorithm (HVEGM) and Algorithm (PCEGM) will fail without knowing the prior information of the Lipschitz constant of the mapping due to the fact that they are fixed-step algorithms. However, the suggested iterative schemes can automatically update the step size by performing a simple calculation with previously known information, thus making them more useful and effective.

(3) Note that Example 4.2 is implemented in an infinite-dimensional space. The methods proposed in this paper acquire strong convergence in an infinite-dimensional Hilbert space, while the algorithms introduced in [10, 32, 33] only obtain weak convergence results. Furthermore, we also noticed that Algorithm (PCEGM) and Algorithm 3 may require more execution time in an infinite-dimensional space under performing the same number of iterations because they need to compute the values of d_n and δ_n. However, they can achieve higher accuracy than the other algorithms and take less time to achieve the same accuracy.

4.2. **Optimal control problems.** Setting $U = I$, $G(x) = x - f(x)$ and $\mu = 1$ in Algorithms 1–3, where $f : \mathcal{H} \to \mathcal{H}$ is a contraction, we can obtain three new iterative schemes for solving variational inequalities that appears in optimal control problems. We recommend readers to refer to [23, 37] for detailed description of the problem. In all algorithms, we set $N = 100$, $\theta_n = 10^{-4}/(n+1)$, $\lambda_1 = 0.4$, $\sigma = 0.1$, $\xi_n = 0.1/(n+1)^{1.1}$ and $f(x) = 0.1x$. Take $\phi = 1.5$ in Algorithm 3. The initial controls $p_0(t) = p_1(t)$ are randomly generated in $[-1, 1]$. The stopping criterion is either $D_n = \|p_{n+1} - p_n\| \le 10^{-4}$, or maximum number of iterations which is set to 1000.

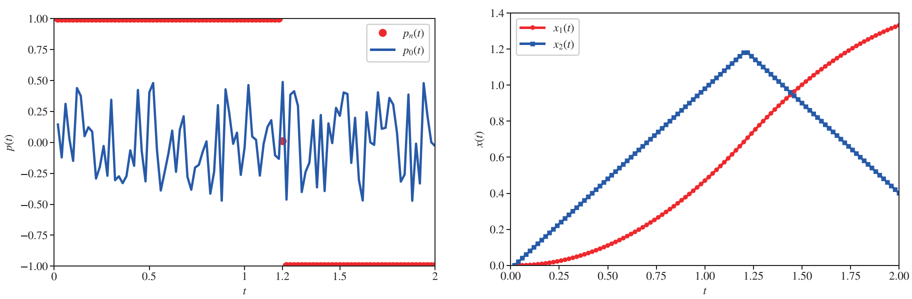
Example 4.4 (Control of a harmonic oscillator, see [24])**.**

$$
\begin{aligned}
\text{minimize}\quad & x_2(3\pi) \\
\text{subject to}\quad & \dot{x}_1(t) = x_2(t)\,, \\
& \dot{x}_2(t) = -x_1(t) + p(t), \quad \forall t \in [0, 3\pi]\,, \\
& x(0) = 0\,, \\
& p(t) \in [-1, 1]\,.
\end{aligned}
$$

The exact optimal control of Example 4.4 is known:

$$
p^*(t) = \begin{cases} 1, & \text{if } \ t \in [0, \pi/2) \cup (3\pi/2, 5\pi/2)\,; \\ -1, & \text{if } \ t \in (\pi/2, 3\pi/2) \cup (5\pi/2, 3\pi]\,. \end{cases}
$$

Figure 3 shows the approximate optimal control and the corresponding trajectories of the stated Algorithm 1.



(a) Initial and optimal controls (b) Optimal trajectories

FIGURE 3. Numerical results of the proposed Algorithm 1 for Example 4.4

We now consider an example in which the terminal function is not linear.

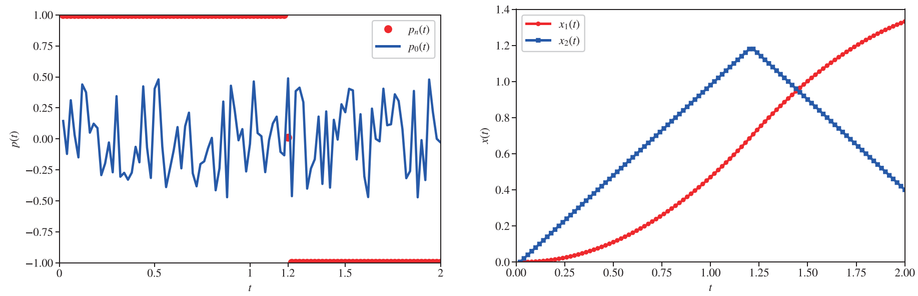
Example 4.5 (see [3])**.**

$$
\begin{aligned}
\text{minimize}\quad & -x_1(2) + (x_2(2))^2\,, \\
\text{subject to}\quad & \dot{x}_1(t) = x_2(t)\,, \\
& \dot{x}_2(t) = p(t), \quad \forall t \in [0, 2]\,, \\
& x_1(0) = 0, \quad x_2(0) = 0\,, \\
& p(t) \in [-1, 1]\,.
\end{aligned}
$$

The exact optimal control of Example 4.5 is

$$p^*(t) = \begin{cases} 1, & \text{if } t \in [0, 1.2)\,; \\ -1, & \text{if } t \in (1.2, 2]\,. \end{cases}$$

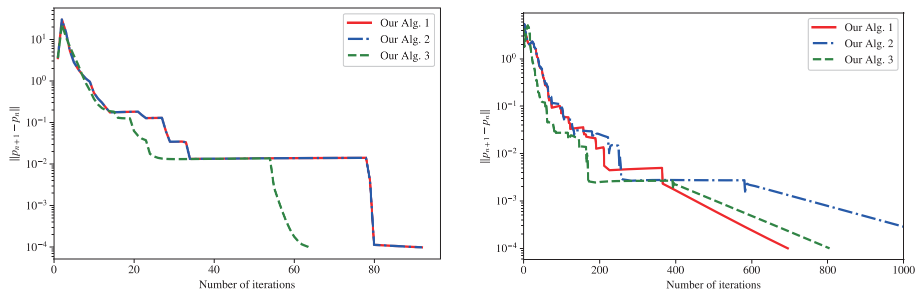
The approximate optimal control and the corresponding trajectories of the suggested Algorithm 3 are plotted in Figure 4.



(a) Initial and optimal controls (b) Optimal trajectories

FIGURE 4. Numerical results of the proposed Algorithm 3 for Example 4.5

Finally, we state the numerical performance of all the algorithms in Examples 4.4–4.5 in Figure 5 and Table 1. Figure 5 presents the numerical behavior of the error estimate $\|p_{n+1} - p_n\|$ with respect to the number of iterations for all the algorithms. In addition, the number of terminated iterations and the execution time of all the algorithms are shown in Table 1.



(a) Example 4.4 (b) Example 4.5

FIGURE 5. Error estimates of all the algorithms for Examples 4.4–4.5

Remark 4.6. We draw the following observations from Examples 4.4–4.5.

(i) The suggested algorithms can be applied to solve optimal control problems, and they perform well when the terminal function is linear or nonlinear.

(ii) As shown in Figure 5 and Table 1, the proposed algorithms perform better when the terminal function is linear than when it is nonlinear, i.e., it requires less execution time and the number of termination iterations in the case where the terminal function is linear.

Table 1. Numerical results of all the algorithms for Examples 4.4–4.5

Algorithms	Example 4.4			Example 4.5		
	Iter.	Time (s)	D_n	Iter.	Time (s)	D_n
Algorithm 1	91	0.064742	9.89E-05	694	0.30101	9.99E-05
Algorithm 2	91	0.051085	9.89E-05	1000	0.33792	2.84E-04
Algorithm 3	63	0.037186	9.80E-05	804	0.29271	9.96E-05

5. Conclusions

In this paper, we developed three new self-adaptive methods to discover common solutions of the variational inequality problem and the fixed point problem in real Hilbert spaces. Our results improve and extend some known related results in the literature [10,15,22,32,33,36] and thus have wider applications. To be more specific we list the advantages next.

(1) Algorithms 1–3 converge strongly in infinite-dimensional Hilbert spaces, improving for example [10,32,33].
(2) In our iterative methods, the monotonicity of mapping A in [15,22,36] is replaced by pseudomonotonicity, and the quasi-nonexpansive property of mapping U in [36] is replaced by the demi-contractive property.
(3) Notice that the algorithms of Maingé [22] and Gibali and Shehu [15] are fixed-step algorithms, i.e., the update of the step size requires advance information about the Lipschitz constant of the mapping. The algorithm of Tong and Tian [36] applies an Armijo-type criterion to update the step size, which increases its computational burden by spending a lot of computation in each iteration to find a suitable step size. However, the step size of the proposed iterative schemes can be updated adaptively without any line search process. Therefore, our algorithms are more practical and efficient.
(4) It should be emphasized that the proposed methods require only one projection onto the feasible set in each iteration, whereas Algorithm (HVEGM) offered by Maingé [22] needs two projections onto the feasible set.
(5) If $U = I$ in Algorithms 1–3, we obtain three new adaptive iterative schemes to solve (VI). Thus, the results gained in this paper enrich and improve some of the existing results of solving (VI) and (VIFPP) in the literature.

Numerical experiments emerging in finite- and infinite-dimensional spaces show the advantages and efficiency of our algorithms over the existing ones.

References

[1] E. Bonacker, A. Gibali and K. H. Küfer, *Nesterov perturbations and projection methods applied to IMRT*, J. Nonlinear Var. Anal. **4** (2020), 63–86.
[2] H. H. Bauschke, P.L. Combettes, *A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces*, Math. Oper. Res. **26** (2001), 248–264.
[3] B. Bressan and B. Piccoli, *Introduction to the mathematical theory of control*, AIMS series on applied mathematics, 2007.
[4] P. Cubiotti and J. C. Yao, *On the Cauchy problem for a class of differential inclusions with applications*, Appl. Anal. **99** (2020), 2543–2554.

[5] T. H. Cuong, J. C. Yao and N. D. Yen, *Qualitative properties of the minimum sum-of-squares clustering problem*, Optimization **69** (2020), 2131–2154.

[6] S. Y. Cho, *A monotone Bregman projection algorithm for fixed point and equilibrium problems in a reflexive Banach space*, Filomat **34** (2020), 1487–1497.

[7] S. Y. Cho, *Strong convergence analysis of a hybrid algorithm for nonlinear operators in a Banach space*, J. Appl. Anal. Comput. **8** (2018), 19–31.

[8] Y. Censor, A. Gibali and S. Reich, *Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space*, Optim. Methods Softw. **26** (2011), 827–845.

[9] Y. Censor, A. Gibali and S. Reich, *Extensions of Korpelevich's extragradient method for solving the variational inequality problem in Euclidean space*, Optimization **61** (2012), 1119–1132.

[10] Y. Censor, A. Gibali and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl. **148** (2011), 318–335.

[11] L. C. Ceng, *Two inertial linesearch extragradient algorithms for the bilevel split pseudomonotone variational inequality with constraints*, J. Appl. Numer. Optim. **2** (2020), 213–233.

[12] G. Fichera, *Sul problema elastostatico di Signorini con ambigue condizioni al contorno*, Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat. **34** (1963), 138–142.

[13] G. Fichera, *Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno*, Atti Accad. Naz. Lincei, Mem., Cl. Sci. Fis. Mat. Nat., Sez. I, VIII. Ser. **7** (1964), 91–140.

[14] A. Gibali and D. V. Hieu, *A new inertial double-projection method for solving variational inequalities*, J. Fixed Point Theory Appl. **21** (2019), Article ID 97.

[15] A. Gibali and Y. Shehu, *An efficient iterative method for finding common fixed point and variational inequalities in Hilbert spaces*, Optimization **68** (2019), 13–32.

[16] G. Cai, Q.-L. Dong and Y. Peng, *Strong convergence theorems for inertial Tseng's extragradient method for solving variational inequality problems and fixed point problems*, Optim. Lett. **15** (2021), 1457–1474.

[17] B. S. He, *A class of projection and contraction methods for monotone variational inequalities*, Appl. Math. Optim. **35** (1997), 69–76.

[18] G. M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Ekonomikai Matematicheskie Metody. **12** (1976), 747–756.

[19] R. Kraikaew and S. Saejung, *Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces*, J. Optim. Theory Appl. **163** (2014), 399–412.

[20] L. Liu, *A hybrid steepest descent method for solving split feasibility problems involving nonexpansive mappings*, J. Nonlinear Convex Anal. **20** (2019), 471–488.

[21] H. Liu and J. Yang, *Weak convergence of iterative methods for solving quasimonotone variational inequalities*, Comput. Optim. Appl. **77** (2020), 491–508.

[22] P. E. Maingé, *A hybrid extragradient-viscosity method for monotone operators and fixed point problems*, SIAM J. Control Optim. **47** (2008), 1499–1515.

[23] J. Preininger and P.T. Vuong, *On the convergence of the gradient projection method for convex optimal control problems with bang-bang solutions*, Comput. Optim. Appl. **70** (2018), 221–238.

[24] A. Pietrus, T. Scarinci and V. M. Veliov, *High order discrete approximations to Mayer's problems for linear systems*, SIAM J. Control Optim. **56** (2018), 102–119.

[25] G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convexes.* C. R. Acad. Sci., Paris **258** (1964), 4413–4416.

[26] Y. Shehu, X. H. Li and Q.-L. Dong, *An efficient projection-type method for monotone variational inequalities in Hilbert spaces*, Numer. Algorithms **84** (2020), 365–388.

[27] Y. Shehu and O. S. Iyiola, *Projection methods with alternating inertial steps for variational inequalities: Weak and linear convergence*, Appl. Numer. Math. **157** (2020), 315–337.

[28] Y. Shehu, Q.-L. Dong and D. Jiang, *Single projection method for pseudo-monotone variational inequality in Hilbert spaces*, Optimization **68** (2018), 385–409.

[29] D. R. Sahu, J.C. Yao, M. Verma and K.K. Shukla, *Convergence rate analysis of proximal gradient methods with applications to composite minimization problems*, Optimization **70** (2021), 75–100.

[30] B. Tan, X. Qin and J.C. Yao, *Strong convergence of self-adaptive inertial algorithms for solving split variational inclusion problems with applications*, J. Sci. Comput. **87** (2021): Article ID 20.

[31] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim. **38** (2000), 431–446.

[32] D. V. Thong and D.V. Hieu, *Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems*, Numer. Algorithm **80** (2019), 1283–1307.

[33] D. V. Thong and D.V. Hieu, *Mann-type algorithms for variational inequality problems and fixed point problems*, Optimization **69** (2020), 2305–2326.

[34] D. V. Thong and D.V. Hieu, *Modified subgradient extragradient algorithms for variational inequality problems and fixed point problems*, Optimization **67** (2018), 83–102.

[35] D. V. Thong and D.V. Hieu, *Some extragradient-viscosity algorithms for solving variational inequality problems and fixed point problems*, Numer. Algorithms **82** (2019), 761–789.

[36] M. Y. Tong and M. Tian, *Strong convergence of the Tseng extragradient method for solving variational inequalities*, Appl. Set-Valued Anal. Optim. **2** (2020), 19–33.

[37] P. T. Vuong and Y. Shehu, *Convergence of an extragradient-type method for variational inequality with applications to optimal control problems*, Numer. Algorithms **81** (2019), 269–291.

[38] I. Yamada, *The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings*, Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications. **8** (2001), 473–504.

[39] H. Zhou and X. Qin, *Fixed Points of Nonlinear Operators: Iterative Methods*, De Gruyter, Berlin, 2020.

Manuscript received March 9 2021
revised July 6 2021

B. Tan
Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu, China;
School of Sciences, Southwest Petroleum University, Chengdu 610500, China
E-mail address: bingtan72@gmail.com

A. Gibali
Department of Mathematics, ORT Braude College, Karmiel, Israel;,
The Center for Mathematics and Scientific Computation, U. Haifa, Mt. Carmel, Haifa, Israel;
Braude College of Engineering, P.O. Box 78, Karmiel 2161002, Israel
E-mail address: avivg@braude.ac.il

X. Qin
Department of Mathematics, Zhejiang Normal University, Zhejiang, China
E-mail address: qxlxajh@163.com