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Research Paper

Extragradient algorithms for solving equilibrium problems on Hadamard manifolds

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ABSTRACT

In this paper, we introduce three adaptive extragradient-based algorithms for solving equilibrium problems in Hadamard manifolds. The proposed algorithms can work adaptively without requiring the prior information about the Lipschitz constants of the bifunctions involved. Moreover, the iterative sequences generated by the suggested algorithms converge to the solutions of the equilibrium problems when the bifunctions are pseudomonotone and Lipschitz continuous. We also establish the global error bounds and *R*-linear convergence rates of the proposed algorithms in the case that the bifunctions involved are strongly pseudomonotone. Finally, a fundamental numerical example is given to illustrate the theoretical findings.

1. Introduction

The goal of this paper is to introduce several adaptive extragradient type algorithms for solving an equilibrium problem in the setting of Hadamard manifolds. The equilibrium problem, which is known as a unified framework for solving many problems, has been applied to various fields, such as operations research, economics, transportation regulation, optimal control problems and so on; see, e.g., [1–3]. In the last few decades, variational inequality problems and their algorithms in finite- and infinite-dimensional spaces were investigated extensively (see [4–9] and the extensive references therein). However, numerous problems in applied sciences are considered in nonlinear rather than linear spaces, for example, image processing and medical imaging problems on Riemannian manifolds (see, e.g., [10–13]). In addition, the extension of numerical optimization algorithms from Euclidean spaces to Riemannian manifolds also has significant advantages (see [13, Section 1] for more details). For example, it is possible to convert a non-convex (resp., non-monotone) optimization problem to a convex (resp., monotone) optimization problem by introducing a suitable Riemannian metric on a Riemannian manifold; see [14, Section 4] for examples and more information. Therefore, it is necessary to construct equilibrium problems and their algorithms in the context of manifolds.

A complete simply connected Riemannian manifold with nonpositive sectional curvature is called a Hadamard manifold, which has some remarkable properties (see Section 2 for more details) and therefore attracted the interest of scholars. Some examples on Hadamard manifolds associated with optimization problems can be found in [15, Section 1]. In recent decades, many authors paid considerable interest and studied the optimization problems and solution methods on Hadamard manifolds. They constructed

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a number of algorithms for solving variational inequalities [16–20], equilibrium problems [21–27], inclusion problems [28–30], fixed point problems [31–34], and others. Our focus in this paper is on numerical optimization algorithms for solving equilibrium problems on Hadamard manifolds. Throughout this paper, we consider the equilibrium problem (shortly, EP) in Hadamard manifolds introduced by Colao et al. [21]. Let us state the mathematical form of the problem. Assume that $\mathcal{M}$ denotes a Hadamard manifold and C is a nonempty, closed, and convex subset of $\mathcal{M}$. Let $f : C \times C \to \mathbb{R}$ be a real-valued bifunction such that $f(x,x) = 0$ for all $x \in C$. The EP associated with the bifunction f and the feasible set C is stated as follows

$$\text{find } s^* \in C \text{ such that } f(s^*, y) \geq 0 \ (\forall y \in C). \tag{1.1}$$

In the whole paper, we always suppose that the solution set of EP (1.1) is nonempty. It should be noted that the EP (1.1) is a generalization of the classical equilibrium problem on linear Euclidean spaces. Recently, extragradient-based algorithms as explicit iterative schemes have attracted research interest from scholars. Next, we recall some useful results on extragradient-based algorithms in linear spaces, which help us to develop new algorithms on Hadamard manifolds. The projection-based extragradient algorithm, introduced by Korpelevich [35], is now known as an effective tool for solving variational inequalities, equilibrium problems, and other optimization problems in the setting of linear and nonlinear spaces. In 2008, Quoc et al. [36] extended the extragradient algorithm to solve equilibrium problems in Euclidean spaces. However, the fixed step size of Algorithm 1 proposed by Quoc et al. [36] needs to satisfy a Lipschitz-type condition, which may limit the applicability of this fixed step size algorithm in practical applications. To overcome this difficulty, they used an Armijo-type line search step size criterion for their algorithm. It is worth noting that the use of Armijo-type step size criterion allows the algorithm to work adaptively while greatly increasing the computational burden of the algorithm. Recently, Hieu et al. [37] introduced two extragradient algorithms to solve equilibrium problems in Hilbert spaces, which use an adaptive step size criterion that does not involve any line search process to speed up the computational efficiency of the algorithms. Note that the algorithms in [36,37] need to compute the strongly convex optimization problem on the feasible set twice in each iteration. In order to reduce the number of computations of the strongly convex optimization problem in each iteration and improve the computational performance of the algorithm, Hoai et al. [38] extended the golden ratio algorithm introduced by Malitsky [39] for solving variational inequality problems to equilibrium problems. The Algorithms 3.1 and 4.1 proposed by Hoai et al. [38] require computing the strongly convex optimization problem only once in each iteration. However, the step size of their proposed Algorithm 3.1 is related to the prior knowledge of the Lipschitz constant of the bifunction f, and the proposed Algorithm 4.1 uses a non-summable sequence of step sizes. The use of both types of step sizes affects the applicability and computational efficiency of their algorithms. Recently, Yin et al. [40] presented a modified golden ratio algorithm with adaptive step sizes for solving equilibrium problems in Hilbert spaces. Their algorithm uses a non-monotonic step size criterion that can be updated with a simple calculation using some previous information.

It should be mentioned that Hadamard manifolds generally do not have a linear structure, which indicates that properties, techniques as well as algorithms in linear spaces are not available in Hadamard manifolds. Therefore, it is valuable and interesting to generalize algorithms for equilibrium problems from linear spaces to Hadamard manifolds. In recent years, researchers proposed some variant forms of extragradient-based methods to solve equilibrium problems in Hadamard manifolds. Next, we state some of the related results. In [21], Colao et al. proved the existence of solutions to equilibrium problems on Hadamard manifolds when the bifunction satisfies some suitable conditions. Subsequently, Cruz Neto et al. [22] developed an extragradient algorithm with a fixed step size for finding the solutions of the equilibrium problem (1.1) on Hadamard manifolds and proved the global convergence of the algorithm. Inspired by the works of Cruz Neto et al. [22] and Hieu [4,5], Khammahawong et al. [23] provided two extragradient methods with non-increasing and non-summable step sizes to solve equilibrium problems in the framework of Hadamard manifolds. Their algorithms directly extend the results obtained by Hieu [4,5] in Hilbert spaces to Hadamard manifolds. In the case that the bifunction involved is strongly pseudomonotone, they proved that the sequences generated by their proposed algorithms converge to the solution of EP (1.1) under some mild conditions. Motivated by the extragradient algorithms for solving equilibrium problems in Hilbert spaces presented by Hieu et al. [37], Ansari et al. [24] introduced two adaptive explicit extragradient algorithms to find the solutions of EP (1.1) in Hadamard manifolds. Their proposed algorithms apply a simple adaptive step size criterion that can be updated by using known information from previous iterations. This allows these algorithms to perform without the prior knowledge of the Lipschitz constant of the bifunctions involved. Moreover, they demonstrated that the iterative sequences generated by the suggested algorithms converge to the solution of EP (1.1) under the condition that the bifunctions are pseudomonotone, and obtained the linear convergence of the proposed algorithms in the case that the bifunctions are strongly pseudomonotone. Notice that the algorithms in [22–24] involve the computation of the strongly convex optimization problem twice in each iteration. Chen et al. [25] presented a modified golden ratio algorithm that requires the computation of the strongly convex optimization problem only once in each iteration for solving equilibrium problems in Hadamard manifolds. Furthermore, the computational advantage and efficiency of the algorithms proposed by Chen et al. [25] compared with the ones in [22,23] were demonstrated by some numerical experiments. Quite recently, Iusem and Mohebbi [26] and Babu et al. [27] suggested several extragradient algorithms with an Armijo-type step size criterion to discover the solutions of equilibrium problems in Hadamard manifolds. However, their methods are computationally expensive because the strongly convex optimization problem may require to be computed many times in each iteration.

In this paper, inspired and motivated by the above results (in particular [22,24,37,40]), we investigate two adaptive extragradient algorithms and a modified golden ratio algorithm for solving equilibrium problems in Hadamard manifolds. The suggested algorithms use a non-monotonic adaptive step size criterion that does not involve any line search procedure allowing them to solve EPs consisting of pseudomonotone and Lipschitz continuous bifunctions. The sequences generated by the proposed algorithms converge to the solutions of the EP under some mild conditions. In addition, the global error bounds and linear convergence results of the proposed algorithms are established under the condition that the bifunctions involved are strongly pseudomonotone. Finally, we also provide a fundamental numerical example in Nash-Cournot oligopolistic equilibrium model to illustrate the theoretical results of this paper. The algorithms developed in this paper improve the results previously obtained in [22–27] for dealing with EPs on Hadamard manifolds.

The remainder of this paper is organized as follows. In Section 2, we recall some important notations, definitions, properties, and lemmas in Riemannian geometry for subsequent use. In Section 3, we present three adaptive explicit algorithms for solving EPs incorporating pseudomonotone and Lipschitz continuous bifunctions on Hadamard manifolds and analyze their convergence. In Section 4, we show the global error bounds and R-linear convergence of the proposed algorithms when the bifunctions involved are strongly pseudomonotone. A basic computational test occurring on Hadamard manifolds is provided in Section 5 to demonstrate the convergence efficiency of our algorithms. In the last section, we conclude the paper and give an outlook on future work.

2. Theoretical framework

The goal of this section is to state some classical representations and results in Hadamard manifolds, which are necessary to understand the content of this paper. Therefore, we shall introduce some important concepts, definitions, properties, and lemmas in Riemannian manifolds and Hadamard manifolds, which can be found in any book and article related to Riemannian geometry; see, e.g., [18,28,41–44].

2.1. Riemannian manifolds

Let $\mathcal{M}$ be a connected m-dimensional manifold and p be an element on manifold $\mathcal{M}$. We denote $T_p\mathcal{M}$ as the `tangent space` of $\mathcal{M}$ at p and represent the `tangent bundle` of $\mathcal{M}$ by $T\mathcal{M}$, i.e., $T\mathcal{M} = \cup_{p\in\mathcal{M}} T_p\mathcal{M}$. Note that $T\mathcal{M}$ is a manifold and $T_p\mathcal{M}$ is a vector space of the same dimension as $\mathcal{M}$. An inner product $\langle\cdot,\cdot\rangle_p : T_p\mathcal{M}\times T_p\mathcal{M} \to \mathbb{R}_{\geq 0}$ associated with a norm $\|\cdot\|_p$ (i.e., $\|u\|_p = \langle u,u\rangle_p^{1/2}$) is said to be a Riemannian metric of $T_p\mathcal{M}$.

Definition 2.1. If the tensor $\langle\cdot,\cdot\rangle_p$ is a Riemannian metric on $T_p\mathcal{M}$ for all $p \in \mathcal{M}$, then the tensor field $\langle\cdot,\cdot\rangle$ is called a `Riemannian metric` on $\mathcal{M}$. A differentiable manifold $\mathcal{M}$ endowed with a Riemannian metric $\langle\cdot,\cdot\rangle_p$ is called a `Riemannian manifold`.

For simplicity, in the subsequent content, we replace inner product $\langle\cdot,\cdot\rangle_p$ and norm $\|\cdot\|_p$ on $T_p\mathcal{M}$ with $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. That is, we omit the subscript p if there is no confusion.

Definition 2.2. The `length` of a piecewise smooth curve $\gamma : [a,b] \to \mathcal{M}$ connecting x to y (i.e., $\gamma(a) = x$ and $\gamma(b) = y$) is defined as

$$L(\gamma) := \int_a^b \|\gamma'(t)\| \,\mathrm{d}t,$$

where $\gamma'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma(t))$ is a tangent vector in $T_{\gamma(t)}\mathcal{M}$. The minimal length of the set of all such curves connecting x to y is called the `Riemannian distance` from x to y, denoted by $d(x,y)$. That is

$$d(x,y) := \inf\{L(\gamma) : \gamma \text{ joining } x \text{ to } y\}.$$

Note that the topology induced by d on $\mathcal{M}$ coincides with the original topology on $\mathcal{M}$ (see [41, p. 146, Proposition 2.6]).

Definition 2.3. Let ∇ be the Levi-Civita connection associated with the Riemannian metric $\langle\cdot,\cdot\rangle$, and let γ be a smooth curve in $\mathcal{M}$. A vector field X is called `parallel` along γ if $\nabla_{\gamma'} X = \mathbf{0}$, where $\mathbf{0}$ is the zero tangent vector. We say that γ is a `geodesic` if γ' itself is parallel along γ, and in this case $\|\gamma'\|$ is constant. If $\|\gamma'\| = 1$ then γ is said to be `normalized`. A geodesic joining x to y in $\mathcal{M}$ is said to be `minimal` if its length equals $d(x,y)$.

Definition 2.4. The `parallel transport` $\mathrm{P}_{\gamma,\gamma(b),\gamma(a)} : T_{\gamma(a)}\mathcal{M} \to T_{\gamma(b)}\mathcal{M}$ on the tangent bundle $T\mathcal{M}$ along γ: $[a,b] \to \mathbb{R}$ with respect to ∇ is defined by

$$\mathrm{P}_{\gamma,\gamma(b),\gamma(a)}(v) = A(\gamma(b)) \;\; (\forall a,b \in \mathbb{R})(\forall v \in T_{\gamma(a)}\mathcal{M}),$$

where A is the unique vector field such that $\nabla_{\gamma'(t)} A = \mathbf{0}$ for all $t \in [a,b]$ and $A(\gamma(a)) = v$.

If γ is a minimal geodesic joining $\gamma(a)$ to $\gamma(b)$, then we write $\mathrm{P}_{\gamma(b),\gamma(a)}$ instead of $\mathrm{P}_{\gamma,\gamma(b),\gamma(a)}$. For any $a, b, b_1, b_2 \in \mathbb{R}$, we have

$$\mathrm{P}_{\gamma(b_2),\gamma(b_1)} \circ \mathrm{P}_{\gamma(b_1),\gamma(a)} = \mathrm{P}_{\gamma(b_2),\gamma(a)} \quad \text{and} \quad \mathrm{P}^{-1}_{\gamma(b),\gamma(a)} = \mathrm{P}_{\gamma(a),\gamma(b)}.$$

Note that $\mathrm{P}_{\gamma(b),\gamma(a)}$ is an isometry from $T_{\gamma(a)}\mathcal{M}$ to $T_{\gamma(b)}\mathcal{M}$. That is, the parallel transport preserves the inner product

$$\left\langle \mathrm{P}_{\gamma(b),\gamma(a)}(u), \mathrm{P}_{\gamma(b),\gamma(a)}(v) \right\rangle_{\gamma(b)} = \langle u, v \rangle_{\gamma(a)} \ (\forall u \in T_{\gamma(a)}\mathcal{M})(\forall v \in T_{\gamma(a)}\mathcal{M}).$$

Definition 2.5. A Riemannian manifold is said to be `complete` if, for any $x \in \mathcal{M}$, all geodesics emanating from x are defined for all $-\infty < t < +\infty$.

With the Riemannian distance $d : \mathcal{M}\times\mathcal{M} \to \mathbb{R}$, the Riemannian manifold $\mathcal{M}$ is a metric space $(\mathcal{M}, d)$ (see [41, p. 146, Proposition 2.5]). We may investigate the global behavior of a Riemannian manifold $\mathcal{M}$ by using the idea of completeness to observe how geodesics operate on $\mathcal{M}$.

Definition 2.6. Assume that $\mathcal{M}$ is a complete Riemannian manifold, the `exponential map` $\exp_x : T_x\mathcal{M} \to \mathcal{M}$ at x is defined by

$$\exp_x v := \gamma_v(1, x) \ (\forall v \in T_x\mathcal{M}),$$

where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting at x with velocity v (i.e., $\gamma_v(0, x) = x$ and $\gamma_v'(0, x) = v$). Then $\exp_x tv = \gamma_v(t, x)$ for $t \in \mathbb{R}$.

It follows that $\exp_x \mathbf{0} = \gamma_v(0, x) = x$. Note that the mapping $\exp_x$ is differentiable on $T_x\mathcal{M}$ for any $x \in \mathcal{M}$. The exponential map has inverse $\exp_x^{-1} : \mathcal{M} \to T_x\mathcal{M}$. Furthermore, the inverse of the exponential mapping exp and the distance d have the following relationship.

$$\left\|\exp_x^{-1} y\right\| = \left\|\exp_y^{-1} x\right\| = d(x, y) = d(y, x) \ (\forall x \in \mathcal{M})(\forall y \in \mathcal{M}),$$

which can be seen in, e.g., [41, p. 146, Proposition 2.5] and [43, p. 39, Corollary 2.8]).

The following property is well known and can be found in [41, p. 146, Theorem 2.8].

Proposition 2.1. (Hopf-Rinow Theorem) *Let $\mathcal{M}$ be a Riemannian manifold and let $p \in \mathcal{M}$. The following assertions are equivalent:*

(i) $\exp_p$ *is defined on all of* $T_p\mathcal{M}$.
(ii) The closed and bounded sets of $\mathcal{M}$ are compact.
(iii) $\mathcal{M}$ is complete as a metric space.
(iv) $\mathcal{M}$ is geodesically complete.

Furthermore, any of the statements above imply that any pair of points in $\mathcal{M}$ can be connected by a minimal geodesic. That is, for any $q \in \mathcal{M}$, there exists a geodesic γ connecting p to q with $L(\gamma) = d(p, q)$.

2.2. Hadamard manifolds

The concept of sectional curvature in Riemannian manifolds in some sense measures the amount by which a Riemannian manifold deviates from Euclidean. In this paper, we are interested in Riemannian manifolds with nonpositive sectional curvature, whose fundamental properties and geometrical features are collected in the following Propositions 2.2 and 2.3, and Lemmas 2.1 and 2.2. We do not include the technical definition of sectional curvature in this paper; see, e.g., [45, p. 259, Section 1], [42, p. 8, Definition 2.3], and [43, p. 43, Section 3.2] for more information.

Definition 2.7. A `Hadamard manifold` is a complete, simply connected Riemannian manifold with nonpositive sectional curvature.

It is known that the Euclidean space $\mathbb{R}^m$ with its usual metric is a Hadamard manifold with constant sectional curvature equal to 0, and the standard m-dimensional hyperbolic space $\mathbb{H}^m$ is a Hadamard manifold with constant sectional curvature equal to -1. If $\mathcal{M}$ is a Hadamard manifold, then it has two important properties. The first one is that it exists a unique minimal geodesic connecting a pair of points on $\mathcal{M}$ (see Proposition 2.1). The other fact is that it is diffeomorphic to the Euclidean space $\mathbb{R}^m$ (see Proposition 2.2). In the rest of the paper, we use $\mathcal{M}$ to represent an m-dimensional Hadamard manifold and C to denote a nonempty, closed, and convex set in $\mathcal{M}$, unless otherwise stated.

The following result is well-known (see, e.g., [43, p. 221, Theorem 4.1]).

Proposition 2.2. (Hadamard-Cartan Theorem) *Let $\mathcal{M}$ be a Hadamard manifold and $p \in \mathcal{M}$. Then $\exp_p : T_p\mathcal{M} \to \mathcal{M}$ is a diffeomorphism, and for any two points $p, q \in \mathcal{M}$, there exists a unique normalized geodesic joining p to q, which is in fact a minimal geodesic and can be expressed by $\gamma(t) = \exp_p(t \exp_p^{-1} q)$, $\forall t \in [0, 1]$.*

From Proposition 2.2, it follows that $\mathcal{M}$ has the same topology and differential structure as $\mathbb{R}^m$. Next, we state some geometric properties in Hadamard manifolds, which are similar to those in Euclidean spaces.

Let p_1, p_2, and p_3 be points on a Riemannian manifold. Let $\Delta\left(p_1, p_2, p_3\right)$ denote a geodesic triangle on a Riemannian manifold which consists of three minimal geodesics γ_i connecting p_i to p_{i+1}, where $i = 1, 2, 3 \pmod 3$. The following Proposition 2.3 is known as the comparison theorem for triangles (see [43, p. 223, Proposition 4.5] and [45, Theorem 2.2]), which is essential for our main results.

Proposition 2.3. *Let $\Delta\left(p_1, p_2, p_3\right)$ be a geodesic triangle on a Hadamard manifold $\mathcal{M}$. We use $\gamma_i:\left[0, l_i\right] \rightarrow \mathcal{M}$ to represent the geodesic connecting p_i to p_{i+1} for each $i=1,2,3 \pmod 3$. Set $l_i:=L\left(\gamma_i\right)$ and $\alpha_i:=\angle\left(\gamma_i^{\prime}(0),-\gamma_{i-1}^{\prime}\left(l_{i-1}\right)\right)$. Then*

(i) $\alpha_1+\alpha_2+\alpha_3 \leq \pi$.
(ii) $l_i^2+l_{i+1}^2-2 l_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2$.
(iii) $l_{i+1} \cos \alpha_{i+2}+l_i \cos \alpha_i \geq l_{i+2}$.

Remark 2.1. By using exponential mapping and distance on $\mathcal{M}$, we have the following findings.

- Since
$$\left\langle\exp _{p_{i+1}}^{-1} p_i, \exp _{p_{i+1}}^{-1} p_{i+2}\right\rangle=d\left(p_i, p_{i+1}\right) d\left(p_{i+1}, p_{i+2}\right) \cos \alpha_{i+1}, \tag{2.1}$$
then the inequality (ii) of Proposition 2.3 can be rewritten as (cf. [28, Eq. (2.3)])
$$d^2\left(p_i, p_{i+1}\right)+d^2\left(p_{i+1}, p_{i+2}\right)-2\left\langle\exp _{p_{i+1}}^{-1} p_i, \exp _{p_{i+1}}^{-1} p_{i+2}\right\rangle \leq d^2\left(p_{i+2}, p_i\right). \tag{2.2}$$
- From the inequality (iii) of Proposition 2.3, we have
$$l_{i+2}^2 \leq l_{i+2} l_{i+1} \cos \alpha_{i+2}+l_{i+2} l_i \cos \alpha_i,$$
which together with (2.1) deduces that
$$\begin{aligned} d^2\left(p_i, p_{i+2}\right) & \leq d^2\left(p_i, p_{i+2}\right) d^2\left(p_{i+1}, p_{i+2}\right) \cos \alpha_{i+2}+d^2\left(p_i, p_{i+2}\right) d^2\left(p_i, p_{i+1}\right) \cos \alpha_i \\ & =\left\langle\exp _{p_{i+2}}^{-1} p_i, \exp _{p_{i+2}}^{-1} p_{i+1}\right\rangle+\left\langle\exp _{p_i}^{-1} p_{i+1}, \exp _{p_i}^{-1} p_{i+2}\right\rangle . \end{aligned} \tag{2.3}$$
Note that inequality (2.3) can also be derived from the well-known "law of cosines" in $\mathbb{R}^2$ and inequality (2.2) (see, e.g., [29, Eq. (9)] and [32, p. 280, Proposition 14.16]).
- By letting $p_{i+2}=p_i$ in (2.1), one obtains
$$\left\langle\exp _{p_{i+1}}^{-1} p_i, \exp _{p_{i+1}}^{-1} p_i\right\rangle=d^2\left(p_{i+1}, p_i\right)=\left\|\exp _{p_{i+1}}^{-1} p_i\right\|^2=\left\|\exp _{p_i}^{-1} p_{i+1}\right\|^2 .$$

Let the triangle $\Delta\left(p^{\prime}, q^{\prime}, r^{\prime}\right)$ denote the comparison triangle of the geodesic triangle $\Delta(p, q, r)$. Notice that the comparison triangle is unique within isometry of $\mathcal{M}$. The following two results demonstrate some interesting findings of comparison triangles. The first one is the existence of comparison triangles in $\mathbb{R}^2$, and the second one shows the angular relationship of points between a geodesic triangle and its comparison triangle.

Lemma 2.1. ([46, p. 24, Lemma 2.14]) *Let $\Delta(p, q, r)$ be a geodesic triangle in a Hadamard manifold $\mathcal{M}$. Then, there exists $p^{\prime}, q^{\prime}, r^{\prime} \in \mathbb{R}^2$ such that*
$$d(p, q)=\left\|p^{\prime}-q^{\prime}\right\|, \quad d(q, r)=\left\|q^{\prime}-r^{\prime}\right\|, \quad d(r, p)=\left\|r^{\prime}-p^{\prime}\right\| .$$

Lemma 2.2. ([31, Lemma 3.5]) *Let $\Delta(p, q, r)$ be a geodesic triangle in a Hadamard manifold $\mathcal{M}$ and $\Delta\left(p^{\prime}, q^{\prime}, r^{\prime}\right)$ be its comparison triangle.*

(i) Let α, β, γ (resp., $\alpha^{\prime}, \beta^{\prime}, \gamma^{\prime}$) be the angles of $\Delta(p, q, r)$ (resp., $\Delta\left(p^{\prime}, q^{\prime}, r^{\prime}\right)$) at the vertices p, q, r (resp., $p^{\prime}, q^{\prime}, r^{\prime}$). Then, the following inequalities hold
$$\alpha^{\prime} \geq \alpha, \quad \beta^{\prime} \geq \beta, \quad \gamma^{\prime} \geq \gamma .$$
(ii) Let z be a point in the geodesic joining p to q, and $z^{\prime} \in\left[p^{\prime}, q^{\prime}\right]$ is the comparison point, if $d(z, p)=\left\|z^{\prime}-p^{\prime}\right\|, d(z, q)=\left\|z^{\prime}-q^{\prime}\right\|$, then
$$d(z, r) \leq\left\|z^{\prime}-r^{\prime}\right\| .$$

2.3. Convex analysis

In this subsection, we recall some concepts about convexity and monotonicity in Hadamard manifolds.

Definition 2.8. ([42, p. 59, Definition 1.3]) Let $\mathcal{M}$ be a Hadamard manifold. A subset $C \subset \mathcal{M}$ is said to be (geodesic) convex if for any two points $p, q \in C$, the geodesic joining p to q is contained in C. That is, if $\gamma : [a, b] \to \mathcal{M}$ is a geodesic such that $p = \gamma(a)$ and $q = \gamma(b)$, then $\gamma((1-t)a + tb) \in C$ for all $t \in [0, 1]$.

Definition 2.9. ([43, p. 172, Definition 5.9]) A real-valued function $f : \mathcal{M} \to \mathbb{R}$ is said to be convex if the composition $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is convex for any geodesic γ of $\mathcal{M}$, which implies that $f \circ \gamma(ta + (1-t)b) \le t f(\gamma(a)) + (1-t) f(\gamma(b))$ holds for any $a, b \in \mathbb{R}$ and $t \in [0, 1]$.

Definition 2.10. ([16, Definition 7], [47, Definition 2.3]) Let C be a nonempty geodesic convex subset of a Hadamard manifold $\mathcal{M}$. A function $f : C \to \mathbb{R}$ is said to be (geodesic) hemicontinuous if for any geodesic $\gamma : [0, 1] \to C$, the function $t \mapsto f(\gamma(t))$ defined on $[0, 1]$ is continuous. That is, $f(\gamma(t)) \to f(\gamma(0))$ as $t \to 0$.

Definition 2.11. ([45]) Let $f : \mathcal{M} \to \mathbb{R}$ be a convex function and $x \in \mathcal{M}$. The vector $s \in T_x\mathcal{M}$ is said to be subgradient of f at $x \in \mathcal{M}$ if

$$f(y) \ge f(x) + \left\langle s, \exp_x^{-1} y \right\rangle \quad (\forall y \in \mathcal{M}).$$

The set of all subgradients of f at x is called the subdifferential of f at x and is denoted by $\partial f(x)$.

The following lemma shows that the subdifferential of a convex function is nonempty.

Lemma 2.3. ([45, Theorem 3.3]) *Let $f : \mathcal{M} \to \mathbb{R}$ be a convex function. Then, for any $x \in \mathcal{M}$, there is $s \in T_x\mathcal{M}$ such that*

$$f(y) \ge f(x) + \left\langle s, \exp_x^{-1} y \right\rangle \quad (\forall y \in \mathcal{M}).$$

That is, the subdifferential $\partial f(x)$ of f at $x \in \mathcal{M}$ is nonempty.

Definition 2.12. Let C be a nonempty, closed, and convex subset of a Hadamard manifold $\mathcal{M}$, $f : C \to \mathbb{R}$ be a convex function. Take $\alpha > 0$. The proximal mapping of f is defined by

$$\operatorname{prox}_{\alpha f}(x) := \underset{y \in C}{\arg\min} \left\{ \alpha f(y) + \frac{1}{2} d^2(x, y) \right\} \quad (\forall x \in \mathcal{M}).$$

Lemma 2.4. ([45, Theorem 5.1]) *Let $f : \mathcal{M} \to \mathbb{R}$ be a convex function. The sequence $\{s_n\}$ generated by the proximal point algorithm*

$$\boxed{s_{n+1} = \underset{y \in \mathcal{M}}{\arg\min} \left\{ f(y) + \frac{1}{2\tau_n} d^2\left(s_n, y\right) \right\}, \text{ where starting point } s_0 \in \mathcal{M},\ \tau_n \in (0, \infty),}$$

is well defined, and characterized by $\frac{1}{\tau_n} \exp_{s_{n+1}}^{-1} s_n \in \partial f\left(s_{n+1}\right)$.

Remark 2.2. From [45, Lemma 4.2], it is known that $\operatorname{prox}_{\alpha f}(x)$ is a single-valued. By the definition of $\partial f\left(s_{n+1}\right)$ and Lemma 2.4, we have

$$\frac{1}{\tau_n} \left\langle \exp_{s_{n+1}}^{-1} s_n, \exp_{s_{n+1}}^{-1} x \right\rangle \le f(x) - f\left(s_{n+1}\right) \quad (\forall x \in \mathcal{M}).$$

The following definition is modified from [48, p. 294, Theorem 3.1(v)] and [36, p. 754, Theorem 3.2(ii)].

Definition 2.13. Let C be a nonempty, closed, and convex subset of a Hadamard manifold $\mathcal{M}$. A bifunction $f : C \times C \to \mathbb{R}$ is said to satisfy a Lipschitz-type condition on C if there exists a positive constant L such that

$$f(x, z) - f(x, y) - f(y, z) \le L d(x, y) d(y, z) \quad (\forall x, y, z \in C). \tag{2.4}$$

Remark 2.3. Note that (2.4) implies that

$$f(x, z) - f(x, y) - f(y, z) \le \gamma_1 d^2(x, y) + \gamma_2 d^2(y, z) \quad (\forall x, y, z \in C), \tag{2.5}$$

where $\gamma_1 = \gamma_2 = \frac{L}{2}$. The Lipschitz-type condition (2.5) is used by many papers solving equilibrium problems on Hadamard manifolds; see, e.g., [22–25].

Definition 2.14. A bifunction $f : C \times C \to \mathbb{R}$ is said to be

(1) `strongly monotone` if, for all $x, y \in C$, there exists a positive constant γ such that

$$f(x,y) + f(y,x) \le -\gamma d^2(x,y).$$

(2) `monotone` if, for all $x, y \in C$,

$$f(x,y) + f(y,x) \le 0.$$

(3) `strongly pseudomonotone` if, for all $x, y \in C$, there exists a constant σ such that

$$f(x,y) \ge 0 \quad \Longrightarrow \quad f(y,x) \le -\sigma d^2(x,y).$$

(4) `pseudomonotone` if, for all $x, y \in C$,

$$f(x,y) \ge 0 \quad \Longrightarrow \quad f(y,x) \le 0.$$

According to Definition 2.14 it is easy to check the following relations: (1) ⇒ (2) ⇒ (4) and (1) ⇒ (3) ⇒ (4).

Definition 2.15. ([45, p. 268, Eq. (25)]) Let C be a nonempty set on a complete metric space X. A sequence $\{s_n\} \subset X$ is called Fejér convergent with respect to C if

$$d\left(s_{n+1}, y\right) \le d\left(s_n, y\right) \quad (\forall y \in C)(\forall n \ge 0).$$

Lemma 2.5. ([45, Lemma 6.1], [17, Lemma 7.2]) *Let C be a nonempty set on a complete metric space X. If $\{s_n\} \subset X$ is Fejér convergent to C, then $\{s_n\}$ is bounded. In addition, if any cluster point of $\{s_n\}$ belongs to C, then $\{s_n\}$ converges to a point of C.*

3. Main results

In this section, we present two modified extragradient type algorithms and a modified golden ratio algorithm for solving the equilibrium problem (1.1) in Hadamard manifolds. The three proposed algorithms can work without the prior knowledge of the Lipschitz constant of the bifunction f involved. The first two extragradient-based algorithms proposed in this paper are motivated by the work of Hieu et al. [37] for solving equilibrium problems in Hilbert spaces. Now, we are in a position to introduce our Algorithm 3.1.

Algorithm 3.1

Initialization: Take $\tau_0 > 0$, $\delta \in (0,1)$, and $\chi \in (0, 2/(1+\delta))$. Let $\{\xi_n\}$ and $\{\sigma_n\}$ satisfy the following Condition (C3). Let $s_0 \in C$ be an initial point and set $n = 0$.

Iterative Steps: Assume that $s_n \in C$ is known, calculate s_{n+1} as follows:

Step 1. Compute

$$t_n = \arg\min_{y \in C} \left\{ f\left(s_n, y\right) + \frac{1}{2\tau_n} d^2\left(s_n, y\right) \right\} = \operatorname{prox}_{\tau_n f(s_n,\cdot)}\left(s_n\right).$$

If $s_n = t_n$, then stop the iterative process and s_n is a solution of EP (1.1); Otherwise, go to *Step 2*.

Step 2. Compute

$$s_{n+1} = \arg\min_{y \in C} \left\{ f\left(t_n, y\right) + \frac{1}{2\chi\tau_n} d^2\left(s_n, y\right) \right\} = \operatorname{prox}_{\chi\tau_n f(t_n,\cdot)}\left(s_n\right).$$

Update the next step size by

$$\tau_{n+1} = \begin{cases} \min\left\{ \dfrac{\delta d\left(s_n, t_n\right) d\left(s_{n+1}, t_n\right)}{\Delta_n}, \xi_n \tau_n + \sigma_n \right\}, & \text{if } \Delta_n > 0; \\ \xi_n \tau_n + \sigma_n, & \text{otherwise,} \end{cases} \tag{3.1}$$

where $\Delta_n := f\left(s_n, s_{n+1}\right) - f\left(s_n, t_n\right) - f\left(t_n, s_{n+1}\right)$.

Set $n := n+1$ and go to *Step 1*.

We assume that the bifunction f satisfies the following four conditions.

(A1) $f : C \times C \to \mathbb{R}$ is a pseudomonotone bifunction and $f(x,x) = 0$ for all $x \in C$;
(A2) f satisfies Lipschitz-type condition (2.4);
(A3) $f(x,\cdot)$ is convex and subdifferentiable on C for each $x \in C$;
(A4) $f(\cdot, y)$ is upper semicontinuous on C for each $y \in C$, i.e., $\limsup_{n\to\infty} f\left(s_n, y\right) \le f(x,y)$ for each $y \in C$ and each $\{s_n\} \subset C$ with $s_n \to x$.

Let the proposed Algorithm 3.1 satisfy the following three conditions in order to perform its convergence analysis.

(C1) The feasible set C is a nonempty, closed, and convex subset of Hadamard manifold $\mathcal{M}$.
(C2) The solution set Ω of EP (1.1) is assumed to be nonempty, that is, $\Omega \neq \emptyset$.
(C3) Let $\{\xi_n\} \subset [1, \infty)$ and $\{\sigma_n\} \subset [0, \infty)$ be two sequences such that $\sum_{n=1}^{\infty}(\xi_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$.

Remark 3.1. From the definition of t_n in Algorithm 3.1 and Remark 2.2, one has

$$\tau_n\left(f\left(s_n, y\right) - f\left(s_n, t_n\right)\right) \geq \left\langle \exp_{t_n}^{-1} s_n, \exp_{t_n}^{-1} y \right\rangle \ (\forall y \in C).$$

If $s_n = t_n$ for some $n \in \mathbb{N}$, then we obtain that $f\left(s_n, y\right) \geq 0$ for all $y \in C$ since $\tau_n > 0$. This implies that $s_n \in \Omega$, i.e., s_n is a solution of EP (1.1). Therefore the iterations of Algorithm 3.1 terminate when $s_n = t_n$.

We begin the convergence analysis of Algorithm 3.1 by showing that the step size τ_n generated by (3.1) is well defined.

Lemma 3.1. *Let step size $\{\tau_n\}$ be a sequence formed by (3.1) and Conditions (A2) and (C3) hold. Then $\{\tau_n\}$ is well defined and $\lim_{n\to\infty} \tau_n$ exists.*

Proof. Since f satisfies the Lipschitz-type condition (2.4), in the case of $\Delta_n > 0$, one obtains

$$\frac{\delta d\left(s_n, t_n\right) d\left(s_{n+1}, t_n\right)}{f\left(s_n, s_{n+1}\right) - f\left(s_n, t_n\right) - f\left(t_n, s_{n+1}\right)} \geq \frac{\delta d\left(s_n, t_n\right) d\left(s_{n+1}, t_n\right)}{L d(s_n, t_n) d(s_{n+1}, t_n)} = \frac{\delta}{L}.$$

This combining with (3.1) yields $\tau_{n+1} \geq \min\{\tau_n, \frac{\delta}{L}\}$. By induction, one finds that $\tau_n \geq \min\{\tau_0, \frac{\delta}{L}\}$. On the other hand, it can be seen from (3.1) that $\tau_{n+1} \leq \xi_n \tau_n + \sigma_n$ for any $n \geq 0$. In view of Condition (C3) and [49, Lemma 1], it can be concluded that $\lim_{n\to\infty} \tau_n$ exists. Since $\{\tau_n\}$ has a lower bound $\min\{\tau_0, \frac{\delta}{L}\}$, we have $\lim_{n\to\infty} \tau_n := \tau > 0$. □

Lemma 3.2. *Assume that Conditions (A1)–(A4) and (C1)–(C3) hold. Let $\{s_n\}$ and $\{t_n\}$ be generated by Algorithm 3.1. Fix $p \in \Omega$. Then*

$$d^2\left(s_{n+1}, p\right) \leq d^2\left(s_n, p\right) - \chi_n^*\left(d^2\left(s_n, t_n\right) + d^2\left(s_{n+1}, t_n\right)\right), \tag{3.2}$$

where

$$\chi_n^* := \begin{cases} 2 - \chi - \frac{\chi\delta\tau_n}{\tau_{n+1}}, & \text{if } \chi \in [1, 2/(1+\delta)); \\ \chi(1 - \frac{\delta\tau_n}{\tau_{n+1}}), & \text{if } \chi \in (0, 1). \end{cases}$$

Furthermore, $\{s_n\}$ is Fejér monotone with respect to the solution set Ω. Both $\{s_n\}$ and $\{t_n\}$ are bounded.

Proof. According to Lemma 2.4 and the definition of s_{n+1} in Algorithm 3.1, we obtain

$$\chi\tau_n\left(f\left(t_n, y\right) - f\left(t_n, s_{n+1}\right)\right) \geq \left\langle \exp_{s_{n+1}}^{-1} s_n, \exp_{s_{n+1}}^{-1} y \right\rangle \ (\forall y \in C). \tag{3.3}$$

Similarly, by means of Lemma 2.4 and the definition of t_n in Algorithm 3.1, we have

$$\tau_n\left(f\left(s_n, y\right) - f\left(s_n, t_n\right)\right) \geq \left\langle \exp_{t_n}^{-1} s_n, \exp_{t_n}^{-1} y \right\rangle \ (\forall y \in C). \tag{3.4}$$

Using $y = s_{n+1} \in C$ in (3.4) yields

$$\tau_n\left(f\left(s_n, s_{n+1}\right) - f\left(s_n, t_n\right)\right) \geq \left\langle \exp_{t_n}^{-1} s_n, \exp_{t_n}^{-1} s_{n+1} \right\rangle. \tag{3.5}$$

Let $\Delta\left(s_n, s_{n+1}, y\right)$ be a geodesic triangle. From (2.2), one sees that

$$2\left\langle \exp_{s_{n+1}}^{-1} s_n, \exp_{s_{n+1}}^{-1} y \right\rangle \geq d^2\left(s_n, s_{n+1}\right) + d^2\left(s_{n+1}, y\right) - d^2\left(s_n, y\right) \ (\forall y \in C). \tag{3.6}$$

Similarly, let $\Delta\left(s_n, t_n, s_{n+1}\right)$ be a geodesic triangle. It follows from (2.2) that

$$2\left\langle \exp_{t_n}^{-1} s_n, \exp_{t_n}^{-1} s_{n+1} \right\rangle \geq d^2\left(s_n, t_n\right) + d^2\left(s_{n+1}, t_n\right) - d^2\left(s_n, s_{n+1}\right). \tag{3.7}$$

Combining (3.3), (3.5), (3.6), and (3.7), we have

$$
\begin{aligned}
&2\chi\tau_n\left(f\left(s_n,s_{n+1}\right)-f\left(s_n,t_n\right)-f\left(t_n,s_{n+1}\right)\right)\\
&\geq 2\chi\left\langle \exp_{t_n}^{-1}s_n,\exp_{t_n}^{-1}s_{n+1}\right\rangle+2\left\langle \exp_{s_{n+1}}^{-1}s_n,\exp_{s_{n+1}}^{-1}y\right\rangle-2\chi\tau_n f\left(t_n,y\right)\\
&\geq \chi d^2\left(s_n,t_n\right)+\chi d^2\left(s_{n+1},t_n\right)-\chi d^2\left(s_n,s_{n+1}\right)\\
&\quad+d^2\left(s_n,s_{n+1}\right)+d^2\left(s_{n+1},y\right)-d^2\left(s_n,y\right)-2\chi\tau_n f\left(t_n,y\right)\ \ (\forall y\in C).
\end{aligned}
$$

That is,

$$
\begin{aligned}
d^2\left(s_{n+1},y\right)\leq{}& d^2\left(s_n,y\right)-\chi d^2\left(s_n,t_n\right)-\chi d^2\left(s_{n+1},t_n\right)-(1-\chi)d^2\left(s_n,s_{n+1}\right)\\
&+2\chi\tau_n\left(f\left(s_n,s_{n+1}\right)-f\left(s_n,t_n\right)-f\left(t_n,s_{n+1}\right)\right)\\
&+2\chi\tau_n f\left(t_n,y\right)\ \ (\forall y\in C).
\end{aligned}
\tag{3.8}
$$

By the definition of τ_{n+1} in (3.1), one sees that

$$
\begin{aligned}
f\left(s_n,s_{n+1}\right)-f\left(s_n,t_n\right)-f\left(t_n,s_{n+1}\right)&\leq\frac{\delta}{\tau_{n+1}}\left(d\left(s_n,t_n\right)d\left(s_{n+1},t_n\right)\right)\\
&\leq\frac{\delta}{2\tau_{n+1}}\left(d^2\left(s_n,t_n\right)+d^2\left(s_{n+1},t_n\right)\right).
\end{aligned}
\tag{3.9}
$$

Combining (3.8) and (3.9), we deduce that

$$
\begin{aligned}
d^2\left(s_{n+1},y\right)\leq{}& d^2\left(s_n,y\right)-\chi d^2\left(s_n,t_n\right)-\chi d^2\left(s_{n+1},t_n\right)-(1-\chi)d^2\left(s_n,s_{n+1}\right)\\
&+\frac{\chi\delta\tau_n}{\tau_{n+1}}\left(d^2\left(s_n,t_n\right)+d^2\left(s_{n+1},t_n\right)\right)+2\chi\tau_n f\left(t_n,y\right)\ \ (\forall y\in C).
\end{aligned}
\tag{3.10}
$$

Letting $y=p$ in (3.10). Since $p\in\Omega$ and $t_n\in C$, one obtains $f\left(p,t_n\right)\geq 0$. This together with the pseudomonotonicity of f yields that $f\left(t_n,p\right)\leq 0$. Then from (3.10) we obtain

$$
\begin{aligned}
d^2\left(s_{n+1},p\right)\leq{}& d^2\left(s_n,p\right)-(1-\chi)d^2\left(s_n,s_{n+1}\right)\\
&-\chi\left(1-\frac{\delta\tau_n}{\tau_{n+1}}\right)\left(d^2\left(s_n,t_n\right)+d^2\left(s_{n+1},t_n\right)\right).
\end{aligned}
\tag{3.11}
$$

By the triangle inequality and the inequality $(a+b)^2\leq 2(a^2+b^2)$, $\forall a,b\in\mathbb{R}$, we have

$$
d^2\left(s_n,s_{n+1}\right)\leq\left(d\left(s_n,t_n\right)+d\left(s_{n+1},t_n\right)\right)^2\leq 2\left(d^2\left(s_n,t_n\right)+d^2\left(s_{n+1},t_n\right)\right).
\tag{3.12}
$$

Combining (3.11) and (3.12), and considering two cases of χ, we obtain that the inequality (3.2) required in Lemma 3.2.

From Lemma 3.1, one has

$$
\lim_{n\to\infty}\chi_n^*=\begin{cases}2-\chi-\chi\delta, & \text{if } \chi\in[1,2/(1+\delta));\\ \chi-\chi\delta, & \text{if } \chi\in(0,1).\end{cases}
$$

Since $\delta\in(0,1)$, we conclude that $\lim_{n\to\infty}\chi_n^*>0$ for all $\chi\in(0,2/(1+\delta))$. Therefore, there exists $n_0>0$ such that $\chi_n^*>0$ for all $n\geq n_0$. Consequently,

$$
d\left(s_{n+1},p\right)\leq d\left(s_n,p\right)\ \ (\forall p\in\Omega)(\forall n\geq n_0).
$$

This means that $\{s_n\}$ is Fejér monotone with respect to the solution set Ω of EP (1.1). Thus $\left\{s_n\right\}$ is bounded and $\lim_{n\to\infty}d\left(s_n,p\right)$ exists. By letting the limit $n\to\infty$ in (3.2), one arrives at

$$
\lim_{n\to\infty}d\left(s_n,t_n\right)=0\quad\text{and}\quad\lim_{n\to\infty}d\left(s_{n+1},t_n\right)=0.
\tag{3.13}
$$

As a result, $\{t_n\}$ is also bounded. It follows from (3.12) and (3.13) that

$$
\lim_{n\to\infty}d\left(s_n,s_{n+1}\right)=0.
\tag{3.14}
$$

This completes the proof. □

Theorem 3.1. *Let $\{s_n\}$ be generated by Algorithm 3.1 and satisfy Conditions (A1)–(A4) and (C1)–(C3). Then $\{s_n\}$ converges to a solution of EP (1.1).*

Proof. From Lemma 3.2, one knows that $\{s_n\}$ is Fejér monotone with respect to the solution set Ω. To show that $\{s_n\}$ converges to a solution of EP (1.1), it is left to prove that any cluster point of $\{s_n\}$ belongs to Ω by means of Lemma 2.5. Let s^* be a cluster point of $\{s_n\}$. According to $\{s_n\}$ is bounded, there exists a subsequence $\left\{s_{n_k}\right\}$ of $\{s_n\}$ satisfies $\lim_{k\to\infty}s_{n_k}=s^*$. We also have $\lim_{k\to\infty}t_{n_k}=s^*$ and $s^*\in C$ due to $\lim_{n\to\infty}d\left(s_n,t_n\right)=0$. In view of (3.10), one sees that

$$
\begin{aligned}
2\chi\tau_n f\left(t_n, y\right) \geq\; & d^2\left(s_{n+1}, y\right)-d^2\left(s_n, y\right)+(1-\chi) d^2\left(s_n, s_{n+1}\right) \\
& -\chi\left(\frac{\delta\tau_n}{\tau_{n+1}}-1\right)\left(d^2\left(s_n, t_n\right)+d^2\left(s_{n+1}, t_n\right)\right) \quad (\forall y \in C).
\end{aligned} \tag{3.15}
$$

From the triangle inequality, we have

$$
\left|d^2\left(s_{n+1}, y\right)-d^2\left(s_n, y\right)\right| \leq d\left(s_{n+1}, s_n\right)\left(d\left(s_{n+1}, y\right)+d\left(s_n, y\right)\right) \quad (\forall y \in C),
$$

which combining with (3.14) and the boundedness of $\left\{s_n\right\}$ implies that

$$
\lim_{n \rightarrow \infty}\left|d^2\left(s_{n+1}, y\right)-d^2\left(s_n, y\right)\right|=0 \quad (\forall y \in C). \tag{3.16}
$$

Replacing n in (3.15) with n_k and letting $k \rightarrow +\infty$, we obtain that the right-hand side of inequality (3.15) tends to 0 according to (3.13), (3.14), (3.16), and $\lim_{k \rightarrow \infty} \tau_{n_k}=\tau>0$. From Condition (A4), we have

$$
f\left(s^*, y\right) \geq \limsup_{k \rightarrow \infty} f\left(t_{n_k}, y\right) \geq 0 \quad (\forall y \in C).
$$

This follows that $s^* \in \Omega$, as required. □

Remark 3.2. The step size (3.1) defined in the proposed Algorithm 3.1 can be replaced by the following expression

$$
\tau_{n+1}= \begin{cases}\min \left\{\dfrac{\delta\left(d^2\left(s_n, t_n\right)+d^2\left(s_{n+1}, t_n\right)\right)}{2 \Delta_n}, \xi_n \tau_n+\sigma_n\right\}, & \text { if } \Delta_n>0; \\ \xi_n \tau_n+\sigma_n, & \text { otherwise, }\end{cases} \tag{3.17}
$$

where Δ_n is defined in (3.1). To see this, it is sufficient to verify that

$$
\begin{aligned}
\frac{\delta\left(d^2\left(s_n, t_n\right)+d^2\left(s_{n+1}, t_n\right)\right)}{2\left(f\left(s_n, s_{n+1}\right)-f\left(s_n, t_n\right)-f\left(t_n, s_{n+1}\right)\right)} & \geq \frac{\delta\left(d^2\left(s_n, t_n\right)+d^2\left(s_{n+1}, t_n\right)\right)}{2 L d(s_n, t_n) d(s_{n+1}, t_n)} \\
& \geq \frac{\delta\left(d^2\left(s_n, t_n\right)+d^2\left(s_{n+1}, t_n\right)\right)}{L\left(d^2\left(s_n, t_n\right)+d^2\left(s_{n+1}, t_n\right)\right)}=\frac{\delta}{L}.
\end{aligned}
$$

Obviously, the inequality (3.9) still holds. Therefore, the result of Theorem 4.1 can also be obtained by Algorithm 3.1 incorporating the step size criterion (3.17).

Notice that the suggested Algorithm 3.1 requires that the values of the bifunction f at s_n and t_n be computed in each iteration. To improve the computational efficiency of the algorithm, we propose a method that only needs to compute the value of f at t_n in each iteration. The second iterative scheme proposed in this paper is shown in Algorithm 3.2 below.

Algorithm 3.2

Initialization: Take $\tau_0>0$, $\delta \in (0,1/3)$, and $\chi \in (0, 2/(1+3\delta))$. Let $\{\xi_n\}$ and $\{\sigma_n\}$ satisfy Condition (C3). Let $t_{-1}, t_0, s_0 \in C$ be three initial points and set $n=0$.
Iterative Steps: Assume that $t_{n-1}, t_n, s_n \in C$ are known. Calculate s_{n+1} and t_{n+1} as follows.
Step 1. Compute

$$
s_{n+1}=\underset{y \in C}{\arg\min}\left\{f\left(t_n, y\right)+\frac{1}{2 \chi \tau_n} d^2\left(s_n, y\right)\right\}=\operatorname{prox}_{\chi \tau_n f\left(t_n, \cdot\right)}\left(s_n\right).
$$

If $s_{n+1}=s_n=t_n$, then stop the iterative process and s_n is a solution of EP (1.1); Otherwise, go to *Step 2*.
Step 2. Compute

$$
\tau_{n+1}= \begin{cases}\min \left\{\dfrac{\delta d\left(t_{n-1}, t_n\right) d\left(s_{n+1}, t_n\right)}{\Delta_n}, \xi_n \tau_n+\sigma_n\right\}, & \text { if } \Delta_n>0; \\ \xi_n \tau_n+\sigma_n, & \text { otherwise, }\end{cases} \tag{3.18}
$$

where $\Delta_n := f\left(t_{n-1}, s_{n+1}\right)-f\left(t_{n-1}, t_n\right)-f\left(t_n, s_{n+1}\right)$. Update t_{n+1} via

$$
t_{n+1}=\underset{y \in C}{\arg\min}\left\{f\left(t_n, y\right)+\frac{1}{2 \tau_{n+1}} d^2\left(s_{n+1}, y\right)\right\}=\operatorname{prox}_{\tau_{n+1} f\left(t_n, \cdot\right)}\left(s_{n+1}\right).
$$

Set $n := n+1$ and go to *Step 1*.

Remark 3.3. According to Remark 2.2 and the definition of s_{n+1} in Algorithm 3.2, one sees that

$$
\chi \tau_n\left(f\left(t_n, y\right)-f\left(t_n, s_{n+1}\right)\right) \geq\left\langle\exp_{s_{n+1}}^{-1} s_n, \exp_{s_{n+1}}^{-1} y\right\rangle \quad (\forall y \in C).
$$

If $s_{n+1} = s_n = t_n$ for some $n \in \mathbb{N}$, then we obtain that $f(t_n, y) \geq 0$ for all $y \in C$ since $\chi\tau_n > 0$. This means that t_n solves EP (1.1). Consequently the iterations of Algorithm 3.2 terminate when $s_{n+1} = s_n = t_n$.

Lemma 3.3. *Let step size $\{\tau_n\}$ be a sequence created by (3.18) and Conditions (A2) and (C3) hold. Then $\{\tau_n\}$ is well defined and $\lim_{n\to\infty} \tau_n$ exists.*

Proof. This proof is similar to the one of Lemma 3.1 and therefore is omitted. □

Lemma 3.4. *Let $\{s_{n+1}\}$ and $\{t_{n+1}\}$ be generated by Algorithm 3.2. Fix $p \in \Omega$. Then $\{s_n\}$ and $\{t_n\}$ are bounded and $\lim_{n\to\infty} d^2(s_n, p)$ exists.*

Proof. From the definition of s_{n+1} in Algorithm 3.2 and Lemma 2.4, we have

$$\chi\tau_n\left(f(t_n, y) - f(t_n, s_{n+1})\right) \geq \left\langle \exp^{-1}_{s_{n+1}} s_n, \exp^{-1}_{s_{n+1}} y \right\rangle \quad (\forall y \in C). \tag{3.19}$$

Similarly, by the definition of t_{n+1} in Algorithm 3.2 and Lemma 2.4, we obtain

$$\tau_{n+1}\left(f(t_n, y) - f(t_n, t_{n+1})\right) \geq \left\langle \exp^{-1}_{t_{n+1}} s_{n+1}, \exp^{-1}_{t_{n+1}} y \right\rangle \quad (\forall y \in C). \tag{3.20}$$

Replacing $n+1$ with n in (3.20), one finds that

$$\tau_n\left(f(t_{n-1}, y) - f(t_{n-1}, t_n)\right) \geq \left\langle \exp^{-1}_{t_n} s_n, \exp^{-1}_{t_n} y \right\rangle \quad (\forall y \in C). \tag{3.21}$$

Using $y = s_{n+1} \in C$ in (3.21) yields

$$\tau_n\left(f(t_{n-1}, s_{n+1}) - f(t_{n-1}, t_n)\right) \geq \left\langle \exp^{-1}_{t_n} s_n, \exp^{-1}_{t_n} s_{n+1} \right\rangle. \tag{3.22}$$

Let $\Delta(s_n, s_{n+1}, y)$ be a geodesic triangle. Then from (2.2) we have

$$2\left\langle \exp^{-1}_{s_{n+1}} s_n, \exp^{-1}_{s_{n+1}} y \right\rangle \geq d^2(s_n, s_{n+1}) + d^2(s_{n+1}, y) - d^2(s_n, y) \quad (\forall y \in C). \tag{3.23}$$

Similarly, let $\Delta(s_n, t_n, s_{n+1})$ be a geodesic triangle. It follows from (2.2) that

$$2\left\langle \exp^{-1}_{t_n} s_n, \exp^{-1}_{t_n} s_{n+1} \right\rangle \geq d^2(s_n, t_n) + d^2(s_{n+1}, t_n) - d^2(s_n, s_{n+1}). \tag{3.24}$$

From (3.19), (3.22), (3.23), and (3.24), we conclude that

$$\begin{aligned}
&2\chi\tau_n\left(f(t_{n-1}, s_{n+1}) - f(t_{n-1}, t_n) - f(t_n, s_{n+1})\right) \\
&\geq 2\chi\left\langle \exp^{-1}_{t_n} s_n, \exp^{-1}_{t_n} s_{n+1} \right\rangle + 2\left\langle \exp^{-1}_{s_{n+1}} s_n, \exp^{-1}_{s_{n+1}} y \right\rangle - 2\chi\tau_n f(t_n, y) \\
&\geq \chi d^2(s_n, t_n) + \chi d^2(s_{n+1}, t_n) - \chi d^2(s_n, s_{n+1}) \\
&\quad + d^2(s_n, s_{n+1}) + d^2(s_{n+1}, y) - d^2(s_n, y) - 2\chi\tau_n f(t_n, y) \quad (\forall y \in C),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
d^2(s_{n+1}, y) \leq{}& d^2(s_n, y) - \chi d^2(s_n, t_n) - \chi d^2(s_{n+1}, t_n) - (1-\chi) d^2(s_n, s_{n+1}) \\
&+ 2\chi\tau_n\left(f(t_{n-1}, s_{n+1}) - f(t_{n-1}, t_n) - f(t_n, s_{n+1})\right) \\
&+ 2\chi\tau_n f(t_n, y) \quad (\forall y \in C).
\end{aligned} \tag{3.25}$$

By the definition of τ_{n+1} in (3.18), we have

$$\begin{aligned}
&f(t_{n-1}, s_{n+1}) - f(t_{n-1}, t_n) - f(t_n, s_{n+1}) \\
&\leq \frac{\delta}{\tau_{n+1}}\left(d(t_{n-1}, t_n)\, d(s_{n+1}, t_n)\right) \\
&\leq \frac{\delta}{2\tau_{n+1}}\left(d^2(t_{n-1}, t_n) + d^2(s_{n+1}, t_n)\right).
\end{aligned} \tag{3.26}$$

Using (3.25) and (3.26), we deduce that

$$\begin{aligned}
d^2(s_{n+1}, y) \leq{}& d^2(s_n, y) - \chi d^2(s_n, t_n) - \chi d^2(s_{n+1}, t_n) - (1-\chi) d^2(s_n, s_{n+1}) \\
&+ \frac{\chi\delta\tau_n}{\tau_{n+1}}\left(d^2(t_{n-1}, t_n) + d^2(s_{n+1}, t_n)\right) + 2\chi\tau_n f(t_n, y) \quad (\forall y \in C).
\end{aligned} \tag{3.27}$$

By the triangle inequality and the inequality $(a+b)^2 \leq 2(a^2 + b^2)$, $\forall a, b \in \mathbb{R}$, one can show that

$$d^2\left(t_{n-1},t_n\right)\le\left(d\left(t_{n-1},s_n\right)+d\left(s_n,t_n\right)\right)^2\le 2d^2\left(t_{n-1},s_n\right)+2d^2\left(s_n,t_n\right). \tag{3.28}$$

From (3.27) and (3.28), we have

$$\begin{aligned} d^2\left(s_{n+1},y\right)\le\, & d^2\left(s_n,y\right)-\chi d^2\left(s_n,t_n\right)-\chi d^2\left(s_{n+1},t_n\right)-(1-\chi)d^2\left(s_n,s_{n+1}\right)\\ &+\frac{2\chi\delta\tau_n}{\tau_{n+1}}d^2\left(t_{n-1},s_n\right)+\frac{2\chi\delta\tau_n}{\tau_{n+1}}d^2\left(s_n,t_n\right)\\ &+\frac{\chi\delta\tau_n}{\tau_{n+1}}d^2\left(s_{n+1},t_n\right)+2\chi\tau_n f\left(t_n,y\right)\\ =\,& d^2\left(s_n,y\right)-\chi\left(1-\frac{2\delta\tau_n}{\tau_{n+1}}\right)d^2\left(s_n,t_n\right)-\chi\left(1-\frac{\delta\tau_n}{\tau_{n+1}}\right)d^2\left(s_{n+1},t_n\right)\\ &+\frac{2\chi\delta\tau_n}{\tau_{n+1}}d^2\left(t_{n-1},s_n\right)-(1-\chi)d^2\left(s_n,s_{n+1}\right)+2\chi\tau_n f\left(t_n,y\right). \end{aligned} \tag{3.29}$$

Adding the term $\frac{2\chi\delta\tau_{n+1}}{\tau_{n+2}}d^2\left(s_{n+1},t_n\right)$ to both sides of the last inequality in (3.29), we obtain

$$\begin{aligned} & d^2\left(s_{n+1},y\right)+\frac{2\chi\delta\tau_{n+1}}{\tau_{n+2}}d^2\left(s_{n+1},t_n\right)\\ \le\, & d^2\left(s_n,y\right)+\frac{2\chi\delta\tau_n}{\tau_{n+1}}d^2\left(t_{n-1},s_n\right)-\chi\left(1-\frac{2\delta\tau_n}{\tau_{n+1}}\right)d^2\left(s_n,t_n\right)\\ & -\chi\left(1-\frac{\delta\tau_n}{\tau_{n+1}}-\frac{2\delta\tau_{n+1}}{\tau_{n+2}}\right)d^2\left(s_{n+1},t_n\right)\\ & -(1-\chi)d^2\left(s_n,s_{n+1}\right)+2\chi\tau_n f\left(t_n,y\right)\quad(\forall y\in C). \end{aligned} \tag{3.30}$$

Fix $p\in\Omega$. We obtain $f\left(t_n,p\right)\le 0$ by means of $f\left(p,t_n\right)\ge 0$ and the pseudomonotonicity of f. Letting $y=p$ in (3.30). Then we have

$$\begin{aligned} & d^2\left(s_{n+1},p\right)+\frac{2\chi\delta\tau_{n+1}}{\tau_{n+2}}d^2\left(s_{n+1},t_n\right)\\ \le\, & d^2\left(s_n,p\right)+\frac{2\chi\delta\tau_n}{\tau_{n+1}}d^2\left(t_{n-1},s_n\right)-\chi\left(1-\frac{2\delta\tau_n}{\tau_{n+1}}\right)d^2\left(s_n,t_n\right)\\ & -\chi\left(1-\frac{\delta\tau_n}{\tau_{n+1}}-\frac{2\delta\tau_{n+1}}{\tau_{n+2}}\right)d^2\left(s_{n+1},t_n\right)-(1-\chi)d^2\left(s_n,s_{n+1}\right). \end{aligned} \tag{3.31}$$

Next we consider two cases of χ in (3.31).

Case 1. First, we consider $\chi\in(0,1]$. Then the term $(1-\chi)d^2\left(s_n,s_{n+1}\right)\ge 0$ for all $n\ge 0$. Let $\phi_1\in(0,1-3\delta)$ be a fixed number. By using Lemma 3.3, we have

$$\lim_{n\to\infty}\left(1-\frac{2\delta\tau_n}{\tau_{n+1}}\right)=1-2\delta>1-3\delta>\phi_1>0$$

and

$$\lim_{n\to\infty}\left(1-\frac{\delta\tau_n}{\tau_{n+1}}-\frac{2\delta\tau_{n+1}}{\tau_{n+2}}\right)=1-3\delta>\phi_1>0.$$

Therefore, there exists a positive constant $n_1\in\mathbb{N}$ such that

$$\left(1-\frac{2\delta\tau_n}{\tau_{n+1}}\right)>\phi_1>0\ \ (\forall n\ge n_1)$$

and

$$\left(1-\frac{\delta\tau_n}{\tau_{n+1}}-\frac{2\delta\tau_{n+1}}{\tau_{n+2}}\right)>\phi_1>0\ \ (\forall n\ge n_1).$$

Now, it follows from (3.31) that

$$\begin{aligned} & d^2\left(s_{n+1},p\right)+\frac{2\chi\delta\tau_{n+1}}{\tau_{n+2}}d^2\left(s_{n+1},t_n\right)\\ \le\, & d^2\left(s_n,p\right)+\frac{2\chi\delta\tau_n}{\tau_{n+1}}d^2\left(t_{n-1},s_n\right)-\chi\phi_1\left(d^2\left(s_n,t_n\right)+d^2\left(s_{n+1},t_n\right)\right)\ \ (\forall n\ge n_1). \end{aligned}$$

Setting

$$a_n:=d^2\left(s_n,p\right)+\frac{2\chi\delta\tau_n}{\tau_{n+1}}d^2\left(t_{n-1},s_n\right)$$

and

$$b_n := \chi\phi_1\left(d^2\left(s_n,t_n\right)+d^2\left(s_{n+1},t_n\right)\right).$$

Then we obtain that $a_{n+1}\le a_n-b_n$ for all $n\ge n_1$. Note that $b_n\ge 0$ for all $n\ge n_1$. Thus we have $a_{n+1}\le a_n$, $\forall n\ge n_1$. Then the limit of $\{a_n\}$ exists and hence $\lim_{n\to\infty} b_n=0$. That is

$$\lim_{n\to\infty} d^2\left(s_n,t_n\right)=0 \quad\text{and}\quad \lim_{n\to\infty} d^2\left(s_{n+1},t_n\right)=0. \tag{3.32}$$

Combining the definition of a_n, $\lim_{n\to\infty} a_n$ exists, and (3.32), we deduce that $\lim_{n\to\infty} d^2\left(s_n,p\right)$ exists for all $p\in\Omega$. Consequently, $\{s_n\}$ is bounded. It follows from the boundedness of $\{s_n\}$ and (3.32) that $\{t_n\}$ is also bounded. According to the triangle inequality and (3.32), we have

$$\lim_{n\to\infty} d^2\left(s_n,s_{n+1}\right)=0. \tag{3.33}$$

Case 2. We consider $\chi\in(1,2/(1+3\delta))$. Note that $(1-\chi)d^2\left(s_n,s_{n+1}\right)<0$ for all $n\ge 0$. From the triangle inequality, one has

$$d^2\left(s_n,s_{n+1}\right)\le\left(d\left(s_n,t_n\right)+d\left(s_{n+1},t_n\right)\right)^2\le 2\left(d^2\left(s_n,t_n\right)+d^2\left(s_{n+1},t_n\right)\right). \tag{3.34}$$

Combining (3.31) and (3.34), we have

$$\begin{aligned}
&d^2\left(s_{n+1},p\right)+\frac{2\chi\delta\tau_{n+1}}{\tau_{n+2}}d^2\left(s_{n+1},t_n\right)\\
&\le d^2\left(s_n,p\right)+\frac{2\chi\delta\tau_n}{\tau_{n+1}}d^2\left(t_{n-1},s_n\right)-\left(2-\chi-\frac{2\chi\delta\tau_n}{\tau_{n+1}}\right)d^2\left(s_n,t_n\right)\\
&\quad-\left(2-\chi-\frac{\chi\delta\tau_n}{\tau_{n+1}}-\frac{2\chi\delta\tau_{n+1}}{\tau_{n+2}}\right)d^2\left(s_{n+1},t_n\right).
\end{aligned}$$

Let $\phi_2\in(0,2-\chi-3\chi\delta)$ be a fixed number. Then

$$\lim_{n\to\infty}\left(2-\chi-\frac{2\chi\delta\tau_n}{\tau_{n+1}}\right)=2-\chi-2\chi\delta>\phi_2>0$$

and

$$\lim_{n\to\infty}\left(2-\chi-\frac{\chi\delta\tau_n}{\tau_{n+1}}-\frac{2\chi\delta\tau_{n+1}}{\tau_{n+2}}\right)=2-\chi-3\chi\delta>\phi_2>0.$$

Therefore, there exists a positive number $n_2\in\mathbb{N}$ such that

$$2-\chi-\frac{2\chi\delta\tau_n}{\tau_{n+1}}>\phi_2>0\ (\forall n\ge n_2)$$

and

$$2-\chi-\frac{\chi\delta\tau_n}{\tau_{n+1}}-\frac{2\chi\delta\tau_{n+1}}{\tau_{n+2}}>\phi_2>0\ (\forall n\ge n_2).$$

Consequently, we have

$$\begin{aligned}
&d^2\left(s_{n+1},p\right)+\frac{2\chi\delta\tau_{n+1}}{\tau_{n+2}}d^2\left(s_{n+1},t_n\right)\\
&\le d^2\left(s_n,p\right)+\frac{2\chi\delta\tau_n}{\tau_{n+1}}d^2\left(t_{n-1},s_n\right)-\phi_2\left(d^2\left(s_n,t_n\right)+d^2\left(s_{n+1},t_n\right)\right)\ (\forall n\ge n_2).
\end{aligned}$$

Based on the statements in *Case 1*, we can obtain that $\lim_{n\to\infty} d^2\left(s_n,p\right)$ exists for all $p\in\Omega$, $\{s_n\}$ and $\{t_n\}$ are bounded, and (3.32) and (3.33) hold. This completes the proof. □

Theorem 3.2. *Let $\{s_n\}$ be generated by Algorithm 3.2 and satisfy Conditions (A1)–(A4) and (C1)–(C3). Then $\{s_n\}$ converges to a solution of EP (1.1).*

Proof. In view of Lemma 3.4, we have that $\lim_{n\to\infty} d^2\left(s_n,p\right)$ exists for all $p\in\Omega$ and $\{s_n\}$ is bounded. Next, we show that any cluster point of $\{s_n\}$ belongs to Ω. Let $s^*\in C$ be a cluster point of $\{s_n\}$. Since $\{s_n\}$ is bounded, there exists a subsequence $\left\{s_{n_k}\right\}$ of $\{s_n\}$ such that $\lim_{k\to\infty} s_{n_k}=s^*$. By virtue of (3.32), one sees that $\lim_{k\to\infty} t_{n_k}=s^*$. From (3.30), we have

$$\begin{aligned} 2\chi\tau_n f\left(t_n, y\right) \geq\, & d^2\left(s_{n+1}, y\right)-d^2\left(s_n, y\right)-\frac{2\chi\delta\tau_n}{\tau_{n+1}} d^2\left(t_{n-1}, s_n\right) \\ & +\chi\left(1-\frac{2\delta\tau_n}{\tau_{n+1}}\right) d^2\left(s_n, t_n\right)+\chi\left(1-\frac{\delta\tau_n}{\tau_{n+1}}\right) d^2\left(s_{n+1}, t_n\right) \\ & +(1-\chi) d^2\left(s_n, s_{n+1}\right) \quad(\forall y \in C). \end{aligned} \tag{3.35}$$

From the triangle inequality, one obtains

$$\left|d^2\left(s_{n+1}, y\right)-d^2\left(s_n, y\right)\right| \leq d\left(s_{n+1}, s_n\right)\left(d\left(s_{n+1}, y\right)+d\left(s_n, y\right)\right) \quad(\forall y \in C),$$

which together with (3.33) and the boundedness of $\{s_n\}$ yields that

$$\lim_{n\to\infty}\left|d^2\left(s_{n+1}, y\right)-d^2\left(s_n, y\right)\right|=0 \quad(\forall y \in C). \tag{3.36}$$

Replacing n in (3.35) with n_k and letting $k \to +\infty$, we obtain that the right-hand side of inequality (3.35) tends to 0 by means of (3.32), (3.33), (3.36), and $\lim_{k\to\infty} \tau_{n_k}=\tau>0$. Thus we have $\limsup_{k\to\infty} 2\chi\tau f\left(t_{n_k}, y\right) \geq 0$ for all $y \in C$. From Condition (A4), we have

$$f\left(s^*, y\right) \geq \limsup_{k\to\infty} f\left(t_{n_k}, y\right) \geq 0 \quad(\forall y \in C).$$

It follows that $s^* \in \Omega$. Then $\lim_{n\to\infty} d^2\left(s_n, s^*\right)$ exists. Thus the sequence of positive numbers $\left\{d^2\left(s_n, s^*\right)\right\}$ is convergent and bounded, and it has a subsequence, namely $\left\{d^2\left(s_{n_k}, s^*\right)\right\}$, which converges to 0. Then the whole sequence converges to 0, i.e., $0=\lim_{n\to\infty} d\left(s_n, s^*\right)$ implying $s^*=\lim_{n\to\infty} s_n$. Therefore we conclude that $\{s_n\}$ converges to a solution of EP (1.1). □

Remark 3.4. As similarly stated in Remark 3.2, it is easy to check that the step size criterion (3.18) used in the suggested Algorithm 3.2 can be replaced by the following (3.37)

$$\tau_{n+1}=\begin{cases}\min\left\{\dfrac{\delta\left(d^2\left(t_{n-1}, t_n\right)+d^2\left(s_{n+1}, t_n\right)\right)}{2\Delta_n}, \xi_n\tau_n+\sigma_n\right\}, & \text{if } \Delta_n>0; \\ \xi_n\tau_n+\sigma_n, & \text{otherwise,}\end{cases} \tag{3.37}$$

where Δ_n is defined in (3.18).

To conclude this section, we introduce a modified golden ratio algorithm that requires the computation of the strongly convex optimization problem on the feasible set only once in each iteration. The proposed method is inspired by the work of Yin et al. [40] and extends the algorithm of Yin et al. [40] from Hilbert spaces to Hadamard manifolds. Now, the last iterative scheme proposed in this paper is shown in Algorithm 3.3.

Algorithm 3.3

Initialization: Choose $\tau_{-1}=\tau_0>0$, $\delta\in(0,1)$, and $\mu\in(1/(2-\delta),1)$. Let $\{\xi_n\}$ and $\{\sigma_n\}$ satisfy Condition (C3). Let $t_{-1}, t_0, s_{-1}\in C$ be initial points and set $n=0$.
Iterative Steps: Given the current iterates s_{n-1}, t_{n-1}, t_n, calculate s_n and t_{n+1} as follows.
Step 1. Compute

$$s_n=\exp_{t_n}\left(\chi_n \exp_{t_n}^{-1} s_{n-1}\right),$$

where

$$\chi_n=\min\left\{\frac{1}{2}\sqrt{1+4\mu\frac{\tau_n}{\tau_{n-1}}}-\frac{1}{2}, 1\right\}.$$

Step 2. Compute

$$t_{n+1}=\arg\min_{y\in C}\left\{f\left(t_n, y\right)+\frac{1}{2\tau_n} d^2\left(s_n, y\right)\right\}=\operatorname{prox}_{\tau_n f\left(t_n,\cdot\right)}\left(s_n\right).$$

If $s_n=t_n=t_{n+1}$, then stop the iterative process and s_n is a solution of EP (1.1); Otherwise, go to *Step 3*.
Step 3. Compute the next step size by

$$\tau_{n+1}=\begin{cases}\min\left\{\dfrac{\delta d\left(t_{n-1}, t_n\right) d\left(t_{n+1}, t_n\right)}{2\chi_n\Delta_n}, \xi_n\tau_n+\sigma_n\right\}, & \text{if } \Delta_n>0; \\ \xi_n\tau_n+\sigma_n, & \text{otherwise.}\end{cases} \tag{3.38}$$

where $\Delta_n:=f\left(t_{n-1}, t_{n+1}\right)-f\left(t_{n-1}, t_n\right)-f\left(t_n, t_{n+1}\right)$.
Set $n:=n+1$ and go to *Step 1*.

Remark 3.5. From the definition of t_{n+1} in Algorithm 3.3 and Remark 2.2, we have

$$\tau_n\left(f\left(t_n,y\right)-f\left(t_n,t_{n+1}\right)\right)\geq\left\langle \exp_{t_{n+1}}^{-1} s_n,\exp_{t_{n+1}}^{-1} y\right\rangle \ (\forall y\in C).$$

If $t_{n+1}=s_n=t_n$ for some $n\in\mathbb{N}$, then we obtain that $f\left(t_n,y\right)\geq 0$ for all $y\in C$ since $\tau_n>0$. That is, $t_n\in\Omega$. Thus the iterations of Algorithm 3.3 terminate when $t_{n+1}=s_n=t_n$.

Lemma 3.5. *Let step size $\{\tau_n\}$ be a sequence generated by (3.38) and Conditions (A2) and (C3) hold. Then $\{\tau_n\}$ is well defined and $\lim_{n\to\infty}\tau_n$ exists.*

Proof. By the definition of χ_n, it follows that $\chi_n\leq 1$ for all $n\geq 0$. This combining the fact that f satisfies the Lipschitz-type condition (2.4), in the case of $\Delta_n>0$, one obtains

$$\begin{aligned}\frac{\delta d\left(t_{n-1},t_n\right)d\left(t_{n+1},t_n\right)}{2\chi_n\left(f\left(t_{n-1},t_{n+1}\right)-f\left(t_{n-1},t_n\right)-f\left(t_n,t_{n+1}\right)\right)}&\geq\frac{\delta d\left(t_{n-1},t_n\right)d\left(t_{n+1},t_n\right)}{2\chi_n L d\left(t_{n-1},t_n\right)d\left(t_{n+1},t_n\right)}\\&\geq\frac{\delta}{2L}.\end{aligned}$$

The rest of the proof is the same as Lemma 3.1 and is therefore omitted. □

Remark 3.6. From Lemma 3.5, one sees that the limit of χ_n exists, denoted as χ. That is, $\lim_{n\to\infty}\chi_n=\chi$. From $\delta\in(0,1)$ and $\mu\in(1/(2-\delta),1)$, it follows that $0<\chi_n\leq 1$ for all $n\geq 0$.

Lemma 3.6. *Let $\{s_n\}$ and $\{t_{n+1}\}$ be created by Algorithm 3.3. Take $p\in\Omega$. Then $\{s_n\}$ and $\{t_n\}$ are bounded and $\lim_{n\to\infty}d^2\left(s_n,p\right)$ exists.*

Proof. From the definition of t_{n+1} in Algorithm 3.3 and Lemma 2.4, one has

$$\tau_n\left(f\left(t_n,y\right)-f\left(t_n,t_{n+1}\right)\right)\geq\left\langle \exp_{t_{n+1}}^{-1} s_n,\exp_{t_{n+1}}^{-1} y\right\rangle \ (\forall y\in C). \tag{3.39}$$

Letting $n=n-1$ in (3.39), one sees that

$$\tau_{n-1}\left(f\left(t_{n-1},y\right)-f\left(t_{n-1},t_n\right)\right)\geq\left\langle \exp_{t_n}^{-1} s_{n-1},\exp_{t_n}^{-1} y\right\rangle \ (\forall y\in C). \tag{3.40}$$

Putting $y=t_{n+1}\in C$ in (3.40), we have

$$\tau_{n-1}\left(f\left(t_{n-1},t_{n+1}\right)-f\left(t_{n-1},t_n\right)\right)\geq\left\langle \exp_{t_n}^{-1} s_{n-1},\exp_{t_n}^{-1} t_{n+1}\right\rangle . \tag{3.41}$$

By using $s_n=\exp_{t_n}\left(\chi_n\exp_{t_n}^{-1} s_{n-1}\right)$, one obtains

$$\exp_{t_n}^{-1} s_n=\chi_n\exp_{t_n}^{-1} s_{n-1}. \tag{3.42}$$

Combining (3.41), (3.42), and $\tau_n>0$, we deduce that

$$\tau_n\left(f\left(t_{n-1},t_{n+1}\right)-f\left(t_{n-1},t_n\right)\right)\geq\frac{1}{\chi_n}\frac{\tau_n}{\tau_{n-1}}\left\langle \exp_{t_n}^{-1} s_n,\exp_{t_n}^{-1} t_{n+1}\right\rangle . \tag{3.43}$$

Let $\Delta\left(s_n,t_{n+1},y\right)$ be a geodesic triangle. From (2.2), one sees that

$$2\left\langle \exp_{t_{n+1}}^{-1} s_n,\exp_{t_{n+1}}^{-1} y\right\rangle\geq d^2\left(s_n,t_{n+1}\right)+d^2\left(t_{n+1},y\right)-d^2\left(s_n,y\right)\ (\forall y\in C). \tag{3.44}$$

Similarly, let $\Delta\left(s_n,t_n,t_{n+1}\right)$ be a geodesic triangle. Then by (2.2) we obtain

$$2\left\langle \exp_{t_n}^{-1} s_n,\exp_{t_n}^{-1} t_{n+1}\right\rangle\geq d^2\left(s_n,t_n\right)+d^2\left(t_{n+1},t_n\right)-d^2\left(s_n,t_{n+1}\right). \tag{3.45}$$

From (3.39), (3.43), (3.44), and (3.45), we have

$$\begin{aligned}&2\tau_n\left(f\left(t_{n-1},t_{n+1}\right)-f\left(t_{n-1},t_n\right)-f\left(t_n,t_{n+1}\right)\right)+2\tau_n f\left(t_n,y\right)\\&\geq 2\frac{1}{\chi_n}\frac{\tau_n}{\tau_{n-1}}\left\langle \exp_{t_n}^{-1} s_n,\exp_{t_n}^{-1} t_{n+1}\right\rangle+2\left\langle \exp_{t_{n+1}}^{-1} s_n,\exp_{t_{n+1}}^{-1} y\right\rangle\\&\geq\frac{1}{\chi_n}\frac{\tau_n}{\tau_{n-1}}\left(d^2\left(s_n,t_n\right)+d^2\left(t_{n+1},t_n\right)-d^2\left(s_n,t_{n+1}\right)\right)\\&\quad+d^2\left(s_n,t_{n+1}\right)+d^2\left(t_{n+1},y\right)-d^2\left(s_n,y\right)\ (\forall y\in C).\end{aligned} \tag{3.46}$$

It follows from the definition of τ_{n+1} that

$$\begin{aligned}&2\tau_n\left(f\left(t_{n-1},t_{n+1}\right)-f\left(t_{n-1},t_n\right)-f\left(t_n,t_{n+1}\right)\right)\\&\le\frac{1}{\chi_n}\frac{\delta\tau_n}{\tau_{n+1}}d\left(t_{n-1},t_n\right)d\left(t_{n+1},t_n\right)\\&\le\frac{1}{2\chi_n}\frac{\delta\tau_n}{\tau_{n+1}}\left(d^2\left(t_{n-1},t_n\right)+d^2\left(t_{n+1},t_n\right)\right).\end{aligned}\tag{3.47}$$

Combining (3.46) and (3.47), we arrive at

$$\begin{aligned}&d^2(t_{n+1},y)+\left(\frac{1}{\chi_n}\frac{\tau_n}{\tau_{n-1}}-\frac{1}{2\chi_n}\frac{\delta\tau_n}{\tau_{n+1}}\right)d^2(t_{n+1},t_n)\\&\le d^2(s_n,y)+\frac{1}{2\chi_n}\frac{\delta\tau_n}{\tau_{n+1}}d^2(t_{n-1},t_n)+2\tau_nf\left(t_n,y\right)\\&\quad+\left(\frac{1}{\chi_n}\frac{\tau_n}{\tau_{n-1}}-1\right)d^2(s_n,t_{n+1})-\frac{1}{\chi_n}\frac{\tau_n}{\tau_{n-1}}d^2(s_n,t_n).\end{aligned}\tag{3.48}$$

Now, we estimate the term $d^2\left(t_{n+1},y\right)$ in (3.48). Recall from Algorithm 3.3 that $s_{n+1}=\exp_{t_{n+1}}\left(\chi_{n+1}\exp^{-1}_{t_{n+1}}s_n\right)$. Then s_{n+1} in the geodesic joining t_{n+1} to s_n. The comparison point of s_{n+1} is $s'_{n+1}=(1-\chi_{n+1})t'_{n+1}+\chi_{n+1}s'_n$, which together with Remark 3.6 implies that $s'_{n+1}\in\left[t'_{n+1},s'_n\right]$. Let $\Delta(s_{n+1},t_{n+1},s_n)$ be a geodesic triangle and $\Delta\left(s'_{n+1},t'_{n+1},s'_n\right)$ be its comparison triangle. By Lemma 2.1 we have

$$d\left(t_{n+1},s_{n+1}\right)=\left\|t'_{n+1}-s'_{n+1}\right\|,\quad d\left(s_n,s_{n+1}\right)=\left\|s'_n-s'_{n+1}\right\|.$$

Let $\Delta(y,t_{n+1},s_n)$ be a geodesic triangle and $\Delta\left(y',t'_{n+1},s'_n\right)$ be its comparison triangle. From Lemma 2.1, one has

$$d\left(t_{n+1},y\right)=\left\|t'_{n+1}-y'\right\|,\quad d\left(s_n,y\right)=\|s'_n-y'\|,\quad d\left(s_n,t_{n+1}\right)=\left\|s'_n-t'_{n+1}\right\|.\tag{3.49}$$

Let $\alpha\in\mathbb{R}$ and $x,y\in\mathbb{R}$. Then

$$\|\alpha x+(1-\alpha)y\|^2=\alpha\|x\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)\|x-y\|^2.$$

It follows from Lemma 2.2(ii) and (3.49) that

$$\begin{aligned}d^2\left(s_{n+1},y\right)&\le\left\|(1-\chi_{n+1})t'_{n+1}+\chi_{n+1}s'_n-y'\right\|^2\\&=\left\|(1-\chi_{n+1})\left(t'_{n+1}-y'\right)+\chi_{n+1}\left(s'_n-y'\right)\right\|^2\\&=(1-\chi_{n+1})\left(t'_{n+1}-y'\right)^2+\chi_{n+1}\left(s'_n-y'\right)^2\\&\quad-\chi_{n+1}(1-\chi_{n+1})\left\|t'_{n+1}-s'_n\right\|^2\\&=(1-\chi_{n+1})d^2\left(t_{n+1},y\right)+\chi_{n+1}d^2\left(s_n,y\right)\\&\quad-\chi_{n+1}(1-\chi_{n+1})d^2\left(t_{n+1},s_n\right).\end{aligned}\tag{3.50}$$

That is

$$d^2\left(t_{n+1},y\right)\ge-\frac{\chi_{n+1}}{1-\chi_{n+1}}d^2\left(s_n,y\right)+\frac{1}{1-\chi_{n+1}}d^2\left(s_{n+1},y\right)+\chi_{n+1}d^2\left(t_{n+1},s_n\right).\tag{3.51}$$

Combining (3.48) and (3.51), we deduce that

$$\begin{aligned}&\frac{1}{1-\chi_{n+1}}d^2\left(s_{n+1},y\right)+\left(\frac{1}{\chi_n}\frac{\tau_n}{\tau_{n-1}}-\frac{1}{2\chi_n}\frac{\delta\tau_n}{\tau_{n+1}}\right)d^2\left(t_{n+1},t_n\right)\\&\le\frac{1}{1-\chi_{n+1}}d^2\left(s_n,y\right)+\frac{1}{2\chi_n}\frac{\delta\tau_n}{\tau_{n+1}}d^2\left(t_{n-1},t_n\right)+2\tau_nf\left(t_n,y\right)\\&\quad+\left(\frac{1}{\chi_n}\frac{\tau_n}{\tau_{n-1}}-1-\chi_{n+1}\right)d^2\left(s_n,t_{n+1}\right)-\frac{1}{\chi_n}\frac{\tau_n}{\tau_{n-1}}d^2\left(s_n,t_n\right).\end{aligned}\tag{3.52}$$

From the definition of χ_n, we obtain that there exists $n_0\ge0$ such that

$$\chi_n=\frac{1}{2}\sqrt{1+4\mu\frac{\tau_n}{\tau_{n-1}}}-\frac{1}{2}\quad(\forall n\ge n_0).$$

That is

$$1+\chi_n-\frac{1}{\chi_n}\frac{\mu\tau_n}{\tau_{n-1}}=0 \quad (\forall n\geq n_0). \tag{3.53}$$

By using $\mu\in(1/(2-\delta),1)$ and Remark 3.6, one sees that

$$\lim_{n\to\infty}\left(\frac{1}{\chi_n}\frac{\tau_n}{\tau_{n-1}}-\frac{1}{\chi_{n+1}}\frac{\mu\tau_{n+1}}{\tau_n}\right)=\frac{1}{\chi}(1-\mu)>0. \tag{3.54}$$

From (3.53) and (3.54), there exists $n_1\left(\geq n_0\right)$ such that

$$\frac{1}{\chi_n}\frac{\tau_n}{\tau_{n-1}}-1-\chi_{n+1}=\frac{1}{\chi_n}\frac{\tau_n}{\tau_{n-1}}-\frac{1}{\chi_{n+1}}\frac{\mu\tau_{n+1}}{\tau_n}>0 \quad (\forall n\geq n_1). \tag{3.55}$$

Note that

$$\begin{aligned} d^2\left(s_n,t_{n+1}\right) &\leq \left(d\left(s_n,t_n\right)+d\left(t_n,t_{n+1}\right)\right)^2 \\ &\leq \left(1+\frac{1}{\eta}\right)d^2\left(s_n,t_n\right)+(1+\eta)d^2\left(t_n,t_{n+1}\right), \end{aligned} \tag{3.56}$$

where $\eta=1-\delta>0$. Combining (3.52), (3.55), and (3.56), we have

$$\begin{aligned} d^2\left(s_{n+1},y\right)+\alpha_n d^2\left(t_{n+1},t_n\right) &\leq d^2\left(s_n,y\right)+\beta_n d^2\left(t_{n-1},t_n\right)-\gamma_n d^2\left(s_n,t_n\right) \\ &\quad +2\left(1-\chi_{n+1}\right)\tau_n f\left(t_n,y\right) \quad (\forall n\geq n_1), \end{aligned} \tag{3.57}$$

where

$$\begin{aligned} \alpha_n &:= \left(1-\chi_{n+1}\right)\left((1+\eta)\frac{1}{\chi_{n+1}}\frac{\mu\tau_{n+1}}{\tau_n}-\frac{\eta}{\chi_n}\frac{\tau_n}{\tau_{n-1}}-\frac{1}{2\chi_n}\frac{\delta\tau_n}{\tau_{n+1}}\right), \\ \beta_n &:= \left(1-\chi_{n+1}\right)\frac{1}{2\chi_n}\frac{\delta\tau_n}{\tau_{n+1}}, \\ \gamma_n &:= \left(1-\chi_{n+1}\right)\left(\left(1+\frac{1}{\eta}\right)\frac{1}{\chi_{n+1}}\frac{\mu\tau_{n+1}}{\tau_n}-\frac{1}{\eta}\frac{1}{\chi_n}\frac{\tau_n}{\tau_{n-1}}\right). \end{aligned}$$

From the facts that $\lim_{n\to\infty}\tau_n=\tau>0$, $\lim_{n\to\infty}\chi_n=\chi\in(0,1)$, $\delta\in(0,1)$, $\mu\in(1/(2-\delta),1)$, and $\eta=1-\delta$, we obtain

$$\begin{aligned} &\lim_{n\to\infty}\alpha_n=(1-\chi)\frac{1}{\chi}\left((1+\eta)\mu-\eta-\frac{\delta}{2}\right)>0, \\ &\lim_{n\to\infty}\beta_n=(1-\chi)\frac{\delta}{2\chi}>0, \\ &\lim_{n\to\infty}\gamma_n=(1-\chi)\frac{1}{\chi}\left(\left(1+\frac{1}{\eta}\right)\mu-\frac{1}{\eta}\right)>0, \end{aligned}$$

and

$$(1-\chi)\frac{1}{\chi}\left((1+\eta)\mu-\eta-\frac{\delta}{2}\right)-(1-\chi)\frac{\delta}{2\chi}>0.$$

According to the deneseness of rational numbers, there exists $\rho>0$ such that

$$(1-\chi)\frac{1}{\chi}\left((1+\eta)\mu-\eta-\frac{\delta}{2}\right)>\rho>(1-\chi)\frac{\delta}{2\chi}.$$

That is, there exists $n_2\left(\geq n_1\right)$ such that

$$\alpha_n>\rho>\beta_n>0 \quad \text{and} \quad \gamma_n>0 \quad (\forall n\geq n_2). \tag{3.58}$$

By means of (3.57) and (3.58), we infer that

$$\begin{aligned} d^2\left(s_{n+1},y\right)+\rho d^2\left(t_{n+1},t_n\right) &\leq d^2\left(s_n,y\right)+\rho d^2\left(t_{n-1},t_n\right)-\gamma_n d^2\left(s_n,t_n\right) \\ &\quad +2\left(1-\chi_{n+1}\right)\tau_n f\left(t_n,y\right) \quad (\forall y\in C)(\forall n\geq n_2). \end{aligned} \tag{3.59}$$

In view of $p\in\Omega$ and $t_n\in C$, one sees that $f\left(p,t_n\right)\geq 0$. This together with the pseudomonotonicity of f yields that $f\left(t_n,p\right)\leq 0$. Letting $y=p\in C$ in (3.59) and setting

$$a_n=d^2\left(s_n,p\right)+\rho d^2\left(t_{n-1},t_n\right) \quad \text{and} \quad b_n=\gamma_n d^2\left(s_n,t_n\right),$$

we deduce that $a_{n+1}\leq a_n-b_n$ for all $n\geq n_2$. This implies that $\left\{a_n\right\}$ is bounded and the limit of $\left\{a_n\right\}$ exists. Thus $\lim_{n\to\infty}b_n=0$. By the definition of b_n and $\lim_{n\to\infty}\gamma_n>0$, one has

$$\lim_{n\to\infty}d(s_n,t_n)=0. \tag{3.60}$$

From (3.42), we have $d(s_n, t_n) = \chi_n d(s_{n-1}, t_n)$. This together with (3.60) yields that

$$\lim_{n\to\infty} d(s_n, t_{n+1}) = 0.$$

Since

$$d\left(s_n, s_{n+1}\right) \leq d\left(s_n, t_{n+1}\right) + d\left(t_{n+1}, s_{n+1}\right),$$

and

$$d\left(t_{n-1}, t_n\right) \leq d\left(t_{n-1}, s_{n-1}\right) + d\left(s_{n-1}, t_n\right),$$

we see that

$$\lim_{n\to\infty} d\left(s_n, s_{n+1}\right) = \lim_{n\to\infty} d\left(t_{n-1}, t_n\right) = 0. \tag{3.61}$$

From the existence of $\lim_{n\to\infty} a_n$ and (3.61), we obtain

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} d^2\left(s_n, p\right).$$

This implies that the sequence $\{d^2\left(s_n, p\right)\}$ is bounded for all $p \in \Omega$ and thus $\left\{s_n\right\}$ is bounded. So $\left\{t_n\right\}$ is bounded by means of (3.60). □

Theorem 3.3. *Let $\{s_n\}$ be formed by Algorithm 3.3 and satisfy Conditions (A1)–(A4) and (C1)–(C3). Then $\{s_n\}$ converges to a solution of EP (1.1).*

Proof. It follows from Lemma 3.6 that $\{s_n\}$ is bounded. Hence there exists a subsequence $\left\{s_{n_k}\right\}$ of $\left\{s_n\right\}$ that converges to some $s^* \in \mathcal{M}$. Consequently, $\lim_{n\to\infty} t_{n_k} = s^*$ and $s^* \in C$ by means of (3.60) and the definition of t_{n+1}. Now we prove that $s^* \in \Omega$. Indeed, it follows from relation (3.59) that

$$\begin{aligned} 2\left(1-\chi_{n+1}\right)\tau_n f\left(t_n, y\right) \geq\ & d^2\left(s_{n+1}, y\right) - d^2\left(s_n, y\right) + \rho d^2\left(t_{n+1}, t_n\right) \\ & - \rho d^2\left(t_{n-1}, t_n\right) + \gamma_n d^2\left(s_n, t_n\right) \quad (\forall y \in C)(\forall n \geq n_2). \end{aligned} \tag{3.62}$$

From the triangle inequality, one obtains

$$\left|d^2\left(s_{n+1}, y\right) - d^2\left(s_n, y\right)\right| \leq d\left(s_{n+1}, s_n\right)\left(d\left(s_{n+1}, y\right) + d\left(s_n, y\right)\right) \ \ (\forall y \in C),$$

which together with (3.61) and the boundedness of $\left\{s_n\right\}$ yields that

$$\lim_{n\to\infty} \left|d^2\left(s_{n+1}, y\right) - d^2\left(s_n, y\right)\right| = 0 \ \ (\forall y \in C). \tag{3.63}$$

Replacing n in (3.62) with n_k and letting $k \to +\infty$, we obtain that the right-hand side of inequality (3.62) tends to 0 by means of (3.60), (3.61), (3.63), and $\lim_{k\to\infty} \tau_{n_k} = \tau > 0$. Thus we have

$$\limsup_{k\to\infty} 2\left(1-\chi_{n_k+1}\right)\tau_{n_k} f\left(t_{n_k}, y\right) = 2(1-\chi)\tau f\left(t_{n_k}, y\right) \geq 0 \ \ (\forall y \in C).$$

From Condition (A4), we have

$$f\left(s^*, y\right) \geq \limsup_{k\to\infty} f\left(t_{n_k}, y\right) \geq 0 \ \ (\forall y \in C).$$

It follows that $s^* \in \Omega$. The remainder of the proof is identical to Theorem 4.2. □

Remark 3.7. It is simple to verify that the step size (3.38) used in the proposed Algorithm 3.3 may be substituted by the following (3.64), as stated identically in Remark 3.2.

$$\tau_{n+1} = \begin{cases} \min\left\{ \dfrac{\delta\left(d^2\left(t_{n-1}, t_n\right) + d^2\left(t_{n+1}, t_n\right)\right)}{4\chi_n \Delta_n}, \xi_n \tau_n + \sigma_n \right\}, & \text{if } \Delta_n > 0; \\ \xi_n \tau_n + \sigma_n, & \text{otherwise,} \end{cases} \tag{3.64}$$

where Δ_n is defined in (3.38).

4. Error bounds and linear convergence for strongly pseudomonotone EPs

4.1. Solution existence for strongly pseudomonotone EPs

In this subsection, we show that the EP (1.1) has a unique solution when the bifunction involved satisfies strong pseudomonotonicity and other conditions. Recall that Colao et al. [21] first introduced the equilibrium problem on Hadamard manifolds and proved the existence of its solution (see Lemma 4.1 below). Recently, Al-Homidan et al. [50] also studied the existence of EP under weaker conditions than [21, Theorem 3.2].

Lemma 4.1. ([21, Theorem 3.2][50, Theorem 3.4]) *Let C be a nonempty, closed, and convex subset of a Hadamard manifold $\mathcal{M}$ and $f : C \times C \to \mathbb{R}$ be a bifunction such that $f(\cdot, y)$ is upper semicontinuous for each $y \in C$ and $f(x, \cdot)$ is lower semicontinuous and convex for each $x \in C$. Suppose that the following coercivity condition holds*

$$\exists \text{ compact set } W : (\forall x \in C \backslash W,\ \exists y \in C \cap W : f(x, y) < 0).$$

Then, the equilibrium problem EP (1.1) has a solution.

We can easily obtain the following consequence by using Proposition 1 obtained from Muu and Quy [51].

Theorem 4.1. *Assume that Conditions (A3), (A4), and (C1) hold. Let $f : C \times C \to \mathbb{R}$ be a β-strongly pseudomonotone on C. Then equilibrium problem (1.1) has a unique solution.*

Proof. Let us start by assuming that C is unbounded. According to Lemma 4.1, it suffices to demonstrate the following coercivity condition:

$$\exists \text{ closed ball } B : (\forall x \in C \backslash B,\ \exists y \in C \cap B : f(x, y) < 0). \tag{4.1}$$

In fact, if not, for every closed ball B_r around 0 with radius r, there is $s^r \in C \backslash B_r$ such that $f(s^r, y) \geq 0$ for all $y \in C \cap B_r$. Take $r_0 > 0$, then for every $r > r_0$, there exists $s^r \in C \backslash B_r$ such that $f\left(s^r, y^0\right) \geq 0$ with $y^0 \in C \cap B_{r_0}$. By using the fact that f is β-strongly pseudomonotone, one has

$$f\left(y^0, s^r\right) + \beta d^2\left(s^r, y^0\right) \leq 0 \quad (\forall r \geq r_0). \tag{4.2}$$

Because of C is convex and $f\left(y^0, \cdot\right)$ is convex on C, it follows from Lemma 2.3 that there exists $s^0 \in C$ such that $\partial_2 f\left(y^0, s^0\right) \neq \emptyset$, where $\partial_2 f\left(y^0, s^0\right)$ stands for the subdifferential of the convex function $f\left(y^0, \cdot\right)$ at s^0. Take $w^* \in \partial_2 f\left(y^0, s^0\right) \in T_{s^0}\mathcal{M}$. Using the definition of subgradient, we have

$$\left\langle w^*, \exp_{s^0}^{-1} x\right\rangle + f\left(y^0, s^0\right) \leq f\left(y^0, x\right) \quad (\forall x \in \mathcal{M}).$$

Putting $x = s^r$ in the above inequality, one has

$$\begin{aligned} f\left(y^0, s^r\right) + \beta d^2\left(s^r, y^0\right) &\geq f\left(y^0, s^0\right) + \left\langle w^*, \exp_{s^0}^{-1} s^r\right\rangle + \beta d^2\left(s^r, y^0\right) \\ &\geq f\left(y^0, s^0\right) - \left\|w^*\right\| d\left(s^r, s^0\right) + \beta d^2\left(s^r, y^0\right). \end{aligned}$$

Letting $r \to +\infty$, since $\|s^r\| \to +\infty$, we have that $f\left(y^0, s^r\right) + \beta d^2\left(s^r, y^0\right) \to +\infty$ which contradicts (4.2). As a result, the coercivity criterion (4.1) must be satisfied. Then the equilibrium problem (1.1) has a solution due to Lemma 4.1.

The assertion follows from Ky Fan's theorem [52] when C is bounded.

Now assume that EP (1.1) has two solutions s^* and y^*. Then

$$f\left(s^*, y^*\right) \geq 0 \tag{4.3}$$

and

$$f\left(y^*, s^*\right) \geq 0. \tag{4.4}$$

By using the strong pseudomonotonicity of f and (4.4), one sees that

$$f\left(s^*, y^*\right) \leq -\beta d^2\left(s^*, y^*\right). \tag{4.5}$$

From (4.3) and (4.5), we have

$$\beta d^2\left(s^*, y^*\right) \leq -f\left(s^*, y^*\right) \leq 0,$$

which leads to $s^* = y^*$. That is, the equilibrium problem (1.1) has a unique solution. □

4.2. Global error bound for strongly pseudomonotone EPs

Theorem 4.2. *Assume that Conditions (A2)–(A4), (C1), and (C3) hold. Let $f : C \times C \to \mathbb{R}$ be a β-strongly pseudomonotone bifunction and s^* be the unique solution of EP (1.1). Let $\{s_n\}$ be generated by Algorithm 3.1. Then we have*

$$\frac{1-\frac{\delta\tau_n}{\tau_{n+1}}}{1+\frac{\delta\tau_n}{\tau_{n+1}}} d\left(t_n, s_n\right) \leq d\left(s_n, s^*\right) \leq \left(1+\frac{1+\frac{\delta\tau_n}{\tau_{n+1}}}{\tau_n\beta}\right) d\left(t_n, s_n\right).$$

Proof. From Lemma 2.4 and the definition of Algorithm 3.1, one obtains

$$\left\langle \exp_{t_n}^{-1} s_n, \exp_{t_n}^{-1} z \right\rangle \leq \tau_n \left(f(s_n, z) - f\left(s_n, t_n\right)\right) \quad (\forall z \in C). \tag{4.6}$$

From $s_{n+1} \to s^*$ and s_{n+1} satisfies the step size criterion (3.1), one obtains

$$f(s_n, s^*) - f(s_n, t_n) \leq f(t_n, s^*) + \frac{\delta}{\tau_{n+1}} d(s_n, t_n) d(t_n, s^*). \tag{4.7}$$

In view of $s^* \in \Omega$ and $t_n \in C$, one obtains $f\left(s^*, t_n\right) \geq 0$, which together with the strong pseudomonotonicity of f yields that

$$f\left(t_n, s^*\right) \leq -\beta d^2(t_n, s^*). \tag{4.8}$$

Combining (4.6), (4.7), and (4.8), we have

$$\begin{aligned} \left\langle \exp_{t_n}^{-1} s_n, \exp_{t_n}^{-1} s^* \right\rangle &\leq \tau_n \left(f\left(t_n, s^*\right) + \frac{\delta}{\tau_{n+1}} d(s_n, t_n) d(t_n, s^*) \right) \\ &\leq \tau_n \left(-\beta d^2(t_n, s^*) + \frac{\delta}{\tau_{n+1}} d(s_n, t_n) d(t_n, s^*) \right). \end{aligned} \tag{4.9}$$

Thus

$$\begin{aligned} \tau_n \beta d^2(t_n, s^*) &\leq \frac{\delta\tau_n}{\tau_{n+1}} d(s_n, t_n) d(t_n, s^*) + \left\langle -\exp_{t_n}^{-1} s_n, \exp_{t_n}^{-1} s^* \right\rangle \\ &\leq \frac{\delta\tau_n}{\tau_{n+1}} d(s_n, t_n) d(t_n, s^*) + d\left(s_n, t_n\right) d\left(t_n, s^*\right). \end{aligned}$$

That is

$$\tau_n \beta d(t_n, s^*) \leq \left(1 + \frac{\delta\tau_n}{\tau_{n+1}}\right) d(s_n, t_n).$$

Then we have

$$d(s_n, s^*) \leq d(s_n, t_n) + d(t_n, s^*) \leq \left(1+\frac{1+\frac{\delta\tau_n}{\tau_{n+1}}}{\tau_n\beta}\right) d(s_n, t_n).$$

The upper error bound is established.

Let $\Delta\left(s_n, t_n, s^*\right)$ be a geodesic triangle. Using (2.3), we have

$$\begin{aligned} \left\langle \exp_{t_n}^{-1} s_n, \exp_{t_n}^{-1} s^* \right\rangle &\geq d^2\left(t_n, s_n\right) - \left\langle \exp_{s_n}^{-1} t_n, \exp_{s_n}^{-1} s^* \right\rangle \\ &\geq d^2\left(t_n, s_n\right) - d\left(t_n, s_n\right) d\left(s_n, s^*\right). \end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10), we have

$$\begin{aligned} d^2\left(t_n, s_n\right) - d\left(t_n, s_n\right) d\left(s_n, s^*\right) &\leq \frac{\delta\tau_n}{\tau_{n+1}} d(s_n, t_n) d(t_n, s^*) \\ &\leq \frac{\delta\tau_n}{\tau_{n+1}} d(s_n, t_n) \left(d(t_n, s_n) + d\left(s_n, s^*\right)\right). \end{aligned}$$

That is

$$d\left(t_n, s_n\right) - d\left(s_n, s^*\right) \leq \frac{\delta\tau_n}{\tau_{n+1}} \left(d(t_n, s_n) + d\left(s_n, s^*\right)\right),$$

which implies that

$$d\left(s_n, s^*\right) \geq \frac{1-\frac{\delta \tau_n}{\tau_{n+1}}}{1+\frac{\delta \tau_n}{\tau_{n+1}}} d\left(t_n, s_n\right).$$

This completes the proof. □

4.3. Linear convergence for the proposed algorithms

In this subsection we establish the R-linear convergence rates of the proposed Algorithms 3.1–3.3 when the involved bifunction f is strongly pseudomonotone. We start by reviewing the definition of R-linear convergence.

Definition 4.1. A sequence $\{s_n\}$ in Hadamard manifold $\mathcal{M}$ is said to be converge R-linearly to s^* with rate $\eta \in [0,1)$ if there exists a constant $c > 0$ such that $d(s_n, s^*) \leq c\eta^n, \forall n \in \mathbb{N}$.

Now we prove the R-linear convergence of the proposed Algorithm 3.1.

Theorem 4.3. *Suppose that Conditions (A1)′, (A2)–(A4), (C1), and (C3) hold. Then the sequence $\{s_n\}$ generated by Algorithm 3.1 converges R-linearly to the unique solution s^* of EP (1.1).*

(A1)′ $f : C \times C \to \mathbb{R}$ is a β-strongly pseudomonotone bifunction and $f(x,x) = 0$ for all $x \in C$.

Proof. Under the above assumptions, the EP (1.1) has a unique solution (see Theorem 4.1), denoted by s^*. Since $s^* \in \Omega$ and $t_n \in C$, one obtains $f\left(s^*, t_n\right) \geq 0$, which together with the β-strong pseudomonotonicity of f implies that $f\left(t_n, s^*\right) \leq -\beta d^2(t_n, s^*)$. Letting $y = s^*$ in (3.10), one has

$$\begin{aligned} d^2\left(s_{n+1}, s^*\right) &\leq d^2\left(s_n, s^*\right) - \chi\left(1-\frac{\delta \tau_n}{\tau_{n+1}}\right)\left(d^2\left(s_n, t_n\right) + d^2\left(s_{n+1}, t_n\right)\right) \\ &\quad - (1-\chi) d^2\left(s_n, s_{n+1}\right) + 2\chi\tau_n f\left(t_n, s^*\right) \\ &\leq d^2\left(s_n, s^*\right) - \chi\left(1-\frac{\delta \tau_n}{\tau_{n+1}}\right)\left(d^2\left(s_n, t_n\right) + d^2\left(s_{n+1}, t_n\right)\right) \\ &\quad - (1-\chi) d^2\left(s_n, s_{n+1}\right) - 2\beta\chi\tau_n d^2(t_n, s^*). \end{aligned} \tag{4.11}$$

We complete the proof by considering two cases of χ.

Case 1. When $\chi \in (0,1]$. Let $\alpha \in \left(0, \frac{1-\delta}{2}\right)$ be a fixed number. Since $\delta \in (0,1)$, one obtains

$$\lim_{n\to\infty}\left(1-\frac{\delta \tau_n}{\tau_{n+1}}\right) = 1-\delta > 2\alpha > 0.$$

Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$1-\frac{\delta \tau_n}{\tau_{n+1}} > 2\alpha > 0 \ (\forall n \geq n_0). \tag{4.12}$$

By (4.11) and (4.12), we have for all $n \geq n_0$,

$$\begin{aligned} d^2\left(s_{n+1}, s^*\right) &\leq d^2\left(s_n, s^*\right) - 2\chi\alpha d^2\left(s_n, t_n\right) - 2\beta\chi\tau_n d^2(t_n, s^*) \\ &\leq d^2\left(s_n, s^*\right) - \min\{\chi\alpha, \beta\chi\tau_n\}\left(2d^2\left(s_n, t_n\right) + 2d^2\left(t_n, s^*\right)\right) \\ &\leq d^2\left(s_n, s^*\right) - \min\{\chi\alpha, \beta\chi\tau_n\} d^2\left(s_n, s^*\right) \\ &= r^2 d^2\left(s_n, s^*\right), \end{aligned}$$

where $r = \sqrt{1-\min\{\chi\alpha, \beta\chi\tau_n\}} \in (0,1)$. Thus we have

$$d\left(s_{n+1}, s^*\right) \leq r d\left(s_n, s^*\right) \ (\forall n \geq n_0). \tag{4.13}$$

By induction of (4.13) we obtain

$$d\left(s_{n+1}, s^*\right) \leq r^{n-n_0+1} d\left(s_{n_0}, s^*\right) \ (\forall n \geq n_0).$$

That is

$$d\left(s_{n+1}, s^*\right) \leq N r^n \ (\forall n \geq n_0),$$

where $N = r^{1-n_0} d\left(s_{n_0}, s^*\right)$. This implies that $\{s_n\}$ is R-linearly convergent when $\chi \in (0,1)$.

Case 2. When $\chi \in (1, 2/(1+\delta))$. From (3.12) and (4.11), we have

$$d^2\left(s_{n+1}, s^*\right) \le d^2\left(s_n, s^*\right) - \left(2 - \chi - \frac{\chi\delta\tau_n}{\tau_{n+1}}\right)\left(d^2\left(s_n, t_n\right) + d^2\left(s_{n+1}, t_n\right)\right)$$
$$- 2\beta\chi\tau_n d^2(t_n, s^*).$$

Let $\alpha \in \left(0, \frac{2-\chi-\chi\delta}{2}\right)$ be a fixed number. Then

$$\lim_{n\to\infty}\left(2 - \chi - \frac{\chi\delta\tau_n}{\tau_{n+1}}\right) = 2 - \chi - \chi\delta > 2\alpha > 0.$$

Therefore, there exists $n_1 \in \mathbb{N}$ such that

$$2 - \chi - \frac{\chi\delta\tau_n}{\tau_{n+1}} > 2\alpha > 0 \ (\forall n \ge n_1).$$

Using a proof similar to that of *Case 1*, we can obtain that $\{s_n\}$ is R-linearly convergent when $\chi \in [1, 2/(1+\delta))$. Thus we prove that the sequence generated $\{s_n\}$ by Algorithm 3.1 is R-linearly convergent when $\chi \in (0, 2/(1+\delta))$. □

Theorem 4.4. *Let Conditions (A1)′, (A2)–(A4), (C1), and (C3) hold. Then the sequence $\{s_n\}$ created by Algorithm 3.2 converges R-linearly to the unique solution s^* of EP (1.1).*

Proof. Using (3.29) with $y = s^*$ and noting that $f\left(t_n, s^*\right) \le -\beta d^2\left(t_n, s^*\right)$, one has

$$\begin{aligned}
&d^2\left(s_{n+1}, s^*\right) \\
&\le d^2\left(s_n, s^*\right) - \chi\left(1 - \frac{2\delta\tau_n}{\tau_{n+1}}\right) d^2\left(s_n, t_n\right) - \chi\left(1 - \frac{\delta\tau_n}{\tau_{n+1}}\right) d^2\left(s_{n+1}, t_n\right) \\
&+ \frac{2\chi\delta\tau_n}{\tau_{n+1}} d^2\left(t_{n-1}, s_n\right) - (1-\chi) d^2\left(s_n, s_{n+1}\right) - 2\beta\chi\tau_n d^2\left(t_n, s^*\right).
\end{aligned} \tag{4.14}$$

We also consider two possible cases of χ.

Case 1. When $\chi \in (0, 1]$. Note that $(1-\chi) d^2\left(s_n, s_{n+1}\right) \ge 0$ for all $n \ge 0$. Let μ and ρ be two real numbers such that

$$\mu \in \left(0, \frac{1-2\delta}{2}\right) \quad \text{and} \quad 1 < \rho < \frac{1}{2}\left(\frac{1}{\delta} - 1\right).$$

Adding the term $\frac{2\chi\delta\rho\tau_{n+1}}{\tau_{n+2}} d^2\left(s_{n+1}, t_n\right)$ to both sides of (4.14), we obtain

$$\begin{aligned}
&d^2\left(s_{n+1}, s^*\right) + \frac{2\chi\delta\rho\tau_{n+1}}{\tau_{n+2}} d^2\left(s_{n+1}, t_n\right) \\
&\le d^2\left(s_n, s^*\right) + \frac{2\chi\delta\tau_n}{\tau_{n+1}} d^2\left(t_{n-1}, s_n\right) - \chi\left(1 - \frac{2\delta\tau_n}{\tau_{n+1}}\right) d^2\left(s_n, t_n\right) \\
&- \chi\left(1 - \frac{\delta\tau_n}{\tau_{n+1}} - \frac{2\delta\rho\tau_{n+1}}{\tau_{n+2}}\right) d^2\left(s_{n+1}, t_n\right) - 2\beta\chi\tau_n d^2\left(t_n, s^*\right).
\end{aligned} \tag{4.15}$$

From $\delta \in (0, \frac{1}{3})$, we have

$$\lim_{n\to\infty}\left(1 - \frac{2\delta\tau_n}{\tau_{n+1}}\right) = 1 - 2\delta > 2\mu > 0$$

and

$$\lim_{n\to\infty}\left(1 - \frac{\delta\tau_n}{\tau_{n+1}} - \frac{2\delta\rho\tau_{n+1}}{\tau_{n+2}}\right) = 1 - (1+2\rho)\delta > 0.$$

It follows that there exists $n_1 \in \mathbb{N}$ satisfying

$$1 - \frac{2\delta\tau_n}{\tau_{n+1}} > 2\mu > 0, \quad (\forall n \ge n_1)$$

and

$$1 - \frac{\delta\tau_n}{\tau_{n+1}} - \frac{2\delta\rho\tau_{n+1}}{\tau_{n+2}} > 0, \quad (\forall n \ge n_1).$$

Thus we have from (4.15) that

$$\begin{aligned} & d^2\left(s_{n+1}, s^*\right)+\frac{2 \chi \delta \rho \tau_{n+1}}{\tau_{n+2}} d^2\left(s_{n+1}, t_n\right) \\ & \leq d^2\left(s_n, s^*\right)+\frac{2 \chi \delta \tau_n}{\tau_{n+1}} d^2\left(t_{n-1}, s_n\right)-2 \mu \chi d^2\left(s_n, t_n\right)-2 \beta \chi \tau_n d^2\left(t_n, s^*\right) . \end{aligned} \tag{4.16}$$

Note that

$$\begin{aligned} 2 \mu \chi d^2\left(s_n, t_n\right)+2 \beta \chi \tau_n d^2\left(t_n, s^*\right) & \geq \min \{\mu \chi, \beta \chi \tau_n\}\left(2 d^2\left(s_n, t_n\right)+2 d^2\left(t_n, s^*\right)\right) \\ & \geq \min \{\mu \chi, \beta \chi \tau_n\} d^2\left(s_n, s^*\right) . \end{aligned} \tag{4.17}$$

From (4.16) and (4.17), one obtains

$$d^2\left(s_{n+1}, s^*\right)+\frac{2 \chi \delta \rho \tau_{n+1}}{\tau_{n+2}} d^2\left(s_{n+1}, t_n\right) \leq \zeta d^2\left(s_n, s^*\right)+\frac{2 \chi \delta \tau_n}{\tau_{n+1}} d^2\left(t_{n-1}, s_n\right), \tag{4.18}$$

where $\zeta=1-\min \{\mu \chi, \beta \chi \tau_n\} \in(0,1)$. Set

$$a_n=d^2\left(s_n, s^*\right), \quad b_n=\frac{2 \chi \delta \rho \tau_n}{\tau_{n+1}} d^2\left(t_{n-1}, s_n\right) .$$

Thus the inequality (4.18) becomes

$$a_{n+1}+b_{n+1} \leq \zeta a_n+\frac{b_n}{\rho} \leq r\left(a_n+b_n\right) \quad (\forall n \geq n_1), \tag{4.19}$$

where $r=\max \left\{\zeta, \frac{1}{\rho}\right\} \in(0,1)$. By induction of (4.19), we have

$$a_{n+1}+b_{n+1} \leq r^{n-n_1+1}\left(a_{n_1}+b_{n_1}\right) \quad (\forall n \geq n_1),$$

which can be reduced to

$$d\left(s_{n+1}, s^*\right) \leq C \eta^n \quad (\forall n \geq n_1),$$

where $C=\sqrt{r^{1-n_1}\left(a_{n_1}+b_{n_1}\right)}$ and $\eta=\sqrt{r}$. This means that $\{s_n\}$ is R-linearly convergent when $\chi \in(0,1]$.

Case 2. When $\chi \in(1,2/(1+3 \delta))$. From (3.34) and (4.14), we have

$$\begin{aligned} & d^2\left(s_{n+1}, s^*\right)+\frac{2 \chi \delta \rho \tau_{n+1}}{\tau_{n+2}} d^2\left(s_{n+1}, t_n\right) \\ & \leq d^2\left(s_n, s^*\right)+\frac{2 \chi \delta \tau_n}{\tau_{n+1}} d^2\left(t_{n-1}, s_n\right)-\left(2-\chi-\frac{2 \chi \delta \tau_n}{\tau_{n+1}}\right) d^2\left(s_n, t_n\right) \\ & \quad -\left(2-\chi-\frac{\chi \delta \tau_n}{\tau_{n+1}}-\frac{2 \chi \delta \rho \tau_{n+1}}{\tau_{n+2}}\right) d^2\left(s_{n+1}, t_n\right)-2 \beta \chi \tau_n d^2\left(t_n, s^*\right) . \end{aligned}$$

Let μ and ρ be two real numbers such that

$$\mu \in\left(0, \frac{2-\chi-2 \chi \delta}{2}\right) \quad \text { and } \quad 1<\rho<\frac{2-\chi-\chi \delta}{2 \chi \delta} .$$

Note that

$$\lim _{n \rightarrow \infty}\left(2-\chi-\frac{2 \chi \delta \tau_n}{\tau_{n+1}}\right)=2-\chi-2 \chi \delta>2 \mu>0$$

and

$$\lim _{n \rightarrow \infty}\left(2-\chi-\frac{\chi \delta \tau_n}{\tau_{n+1}}-\frac{2 \chi \delta \rho \tau_{n+1}}{\tau_{n+2}}\right)=2-\chi-\chi \delta-2 \chi \delta \rho>0 .$$

Thus, there exists $n_2 \geq 1$ such that

$$2-\chi-\frac{2 \chi \delta \tau_n}{\tau_{n+1}}>2 \mu>0, \quad \forall n \geq n_2,$$

and

$$2-\chi-\frac{\chi \delta \tau_n}{\tau_{n+1}}-\frac{2 \chi \delta \rho \tau_{n+1}}{\tau_{n+2}}>0 \quad (\forall n \geq n_2) .$$

Therefore we have

$$\begin{aligned} & d^2\left(s_{n+1}, s^*\right)+\frac{2\chi\delta\rho\tau_{n+1}}{\tau_{n+2}} d^2\left(s_{n+1}, t_n\right) \\ \leq\, & d^2\left(s_n, s^*\right)+\frac{2\chi\delta\tau_n}{\tau_{n+1}} d^2\left(t_{n-1}, s_n\right)-2\mu d^2\left(s_n, t_n\right)-2\beta\chi\tau_n d^2\left(t_n, s^*\right) \\ \leq\, & \zeta d^2\left(s_n, s^*\right)+\frac{2\chi\delta\tau_n}{\tau_{n+1}} d^2\left(t_{n-1}, s_n\right) \quad (\forall n \geq n_2), \end{aligned}$$

where $\zeta=1-\min\{\mu, \beta\chi\tau_n\} \in (0,1)$. The rest of the proof is similar to *Case 1* and is therefore omitted. To this end, we proved that $\{s_n\}$ converges R-linearly to s^* when $\chi \in (0, 2/(1+3\delta))$. This completes the proof. □

Theorem 4.5. *Assume that Conditions (A1)′, (A2)–(A4), (C1), and (C3) hold. Then the sequence $\{s_n\}$ formed by Algorithm 3.3 converges R-linearly to the unique solution s^* of EP (1.1).*

Proof. Combining $s^* \in \Omega$, $t_n \in C$, and the strong pseudomonotonicity of f, we have $f\left(t_n, s^*\right) \leq -\beta d^2(t_n, s^*)$. Letting $y=s^* \in C$ in (3.57), one has

$$\begin{aligned} d^2\left(s_{n+1}, s^*\right)+\alpha_n d^2\left(t_{n+1}, t_n\right) \leq\, & d^2\left(s_n, s^*\right)+\beta_n d^2\left(t_{n-1}, t_n\right)-\gamma_n d^2\left(s_n, t_n\right) \\ & -2\left(1-\chi_{n+1}\right)\tau_n\beta d^2(t_n, s^*) \quad (\forall n \geq n_1). \end{aligned} \tag{4.20}$$

From $\delta \in (0,1)$, $\mu \in (1/(2-\delta), 1)$, $\lim_{n\to\infty} \tau_n = \tau > 0$, we have

$$\lim_{n\to\infty} \chi_n = \frac{1}{2}\sqrt{1+4\mu}-\frac{1}{2} \in \left(\frac{\sqrt{3}-1}{2}, \frac{\sqrt{5}-1}{2}\right).$$

Then

$$\gamma=\lim_{n\to\infty} \gamma_n=(1-\chi)\frac{1}{\chi}\left(\left(1+\frac{1}{\eta}\right)\mu-\frac{1}{\eta}\right) \in (0, \sqrt{3}).$$

Then there exists a constant $n_2 (\geq n_1)$ such that

$$\gamma_n \geq \gamma \quad (\forall n \geq n_3).$$

Note that

$$\begin{aligned} & \gamma d^2\left(s_n, t_n\right)+2\left(1-\chi_{n+1}\right)\tau_n\beta d^2\left(t_n, s^*\right) \\ \geq\, & \min\{\gamma/2, \left(1-\chi_{n+1}\right)\tau_n\beta\}\left(2d^2\left(s_n, t_n\right)+2d^2\left(t_n, s^*\right)\right) \\ \geq\, & \min\{\gamma/2, \left(1-\chi_{n+1}\right)\tau_n\beta\} d^2\left(s_n, s^*\right). \end{aligned} \tag{4.21}$$

From (4.20) and (4.21), one obtains

$$d^2\left(s_{n+1}, s^*\right)+\alpha_n d^2\left(t_{n+1}, t_n\right) \leq \zeta d^2\left(s_n, s^*\right)+\beta_n d^2\left(t_{n-1}, t_n\right), \tag{4.22}$$

where $\zeta=1-\min\{\gamma/2, \left(1-\chi_{n+1}\right)\tau_n\beta\} \in (0,1)$. From (3.58), there exists a constant $n_3 (\geq n_2)$ such that

$$\alpha_n > \rho_1 > \rho_2 > \beta_n > 0 \quad (\forall n \geq n_3). \tag{4.23}$$

Combining (4.22) and (4.23), we have

$$d^2\left(s_{n+1}, s^*\right)+\rho_1 d^2\left(t_{n+1}, t_n\right) \leq \zeta d^2\left(s_n, s^*\right)+\rho_2 d^2\left(t_{n-1}, t_n\right). \tag{4.24}$$

Set

$$a_n=d^2\left(s_n, s^*\right), \quad b_n=\rho_1 d^2\left(t_{n-1}, t_n\right).$$

It follows from (4.24) that

$$a_{n+1}+b_{n+1} \leq \zeta a_n+\frac{\rho_2}{\rho_1} b_n \leq r\left(a_n+b_n\right) \quad (\forall n \geq n_3),$$

where $r=\max\left\{\zeta, \frac{\rho_2}{\rho_1}\right\} \in (0,1)$. Therefore by induction, we have

$$a_{n+1}+b_{n+1} \leq r^{n-n_3+1}\left(a_{n_3}+b_{n_3}\right) \quad (\forall n \geq n_3),$$

which can be reduced to

$$d\left(s_{n+1}, s^*\right) \leq C\eta^n \ (\forall n \geq n_3),$$

where $C = \sqrt{r^{1-n_3}\left(a_{n_3} + b_{n_3}\right)}$ and $\eta = \sqrt{r}$. We get the required result. □

5. Numerical experiments

In this section, we provide a fundamental numerical example to illustrate the computational performance of the algorithms proposed in this paper. All codes were written in MATLAB 2018a and run on a PC with an Intel(R) Core(TM) i5-8250U CPU @ 1.60 GHz 1.80 GHz and 8.00 GB of running memory.

The following example is regarded by many literature on Hadamard manifolds; see, e.g., [23,30,53].

Example 5.1. Let $\mathbb{R}_{++} = \{x \in \mathbb{R} \mid x > 0\}$ and the Riemannian metric $\langle \cdot, \cdot \rangle$ be given by

$$\langle u, v\rangle := \frac{1}{x^2} uv \ (\forall u, v \in T_x\mathcal{M})(\forall x \in \mathcal{M}).$$

Then $\mathcal{M} = \left(\mathbb{R}_{++}, \langle \cdot, \cdot \rangle\right)$ becomes a Riemannian manifold. The Riemannian distance $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$ with respect to $x \in \mathcal{M}$ and $y \in \mathcal{M}$ is defined as

$$d(x, y) = |\ln(x/y)| \ \ (\forall x, y \in \mathcal{M}),$$

see [53, Example 1] for more details. Thus $\mathcal{M}$ is a Hadamard manifold. For each $x \in \mathcal{M}$, the tangent space $T_x\mathcal{M}$ at x equals to $\mathbb{R}$. The unique geodesic $\gamma : \mathbb{R} \to \mathbb{R}_{++}$ with initial conditions $\gamma(0) = x$ and $\gamma'(0) = v$ is given by

$$\gamma(t) = xe^{(v/x)t} \ \ (\forall t \in \mathbb{R}).$$

Thus $\exp_x tv = xe^{(v/x)t}$ for all $x \in \mathcal{M}$, $t \in \mathbb{R}$, and $v \in T_x\mathcal{M}$. In addition, for any $x \in \mathcal{M}$ and $y \in \mathcal{M}$, one can show that

$$y = \exp_x\left(d(x, y)\frac{\exp_x^{-1} y}{d(x, y)}\right) = xe^{\frac{\exp_x^{-1} y}{xd(x,y)} d(x,y)} = xe^{\frac{\exp_x^{-1} y}{x}}.$$

Then the inverse of exponential map is denoted by

$$\exp_x^{-1} y = x \ln\frac{y}{x} \ \ (\forall x, y \in \mathcal{M}).$$

The Nash-Cournot oligopolistic equilibrium model (see [1, Sect. 1.4.3, p. 26]) which assumes that the price and tax-fee functions are affine, and provides the basis for the bifunction f of the equilibrium problem. The following is a description of the test problem: consider that a commodity is produced by m different businesses. Let x represent the vector, in which the value of the element x_j describes the quantity of the commodity created by business j. We use the following notations for the price, profit, and strategy involved in the problem.

- The price $p_j(s)$ is a decreasing affine function of s with $s = \sum_{j=1}^m x_j$, i.e., $p_j(s) = \alpha_j - \beta_j s$, where $\alpha_j > 0, \beta_j > 0$.
- Use $f_j(x) = p_j(s)x_j - c_j\left(x_j\right)$ to calculate the profit produced by business j, where $c_j\left(x_j\right)$ is the tax and fee for producing x_j.
- Let $C_j = \left[x_j^{\min}, x_j^{\max}\right]$ represent the strategy set of business j, where $x_j^{\min}$ and $x_j^{\max}$ denote the lower and upper bounds on the commodity that can be produced by business j, respectively. Then the strategy set of the equilibrium model is $C := C_1 \times C_2 \ldots \times C_m$.

Under the assumption that the production of the other businesses is a parametric input, each company aims to maximize its profit by selecting the appropriate production level. A typical method for solving this problem is based on the well-known Nash equilibrium idea. Recall that if x^* satisfies the model

$$f_j\left(x^*\right) \geq f_j\left(x^*\left[x_j\right]\right) \quad (\forall x_j \in C_j)(\forall j = 1, 2, \ldots, m),$$

where $x^*\left[x_j\right]$ denotes the vector obtained from x^* by replacing x_j^* with x_j, then x^* is said to be an equilibrium point of the model. Choose $f(x, y) := \phi(x, y) - \phi(x, x)$ with $\phi(x, y) := -\sum_{j=1}^m f_j\left(x\left[y_j\right]\right)$. The problem of locating the Nash equilibrium point of the model can be stated as follows:

$$\text{find } x^* \in C, \quad \text{such that } f\left(x^*, x\right) \geq 0 \quad (\forall x \in C).$$

Now assume that for each business j, the tax-fee function $c_j\left(x_j\right)$ is increasing and affine. According to this assumption, the tax and fee for generating a unit both increase as the number of production increases. The bifunction f in such case can be written as follows:

$$f(x, y) = \langle Px + Qy + r, y - x\rangle,$$

Table 1
Some parameter settings of the Nash-Cournot oligopolistic equilibrium model.

Company j	Price $p_j(s)$	Tax-fee $c_j(x_j)$	Strategy set C_j
1	$p_1(s) = 100 - 0.01s$	$c_1(x_1) = 20x_1$	$C_1 = [1000, 2000]$
2	$p_2(s) = 110 - 0.02s$	$c_2(x_2) = 15x_2 + 100$	$C_2 = [500, 2500]$
3	$p_3(s) = 100 - 0.015s$	$c_3(x_3) = 17x_3$	$C_3 = [800, 1500]$
4	$p_4(s) = 115 - 0.05s$	$c_4(x_4) = 20x_4 + 75$	$C_4 = [500, 3000]$

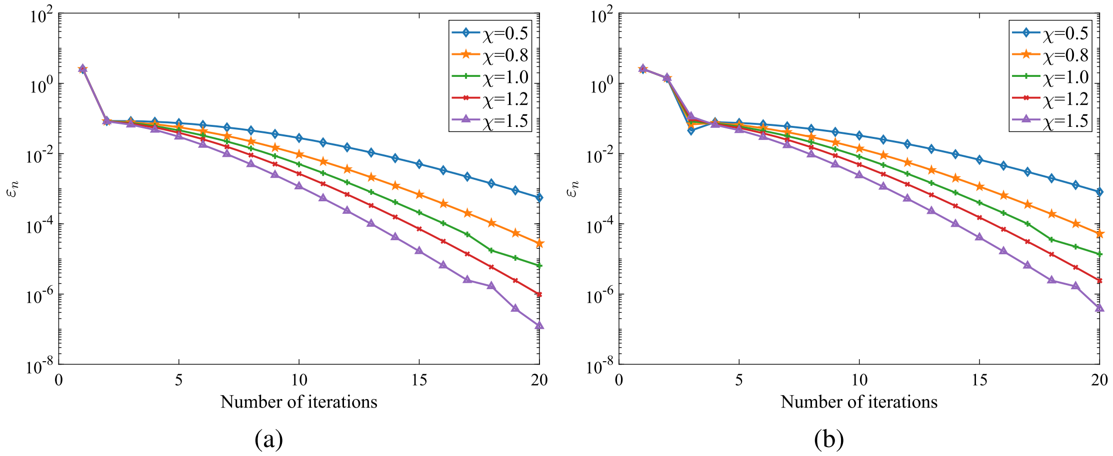

Fig. 1. Numberical behavior of our algorithms with different χ in Example 5.1. (a) Our Algorithm 3.1 and (b) Our Algorithm 3.2.

where $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ are two symmetric matrices, and $r \in \mathbb{R}^n$ is a vector. If Q is positive semidefinite and $Q - P$ is negative semidefinite, then f has the following characteristics (see [36, p. 769]):

- f is monotone (hence pseudomonotone), $f(x, \cdot)$ is differentiable and convex on C, and $f(\cdot, y)$ is continuous on C;
- f satisfies the Lipschitz-type condition (2.4) with constant $L = \|P - Q\|$.

Therefore, the Assumptions (A1)–(A4) all hold. That is, the proposed Algorithms 3.1–3.3 can be used to solve the mentioned problem. Next, we set the following parameters for the price, tax-fee, and strategy involved in the problem (see [23, Sect. 4]) (Table 1).

Now we use the proposed Algorithms 3.1–3.3 to solve Example 5.1 and compare them with the methods presented in [23,24]. Choose the following parameters for these algorithms.

(1) Select $\delta = 0.1$, $\tau_0 = 0.01$, $\xi_n = 0$, and $\sigma_n = 1/(n + 1000)^2$ for the proposed Algorithms 3.1 and 3.2. Take $\tau_{-1} = \tau_0 = 0.01$, $\delta = 0.1$, $\mu = 0.6$, $\xi_n = 0$, and $\sigma_n = 1/(n + 1000)^2$ for the suggested Algorithm 3.3.
(2) For the Algorithm 1 and Algorithm 2 suggested by Khammahawong et al. [23] (shortly KKCYWJ Alg. 1 and KKCYWJ Alg. 2), we set $\tau_n = \frac{1}{n+1}$.
(3) For the Algorithm 4.1 and Algorithm 4.3 proposed by Ansari and Islam [24] (shortly AI Alg. 4.1 and AI Alg. 4.3), we choose $\tau_0 = 0.01$ and $\delta = 0.1$.

Since we do not know the exact solution of Example 5.1, we use $\varepsilon_n = d(s_n, t_n)$, $\varepsilon_n = \max\{d(s_n, t_n), d(s_{n+1}, t_n)\}$, and $\varepsilon_n = \max\{d(s_n, t_n), d(t_{n+1}, t_n)\}$ to measure the iteration error of the proposed Algorithm 3.1, Algorithm 3.2, and Algorithm 3.3 at the n-th step, respectively. For KKCYWJ Alg. 2 and AI Alg. 4.1, we use $\varepsilon_n = d(s_n, t_n)$. For KKCYWJ Alg. 1 and AI Alg. 4.3, we select $\varepsilon_n = \max\{d(s_n, t_n), d(s_{n+1}, t_n)\}$. According to Remarks 3.1, 3.3, and 3.5, s_n can be seen as an approximate solution of the problem when $\varepsilon_n \to 0$. For convenience, we adopt the maximum number of iterations 50 as the common stopping condition of all algorithms. The initial values of all algorithms are randomly generated by MATLAB function `round(500*rand(1,4))+500`, and the optimization problems of all algorithms are solved by the function `fmincon` in the MATLAB optimization toolbox. First, we demonstrate in Fig. 1 the numerical results of the proposed Algorithms 3.1 and 3.2 with different parameters χ for a maximum number of iterations of 20. Next we set $\chi = 1.2$ for the proposed Algorithms 3.1 and 3.2. Table 2 presents the termination iteration errors and execution times in seconds for all algorithms under four different initial values (Case I: $s_0 = [570, 948, 503, 812]$; Case II: $s_0 = [620, 932, 511, 808]$; Case III: $s_0 = [558, 786, 641, 956]$; Case IV: $s_0 = [875, 859, 959, 816]$). Figure 2 shows the convergence behavior of the iteration error ε_n of the proposed Algorithms 3.1–3.3 and the compared methods with respect to the number of iterations. As an example, the convergence behavior of each component of our algorithms under Case IV is depicted in Fig. 3. Finally, we display in Fig. 4 the trends of step size changes for all algorithms under Case I and Case IV.

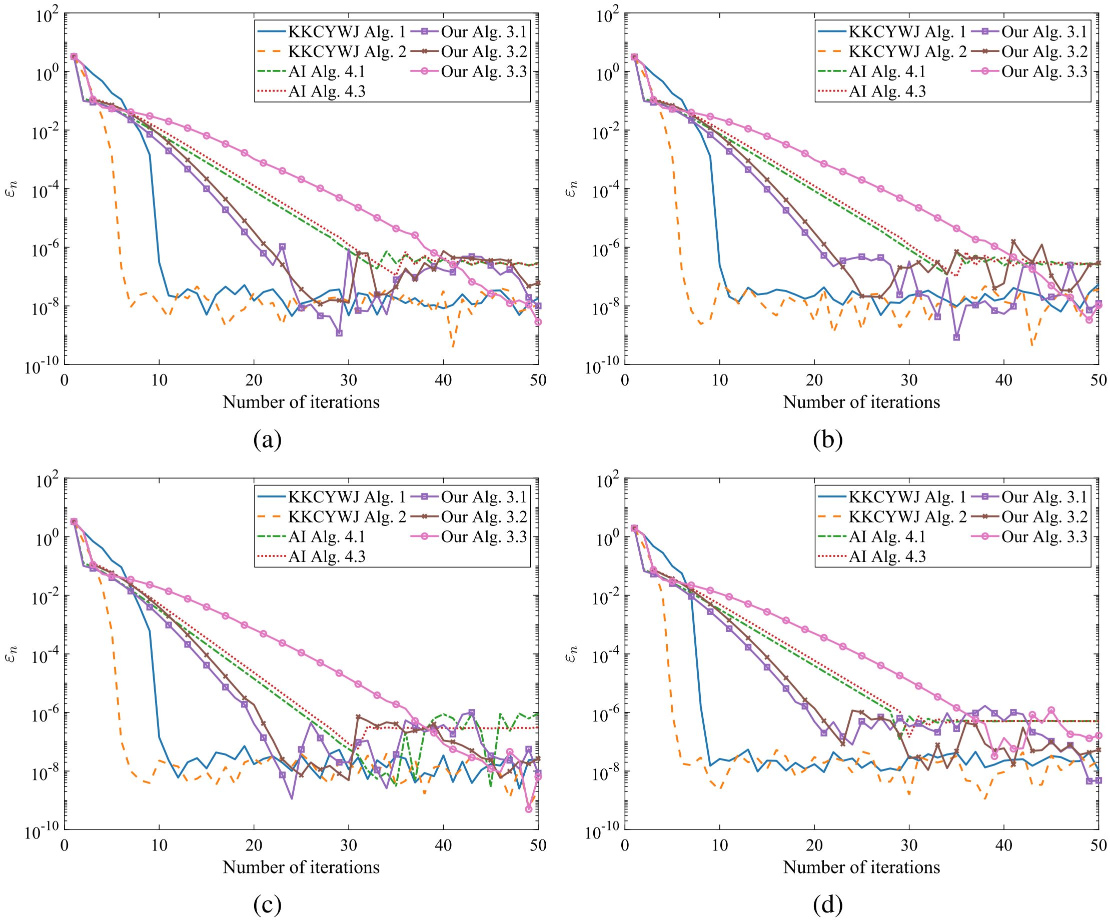

Fig. 2. Convergence behavior of our algorithms with the number of iterations in Example 5.1. (a) Case I, (b) Case II, (c) Case III and (d) Case IV.

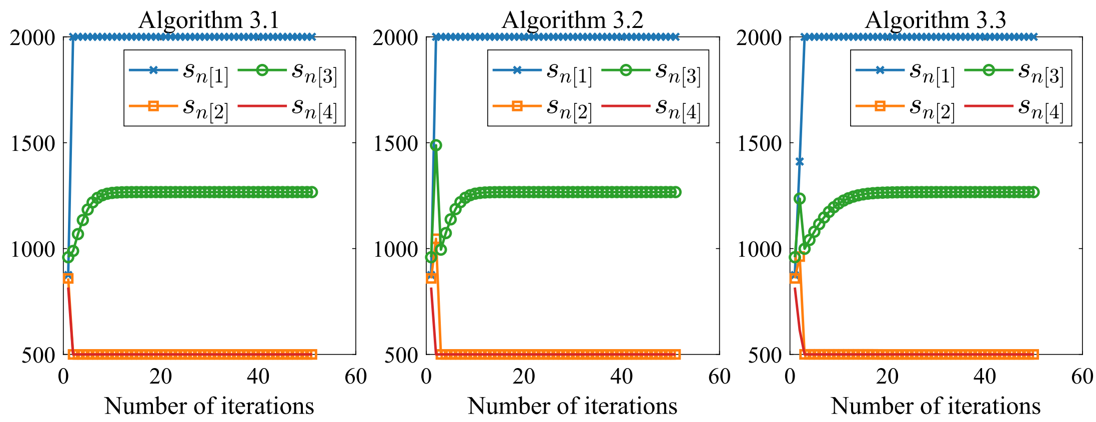

Fig. 3. Convergence behavior of each component of our algorithms in Case IV.

The convergence behavior of Figs. 1 and 2 show that the algorithms suggested in this paper can be used to solve equilibrium problems on Hadamard manifolds since $\varepsilon_n \to 0$. Furthermore, from Fig. 3, it can be observed that each component of the iterative values of our algorithms converges, indicating that the equilibrium solution for this problem is $s^* = [2000, 500, 1267, 500]$. On the other hand, it can be seen from Fig. 1 that our Algorithms 3.1 and 3.2 have a better performance when the parameter χ is properly chosen. The results from Table 2 and Fig. 2 indicate that our algorithm converges faster than some known algorithms in the literature [23,24]. The advantage of our algorithms is that they use different adaptive non-monotonic step sizes in each iteration. Table 2 shows that our Algorithm 3.3 takes the least time, while our Algorithms 3.1 and 3.2 and the compared methods speed more time. This is

Table 2
Numerical results of all algorithms under four different initial values.

Algorithms	Case I		Case II		Case III		Case IV	
	ε_n	Time (s)	ε_n	Time (s)	ε_n	Time (s)	ε_n	Time (s)
KKCYWJ Alg. 1	1.80E-08	3.32	5.41E-08	2.48	2.53E-08	3.41	1.06E-08	3.12
KKCYWJ Alg. 2	1.12E-08	2.43	3.59E-08	1.85	2.22E-09	2.30	2.28E-08	2.51
AI Alg. 4.1	2.88E-07	2.67	2.49E-07	2.41	9.25E-07	3.51	5.05E-07	2.71
AI Alg. 4.2	2.72E-07	2.36	2.85E-07	2.18	3.11E-07	2.26	5.06E-07	2.20
Our Algorithm 3.1	9.97E-09	2.36	1.22E-08	1.90	8.50E-09	2.33	4.79E-09	2.14
Our Algorithm 3.2	6.11E-08	2.43	2.93E-07	1.83	2.68E-08	2.04	5.32E-08	2.33
Our Algorithm 3.3	2.82E-09	1.65	1.05E-08	1.52	6.36E-09	1.67	1.63E-07	1.44

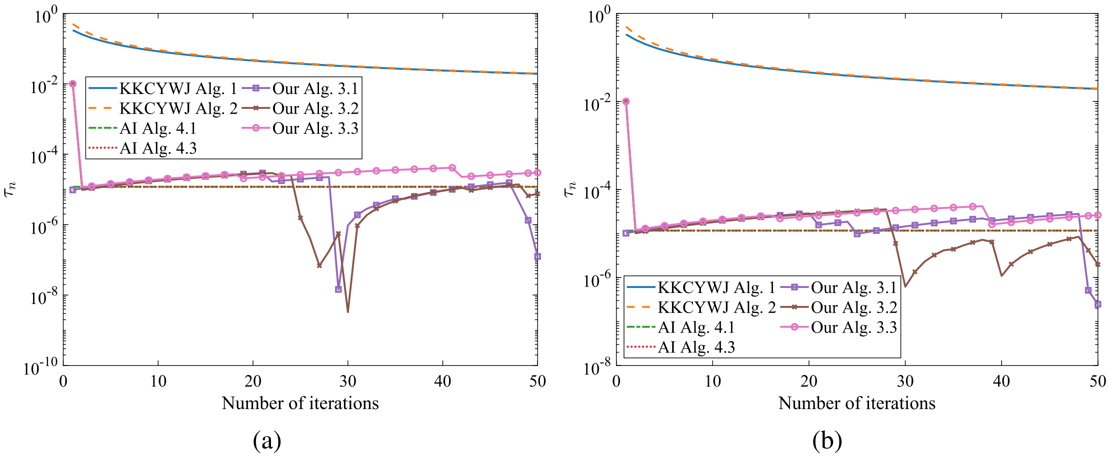

Fig. 4. The trend of step size changes for all algorithms under two cases. (a) Case I and (b) Case IV.

related to the number of optimization problems that need to be computed in the algorithms and also to the number of values of the bifunction f that need to be evaluated. As the theoretical analysis in Section 3 demonstrates, the presented Algorithm 3.3 takes the shortest time as it needs to compute the optimization problem and the value of the bifunction f only once in each iteration. In addition, it is evident from Fig. 2 that the proposed Algorithms 3.1–3.3 converge quickly in the first 20 iterations, and there is some oscillatory characteristic in the subsequent iterations. The primary cause of oscillation is attributed to changes in the step size, as the step size is very small in this example (cf. Fig. 4), and even minor variations can induce oscillations in the iteration process. Reducing this type of oscillation is a potential research direction for future consideration. Figure 4 also demonstrates the fact that the step size sequences generated by our algorithms are non-monotonic. Notice that the step size sequences of the algorithms suggested by Khammahawong et al. [23] are non-summable and the step size sequences of the methods proposed by Ansari and Islam [24] are non-increasing (see Fig. 4).

6. Conclusions

In this paper, we propose three adaptive numerical algorithms to discover solutions of equilibrium problems in Hadamard manifolds. The presented algorithms are inspired by the extragradient method and the golden ratio algorithm. Our approaches employ a step size criterion that can be dynamically adjusted, enabling them to work adaptively. In the case where the bifunction is pseudomonotone and Lipschitz continuous, we proved that the sequences generated by the presented algorithms converge to the solution of the equilibrium problem when the solution exists. Furthermore, we established the global error bounds for our first algorithm and R-linear convergence of the suggested algorithms in the case of the bifunctions governed by strongly pseudomonotone. Finally, a basic computational test demonstrates the efficiency of our algorithms. The results obtained in this paper extended and improved some existing algorithms in the literature for solving equilibrium problems in Hadamard manifolds. It is also interesting to explore the practical applications of the algorithms offered in this paper on Hadamard manifolds.

Declaration of competing interest

The authors declare that they have no conflict of interest.

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