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Strong Convergence of Self-adaptive Inertial Algorithms for Solving Split Variational Inclusion Problems with Applications

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Abstract

In this paper, four self-adaptive iterative algorithms with inertial effects are introduced to solve a split variational inclusion problem in real Hilbert spaces. One of the advantages of the suggested algorithms is that they can work without knowing the prior information of the operator norm. Strong convergence theorems of these algorithms are established under mild and standard assumptions. As applications, the split feasibility problem and the split minimization problem in real Hilbert spaces are studied. Finally, several preliminary numerical experiments as well as an example in the field of compressed sensing are proposed to support the advantages and efficiency of the suggested methods over some existing ones.

Keywords Split variational inclusion problem · Signal processing problem · Strong convergence · Inertial method · Mann method · Viscosity method

Mathematics Subject Classification 65J15 · 68W10 · 65K15 · 47J20 · 90C25

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1 Introduction

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. In this paper, we focus on the following split variational inclusion problem (in short, SVIP):

$$\text{find } x^* \in \mathscr{H}_1 \quad \text{such that} \quad 0 \in F_1(x^*) \text{ and } 0 \in F_2(Ax^*)\,, \tag{SVIP}$$

where operators $F_1 : \mathscr{H}_1 \to 2^{\mathscr{H}_1}$ and $F_2 : \mathscr{H}_2 \to 2^{\mathscr{H}_2}$ are multi-valued maximal monotone, and $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator with its adjoint A^*. The solution set of (SVIP) is denoted by $\Omega := \{x^* \in \mathscr{H}_1 : 0 \in F_1(x^*) \text{ and } 0 \in F_2(Ax^*)\}$. The (SVIP) was first introduced by Moudafi [1]. It is worth noting that (SVIP) is a unified framework for many mathematical problems, including split minimization problem, split feasibility problems, fixed-point problems, linear inverse problems and variational inclusion problems; see, e.g., [2–4]. This formalism is also at the core of the modeling of many inverse problems arising for phase retrieval and other real-world problems; e.g., in sensor networks in computerized tomography and data compression; see [5,6] and the references therein. Thus, implementable and efficient solutions to this problem are of practical significance in many cases.

The goal of this paper is to build some fast and efficient iterative algorithms to solve the (SVIP). In recent years, there has been tremendous interest in solving (SVIP) and many researchers have constructed a large number of methods to solve the problem; see, e.g., [7–13] and the references therein. Next, we first recall some known algorithms for solving (SVIP) in the literature and then propose our methods. Byrne et al. [7, Algorithm 3.1] introduced the following algorithm to find the solution for (SVIP): $x_{n+1} = J_{\lambda F_1}(x_n - \gamma A^*(I - J_{\lambda F_2})Ax_n)$, where $J_{\lambda F_1}$ and $J_{\lambda F_2}$ are the resolvent mappings of F_1 and F_2, respectively (see the definition in Sect. 2) and I is the identity mapping. They proved that the iterative scheme converges weakly to a solution of (SVIP) provided that $\Omega \neq \emptyset$ and stepsize $\gamma \in (0, 2/\|A^*A\|)$. An important method to solve the variational inclusion problem (i.e., find $x^* \in \mathscr{H}_1$ such that $0 \in F_1(x^*)$) is the proximal point method (in short, PPM): $x_{n+1} = J_{\lambda F_1}(x_n)$. In order to speed up the convergence speed of PPM, Alvarez and Attouch [14] considered the following iterative scheme: $x_{n+1} = J_{\lambda F_1}(x_n + \vartheta_n(x_n - x_{n-1}))$, where F_1 is a maximal monotone operator, $\lambda > 0$ and $\vartheta_n \in [0, 1)$. This iterative scheme is now called the inertial proximal point method (in short, IPPM). They proved that the iterative sequence generated by IPPM converges weakly to a zero of F_1 under the condition that $\sum_{n=1}^{\infty} \vartheta_n \|x_n - x_{n-1}\|^2 < \infty$. It should be noted that the inertial is induced by the term $\vartheta_n(x_n - x_{n-1})$ and it can be regarded as a procedure of speeding up the convergence properties; see [15,16]. Recently, the idea of the inertial has been widely studied by many scholars in the optimization community as a technology to build fast iterative algorithms; see, e.g., [17–22] and the references therein. Inspired by the method proposed by Byrne et al. [7], the inertial method [14] and the projection and contraction method [23], Chuang [17] introduced the following hybrid inertial proximal algorithm to solve the (SVIP):

$$\begin{cases} u_n = x_n + \vartheta_n(x_n - x_{n-1})\,, \\ q_n = J_{\lambda_n F_1}[u_n - \gamma_n A^*(I - J_{\lambda_n F_2})Au_n]\,, \\ x_{n+1} = J_{\lambda_n F_1}(u_n - \mu_n c_n)\,, \end{cases} \tag{1.1}$$

where $\lambda_n > 0$, $\{\vartheta_n\}$ is a sequence in $[0, \vartheta] \subset [0, 1)$ satisfying $\sum_{n=1}^{\infty} \vartheta_n \|x_n - x_{n-1}\|^2 < \infty$, sequences $\{\mu_n\}$ and $\{c_n\}$ are defined in (3.3), and $\{\gamma_n\}$ is a real sequence in $[\gamma, \delta/\|A\|^2] \subset (0, \infty)$ satisfying

$$\gamma_n\|A^*(I - J_{\lambda F_2})Au_n - A^*(I - J_{\lambda F_2})Aq_n\| \leq \delta\|u_n - q_n\|,\ 0 < \delta < 1\,.$$

Chuang proved that the iterative sequence generated by (1.1) converges weakly to a solution of (SVIP). Later, Majee and Nahak [18] revised the way of calculating x_{n+1} in the third step of Algorithm (1.1). More precisely, they used $x_{n+1} = u_n - \kappa \mu_n c_n$ to calculate the value of x_{n+1}, where $\kappa \in (0, 2)$. It is worth noting that this method does not need to evaluate the resolvent mapping of F_1 again. They also established the weak convergence of the suggested method under some suitable conditions.

Note that the methods suggested in [7,17,18] all achieve weak convergence in infinite-dimensional spaces. Examples in CT reconstruction and machine learning tell us that strong convergence is preferable to the weak convergence in an infinite-dimensional space. Therefore, a natural question is how to modify the method proposed by Byrne et al. [7, Algorithm 3.1] such that it can achieve the strong convergence in infinite-dimensional spaces. In fact, in the past few years, many scholars have presented various iterative schemes with strong convergence to solve the split variational inclusion problem in real Hilbert spaces; see, e.g., [7,8,10,24–26]. Byrne et al. [7, Algorithm 4.4] used the Halpern method to guarantee the strong convergence of the proposed algorithm. It is known that Halpern-type methods use the initial point x_0 in each iteration, which results in slow convergence. Kazmi and Rizvi [8] applied the viscosity-type method to accelerate the convergence speed of Byrne et al.'s algorithm [7, Algorithm 4.4] and obtained the strong convergence of the suggested method. Sitthithakerngkiet et al. [10] combined the viscosity method and the hybrid steepest descent method to assure the strong convergence of the offered algorithm. Recently, based on the inertial idea, Byrne et al.'s method [7], Majee and Nahak's method [18], Mann method and viscosity method, Thong et al. [24], Long et al. [25] and Anh et al. [26] presented several new inertial strongly convergent algorithms and their numerical experiments show that the new iterative schemes are efficient and easy to implement.

On the other hand, the strongly convergent algorithms mentioned above share a common feature, that is, their step size needs to know the prior information of the operator (matrix) norm $\|A\|$ in advance. It may be difficult to estimate $\|A\|$ in general and thus affecting the implementation of the fixed step size algorithms. To overcome this shortcoming, the construction of self-adaptive step size algorithms has aroused numerous interest among researchers. López et al. [27] introduced a new relaxation algorithm whose iterative process is as follows: $x_{n+1} = P_C(x_n - \gamma_n A^*(I - P_Q)Ax_n)$, where the step size γ_n is computed as

$$\gamma_n = \frac{\varphi_n \frac{1}{2}\|(I - P_Q)Ax\|^2}{\|A^*(I - P_Q)Ax\|^2}, \ 0 < \varphi_n < 4, \ \inf \varphi_n(4 - \varphi_n) > 0 .$$

P_C and P_Q stand for the orthogonal projections on the closed convex sets C and Q, respectively. Recently, there are many algorithms that do not require the prior information of the operator (matrix) norm to solve (SVIP) and other problems; see, e.g., [28–33].

Motivated and stimulated by the above work, in this paper, we propose four self-adaptive inertial algorithms with strong convergence to solve the split variational inclusion problem (SVIP) in real Hilbert spaces. The advantages of the suggested iterative algorithms are that (1) the prior information of the operator (matrix) norm is not required, (2) the strong convergence theorems of the suggested algorithms are established under some weaker conditions, and (3) the inertial term is embedded to accelerate the convergence speed of the algorithms. Furthermore, we also give several theoretical applications of the proposed algorithms. Finally, some numerical experiments are provided to show the advantages of the stated algorithms over the previously existing ones. Our approaches obtained in this paper improve and summarize some results in the literature [7,17,18,24–26,28,34].

The rest of the paper is organized as follows. Some essential definitions and technical lemmas that need to be used are given in the next section. In Sect. 3, we propose several algorithms and analyze their convergence. Section 4 introduces some theoretical applications of the proposed methods. Some numerical experiments to verify our theoretical results are presented in Sect. 5. Finally, the paper ends with a brief summary in Sect. 6, the last section.

2 Preliminaries

Let C be a closed and convex nonempty subset in a real Hilbert space $\mathscr{H}$. The weak convergence and strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x are represented by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. For each $x, y, z \in \mathscr{H}$, we have the following basic facts:

(1) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;
(2) $\|\alpha x+\lambda y+\kappa z\|^2 = \alpha\|x\|^2+\lambda\|y\|^2+\kappa\|z\|^2-\alpha\lambda\|x-y\|^2-\alpha\kappa\|x-z\|^2-\lambda\kappa\|y-z\|^2$, where $\alpha, \lambda, \kappa \in [0, 1]$ with $\alpha+\lambda+\kappa = 1$.

Let $T : \mathscr{H} \to \mathscr{H}$ be a mapping with its fixed point set $\operatorname{Fix}(T) = \{x : Tx = x\} \neq \emptyset$. Recall that a mapping T is said to be:

(1) *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty,\ x - y\rangle, \quad \forall x, y \in \mathscr{H},$$

or equivalently,

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 - \|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in \mathscr{H}.$$

(2) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in \mathscr{H}.$$

Recall that a multi-valued mapping $F : \mathscr{H} \to 2^{\mathscr{H}}$ with domain $\operatorname{Dom}(F) := \{x \in \mathscr{H} : Fx \neq \emptyset\}$ is said to be (i) monotone if, for all $x, y \in \mathscr{H}$, $u \in Fx$ and $v \in Fy$ indicates that $\langle u-v, x-y\rangle \geq 0$; (ii) maximal monotone if it is monotone and if, for any $(x, u) \in \mathscr{H} \times \mathscr{H}$, $\langle u-v, x-y\rangle \geq 0$ for every $(y, v) \in \operatorname{Graph}(F)$ (the graph of mapping F) indicates that $u \in Fx$. Let mapping $F : \mathscr{H} \to 2^{\mathscr{H}}$ be set-valued maximal monotone. Then, for $\forall x \in \mathscr{H}$ and $\gamma > 0$, the resolvent mapping $J_{\gamma F} : \mathscr{H} \to \mathscr{H}$ associated with F is represented as $J_{\gamma F}(x) = (I + \gamma F)^{-1}(x)$, where I stands for the identity operator on $\mathscr{H}$.

For every point $x \in \mathscr{H}$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $P_C(x) := \operatorname{argmin}\{\|x-y\|,\ y \in C\}$. P_C is called the metric or the nearest point projection of $\mathscr{H}$ onto C. The two projection formulas given next will be used in the sequel (see Sect. 5).

(1) The projection of x onto a half-space $H_{u,v} = \{x : \langle u, x\rangle \leq v\}$ is computed by

$$P_{H_{u,v}}(x) = x - \max\left\{[\langle u, x\rangle - v]/\|u\|^2,\ 0\right\}u.$$

(2) The projection of x onto a ball $B[p, q] = \{x : \|x-p\| \leq q\}$ is computed by

$$P_{B[p,q]}(x) = p + \frac{q}{\max\{\|x-p\|,\ q\}}(x-p).$$

The following lemmas are very helpful for the convergence analysis of our algorithms.

Lemma 2.1 ([35,36]) *Assume that mapping $F : \mathscr{H} \to 2^{\mathscr{H}}$ is set-valued maximal monotone and $\lambda > 0$. Then the following statements hold:*

(1) $J_{\lambda F}$ *is a single-valued and firmly nonexpansive mapping.*
(2) $\mathrm{Dom}(J_{\lambda F}) = \mathscr{H}$ *and* $\mathrm{Fix}(J_{\lambda F}) = F^{-1}(0) = \{x \in \mathrm{Dom}(F) : 0 \in Fx\}$.
(3) $(I - J_{\lambda F})$ *is a firmly nonexpansive mapping.*
(4) *Suppose that* $F^{-1}(0) \neq \emptyset$. *Then,* $\langle x - J_{\lambda F}x, J_{\lambda F}x - z\rangle \geq 0$ *for all* $x \in \mathscr{H}, z \in F^{-1}(0)$.

Lemma 2.2 *Assume that* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *are real Hilbert spaces, and* $A : \mathscr{H}_1 \to \mathscr{H}_2$ *is a linear operator with its adjoint* A^*. *Let mapping* $F : \mathscr{H}_2 \to 2^{\mathscr{H}_2}$ *be set-valued maximal monotone and let* $J_{\lambda F}$ *be a resolvent mapping of* F. *For all* $x \in \mathscr{H}_1$, *let* $T : \mathscr{H}_1 \to \mathscr{H}_1$ *and* $V : \mathscr{H}_1 \to \mathscr{H}_1$ *be defined as* $Tx := A^*(I - J_{\lambda F})Ax$ *and* $Vx := x - \gamma_n A^*(I - J_{\lambda F})Ax$, *respectively. Let* $\{\gamma_n\}$ *be a sequence of positive real numbers,* $L = \|A\|^2$ *and* $\lambda > 0$. *Then, for any* $x, y \in \mathscr{H}_1$ *and* $p \in A^{-1}(\mathrm{Fix}(J_{\lambda F}))$, *the following statements hold:*

(1) $\|(I - J_{\lambda F})Ax - (I - J_{\lambda F})Ay\|^2 \leq \langle Tx - Ty, x - y\rangle$.
(2) $\|A^*(I - J_{\lambda F})Ax - A^*(I - J_{\lambda F})Ay\| \leq \|A\|^2\|x - y\|$.
(3) $\|Vx - p\|^2 \leq \|x - p\|^2 - \gamma_n(2 - \gamma_n L)\|(I - J_{\lambda F})Ax\|^2$.

Proof From the fact that $(I - J_{\lambda F})$ is a firmly nonexpansive mapping (Lemma 2.1 (3)), one has

$$\begin{aligned}\langle Tx - Ty, x - y\rangle &= \left\langle A^*(I - J_{\lambda F})Ax - A^*(I - J_{\lambda F})Ay, x - y\right\rangle \\ &= \langle (I - J_{\lambda F})Ax - (I - J_{\lambda F})Ay, Ax - Ay\rangle \\ &\geq \|(I - J_{\lambda F})Ax - (I - J_{\lambda F})Ay\|^2, \quad \forall x, y \in \mathscr{H}_1 .\end{aligned}$$

Moreover, we have

$$\begin{aligned}\|A^*(I - J_{\lambda F})Ax - A^*(I - J_{\lambda F})Ay\|^2 &\leq \|A\|^2 \cdot \|(I - J_{\lambda F})Ax - (I - J_{\lambda F})Ay\|^2 \\ &\leq \|A\|^2 \cdot \|Ax - Ay\|^2 \\ &\leq \|A\|^4 \cdot \|x - y\|^2, \quad \forall x, y \in \mathscr{H}_1 .\end{aligned}$$

Thus, $\|A^*(I - J_{\lambda F})Ax - A^*(I - J_{\lambda F})Ay\| \leq \|A\|^2\|x - y\|$. Finally, we prove that statement (3) is valid. Indeed, since $p \in A^{-1}(\mathrm{Fix}(J_{\lambda F}))$, one has $Ap \in \mathrm{Fix}(J_{\lambda F})$ and thus $A^*(I - J_{\lambda F})Ap = 0$. From statement (1), we have

$$\left\langle A^*(I - J_{\lambda F})Ax - A^*(I - J_{\lambda F})Ap, x - p\right\rangle \geq \|(I - J_{\lambda F})Ax - (I - J_{\lambda F})Ap\|^2 ,$$

which yields $\langle A^*(I - J_{\lambda F})Ax, x - p\rangle \geq \|Ax - J_{\lambda F}Ax\|^2$. This together with Lemma 2.1 (1) obtains

$$\begin{aligned}\|Vx - p\|^2 &= \|x - \gamma_n A^*(I - J_{\lambda F})Ax - p\|^2 \\ &= \|x - p\|^2 + \|\gamma_n A^*(I - J_{\lambda F})Ax\|^2 - 2\gamma_n \left\langle x - p, A^*(I - J_{\lambda F})Ax\right\rangle \\ &\leq \|x - p\|^2 + \gamma_n^2\|A\|^2\|(I - J_{\lambda F})Ax\|^2 - 2\gamma_n\|(I - J_{\lambda F})Ax\|^2 \\ &= \|x - p\|^2 - \gamma_n(2 - \gamma_n L)\|(I - J_{\lambda F})Ax\|^2 .\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.3 ([37]) *Assume that* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *are real Hilbert spaces, and* $A : \mathscr{H}_1 \to \mathscr{H}_2$ *is a linear operator with its adjoint* A^*. *Let* $F_1 : \mathscr{H}_1 \to 2^{\mathscr{H}_1}$ *and* $F_2 : \mathscr{H}_2 \to 2^{\mathscr{H}_2}$ *be two set-valued maximal monotone mappings. Let* $J_{\lambda F_1}$ *and* $J_{\lambda F_2}$ *be the resolvent mapping of* F_1 *and* F_2, *respectively. Suppose that the solution set of the problem (SVIP) is non-empty and* $\lambda > 0, \gamma > 0$. *Then, for any* $z \in \mathscr{H}_1$, z *is a solution of (SVIP) if and only if* $J_{\lambda F_1}(z - \gamma A^*(I - J_{\lambda F_2})Az) = z$.

Lemma 2.4 ([38]) *Let $\{\Upsilon_n\}$ be a sequence of nonnegative real numbers, $\{\zeta_n\}$ be a sequence of real numbers in $(0, 1)$ with $\sum_{n=1}^{\infty} \zeta_n = \infty$, and $\{\Phi_n\}$ be a sequence of real numbers. Assume that*

$$\Upsilon_{n+1} \leq (1-\zeta_n)\Upsilon_n + \zeta_n \Phi_n, \quad \forall n \geq 1 .$$

If $\limsup_{k\to\infty} \Phi_{n_k} \leq 0$ *for every subsequence* $\{\Upsilon_{n_k}\}$ *of* $\{\Upsilon_n\}$ *satisfying* $\liminf_{k\to\infty} (\Upsilon_{n_k+1} - \Upsilon_{n_k}) \geq 0$, *then* $\lim_{n\to\infty} \Upsilon_n = 0$.

3 Main Results

In this section, we propose four self-adaptive inertial algorithms to solve the split variational inclusion problem (SVIP). The advantage of our algorithms is that they do not require the prior information of the operator norm. Before introducing our algorithms, we assume that the following conditions are satisfied.

(C1) The solution set of problem (SVIP) is nonempty, i.e., $\Omega \neq \emptyset$.
(C2) Assume that $\mathscr{H}_1$ and $\mathscr{H}_2$ are real Hilbert spaces, and $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator with its adjoint A^*. Let $F_1 : \mathscr{H}_1 \to \mathscr{H}_1$ and $F_2 : \mathscr{H}_2 \to \mathscr{H}_2$ be two set-valued maximal monotone mappings.
(C3) Let $\{\varpi_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\varpi_n}{\sigma_n} = 0$, where $\{\sigma_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \sigma_n = 0$ and $\sum_{n=1}^{\infty} \sigma_n = \infty$.
(C4) The mapping $f : \mathscr{H}_1 \to \mathscr{H}_1$ is ρ-contractive with constant $\rho \in [0, 1)$.

3.1 The Algorithm 3.1

In this subsection, inspired by the inertial method, Byrne et al.'s method [7], the projection and contraction method and the viscosity-type method, we introduce a self-adaptive inertial projection and contraction method to solve the (SVIP). The details of the first iterative scheme are described in Algorithm 3.1.

Remark 3.1 From Algorithm 3.1, we have the following observations.

(i) It follows from (3.1) that

$$\lim_{n\to\infty} \frac{\vartheta_n}{\sigma_n} \|x_n - x_{n-1}\| = 0 .$$

Indeed, we have $\vartheta_n \|x_n - x_{n-1}\| \leq \varpi_n$ for all n, which together with $\lim_{n\to\infty} \frac{\varpi_n}{\sigma_n} = 0$ implies that

$$\lim_{n\to\infty} \frac{\vartheta_n}{\sigma_n} \|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\varpi_n}{\sigma_n} = 0 .$$

(ii) If $u_n = q_n$ or $c_n = 0$, then $q_n \in \Omega$. Indeed, from the definition of c_n, one obtains

$$\begin{aligned}\|c_n\| &\geq \|u_n - q_n\| - \gamma_n \|A^*(I - J_{\lambda F_2})Au_n - A^*(I - J_{\lambda F_2})Aq_n\| \\ &\geq (1-\delta)\|u_n - q_n\| .\end{aligned}$$

It can be easily proved that $\|c_n\| \leq (1+\delta)\|u_n - q_n\|$. Therefore,

$$(1-\delta)\|u_n - q_n\| \leq \|c_n\| \leq (1+\delta)\|u_n - q_n\| ,$$

Algorithm 3.1 The inertial viscosity-type projection and contraction algorithm for (SVIP)

Initialization: Set $\lambda > 0$, $\vartheta > 0$, $\zeta > 0$, $\chi \in (0,1)$, $\delta \in (0,1)$, $\kappa \in (0,2)$ and let $x_0, x_1 \in \mathscr{H}$.
Iterative Steps: Calculate x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$. Set $u_n = x_n + \vartheta_n (x_n - x_{n-1})$, where

$$\vartheta_n = \begin{cases} \min\left\{ \dfrac{\varpi_n}{\|x_n - x_{n-1}\|}, \vartheta \right\}, & \text{if } x_n \neq x_{n-1}; \\ \vartheta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Compute $q_n = J_{\lambda F_1}[u_n - \gamma_n A^*(I - J_{\lambda F_2})Au_n]$, where $\gamma_n = \zeta \chi^{r_n}$ and r_n is the smallest nonnegative integer such that

$$\gamma_n \|A^*(I - J_{\lambda F_2})Au_n - A^*(I - J_{\lambda F_2})Aq_n\| \leq \delta \|u_n - q_n\| . \tag{3.2}$$

If $u_n = q_n$, then stop and q_n is a solution of the problem (SVIP). Otherwise, go to **Step 3**.
Step 3. Compute $g_n = u_n - \kappa \mu_n c_n$, where

$$\begin{aligned} c_n &= u_n - q_n - \gamma_n [A^*(I - J_{\lambda F_2})Au_n - A^*(I - J_{\lambda F_2})Aq_n] , \\ \mu_n &= \frac{\langle u_n - q_n, c_n \rangle}{\|c_n\|^2} . \end{aligned} \tag{3.3}$$

Step 4. Compute $x_{n+1} = \sigma_n f(x_n) + (1 - \sigma_n) g_n$.
Set $n := n + 1$ and go to **Step 1**.

and thus $u_n = q_n$ iff $c_n = 0$. Hence, if $u_n = q_n$ or $c_n = 0$, then $q_n = J_{\lambda F_1}[q_n - \gamma_n A^*(I - J_{\lambda F_2})Aq_n]$. This implies that $q_n \in \Omega$ by means of Lemma 2.3.

The following lemmas are quite helpful to analyze the convergence of our algorithms.

Lemma 3.1 *The Armijo-like search rule (3.2) is well defined and* $\min\left\{\zeta, \frac{\delta\chi}{L}\right\} \leq \gamma_n \leq \zeta$.

Proof Indeed, using Lemma 2.2 (2), one sees that

$$\|A^*(I - J_{\lambda F_2})Au_n - A^*(I - J_{\lambda F_2})Aq_n\| \leq L\|u_n - q_n\| ,$$

where $L = \|A\|^2$. Obviously, (3.2) holds for all $0 < \gamma_n \leq \delta L^{-1}$. On the other hand, it is easy to see that $\gamma_n \leq \zeta$. If $\gamma_n = \zeta$, then this lemma is proved. Otherwise, if $\gamma_n < \zeta$, then inequality (3.2) will be violated when $\gamma = \gamma_n \chi^{-1}$, which indicates that $\gamma_n \chi^{-1} > \delta L^{-1}$. Hence $\gamma_n \geq \min\{\zeta, \frac{\delta\chi}{L}\}$. □

Lemma 3.2 *Suppose that Conditions (C1)–(C3) hold. Let* $\{u_n\}$, $\{q_n\}$ *and* $\{g_n\}$ *be three sequences created by Algorithm 3.1. Then, for all* $p \in \Omega$,

$$\|g_n - p\|^2 \leq \|u_n - p\|^2 - \frac{2-\kappa}{\kappa}\|g_n - u_n\|^2 ,$$

and

$$\|u_n - q_n\|^2 \leq \frac{(1+\delta)^2}{(1-\delta)^2\kappa^2}\|g_n - u_n\|^2 .$$

Proof Indeed, from the definitions of g_n and μ_n, we get

$$\begin{aligned} \|g_n - p\|^2 &= \|u_n - p\|^2 - 2\kappa\mu_n \langle u_n - p, c_n \rangle + \kappa^2 \mu_n^2 \|c_n\|^2 \\ &= \|u_n - p\|^2 - 2\kappa\mu_n \langle u_n - p, c_n \rangle + \kappa^2 \mu_n \langle u_n - q_n, c_n \rangle . \end{aligned} \tag{3.4}$$

It is easy to see the following equation

$$\langle u_n - p, c_n \rangle = \langle u_n - q_n, c_n \rangle + \langle q_n - p, c_n \rangle . \tag{3.5}$$

Next, we prove that

$$\langle q_n - p, c_n \rangle \geq 0 . \tag{3.6}$$

From the definition of q_n and Lemma 2.1 (4), we obtain

$$\langle u_n - \gamma_n A^*(I - J_{\lambda F_2}) A u_n - q_n, q_n - p \rangle \geq 0 . \tag{3.7}$$

It follows from $p \in \Omega$ that $Ap \in F_2^{-1}(0)$. Using Lemma 2.1 (2), one sees that $Ap \in \mathrm{Fix}(J_{\lambda F_2})$. This indicates that $A^* \, (I - J_{\lambda F_2})Ap = 0$. By Lemma 2.2 (1), one has

$$\left\langle \gamma_n \left(A^*(I - J_{\lambda F_2})Aq_n - A^*(I - J_{\lambda F_2})Ap \right), q_n - p \right\rangle \geq 0 ,$$

which yields $\langle q_n - p, \gamma_n A^*(I - J_{\lambda F_2})Aq_n \rangle \geq 0$. This together with Eq. (3.7) gets

$$\left\langle q_n - p, u_n - q_n - \gamma_n [A^*(I - J_{\lambda F_2})Au_n - A^*(I - J_{\lambda F_2})Aq_n] \right\rangle \geq 0 .$$

Thus, (3.6) was proved by using the definition of c_n. Combining (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned} \|g_n - p\|^2 &\leq \|u_n - p\|^2 - (2\kappa - \kappa^2)\mu_n \langle u_n - q_n, c_n \rangle \\ &= \|u_n - p\|^2 - \frac{2-\kappa}{\kappa} \|g_n - u_n\|^2 . \end{aligned}$$

On the other hand, by using (3.2), we have

$$\begin{aligned} \langle u_n - q_n, c_n \rangle &= \left\langle u_n - q_n, u_n - q_n - \gamma_n [A^*(I - J_{\lambda F_2})Au_n - A^*(I - J_{\lambda F_2})Aq_n] \right\rangle \\ &= \|u_n - q_n\|^2 - \gamma_n \left\langle u_n - q_n, A^*(I - J_{\lambda F_2})Au_n - A^*(I - J_{\lambda F_2})Aq_n \right\rangle \\ &\geq \|u_n - q_n\|^2 - \gamma_n \|u_n - q_n\| \|A^*(I - J_{\lambda F_2})Au_n - A^*(I - J_{\lambda F_2})Aq_n\| \\ &\geq (1-\delta)\|u_n - q_n\|^2 . \end{aligned}$$

From Remark 3.1 (ii), one obtains $\|c_n\|^2 \leq (1+\delta)^2 \|u_n - q_n\|^2$. According to the definition of μ_n, we have

$$\mu_n = \frac{\langle u_n - q_n, c_n \rangle}{\|c_n\|^2} \geq \frac{1-\delta}{(1+\delta)^2} .$$

By the definitions of g_n and μ_n, we get

$$\|u_n - q_n\|^2 \leq \frac{1}{1-\delta} \langle u_n - q_n, c_n \rangle \leq \frac{1}{(1-\delta)\mu_n \kappa^2} \|g_n - u_n\|^2 .$$

Therefore, we conclude that

$$\|u_n - q_n\|^2 \leq \frac{(1+\delta)^2}{(1-\delta)^2 \kappa^2} \|g_n - u_n\|^2 .$$

This completes the proof of the lemma. □

Lemma 3.3 *Assume that the sequences $\{u_n\}$ and $\{q_n\}$ are formed by Algorithm 3.1. If $\left\{u_{n_k}\right\}$ converges weakly to p and $\lim_{n\to\infty} \|u_n - q_n\| = 0$, then $p \in \Omega$.*

Proof Let us assume that $p \in \Omega$. Then $p \in F_1^{-1}(0)$, and thus $p \in \mathrm{Fix}(J_{\lambda F_1})$ by means of Lemma 2.1 (2). Using the definition of q_n and Lemma 2.1 (4), we obtain

$$\langle u_n - \gamma_n A^*(I - J_{\lambda F_2})Au_n - q_n, q_n - p\rangle \geq 0\,. \tag{3.8}$$

It follows from $p \in \Omega$ that $Ap \in F_2^{-1}(0)$. Hence $Ap \in \mathrm{Fix}(J_{\lambda F_2})$. This indicates that $A^*(I - J_{\lambda F_2})Ap = 0$. From Lemma 2.2 (1), one infers that

$$\langle A^*(I - J_{\lambda F_2})Aq_n - A^*(I - J_{\lambda F_2})Ap, q_n - p\rangle \geq \|(I - J_{\lambda F_2})Aq_n\|^2\,. \tag{3.9}$$

Combining (3.8), (3.9) and Lemma 2.2 (2), we have

$$\begin{aligned}\gamma_n\|Aq_n - J_{\lambda F_2}Aq_n\|^2 &\leq \big\langle \gamma_n A^*(I - J_{\lambda F_2})Aq_n, q_n - p\big\rangle \\ &\leq \big\langle u_n - q_n - \gamma_n A^*(I - J_{\lambda F_2})Au_n + \gamma_n A^*(I - J_{\lambda F_2})Aq_n, q_n - p\big\rangle \\ &\leq \|u_n - q_n - \gamma_n A^*(I - J_{\lambda F_2})Au_n + \gamma_n A^*(I - J_{\lambda F_2})Aq_n\|\|q_n - p\| \\ &\leq \big(\|u_n - q_n\| + \gamma_n\|A^*(I - J_{\lambda F_2})Au_n - A^*(I - J_{\lambda F_2})Aq_n\|\big)\|q_n - p\| \\ &\leq \big(1 + \gamma_n\|A\|^2\big)\|u_n - q_n\|\|q_n - p\|\,.\end{aligned}$$

Since $\gamma_n > 0$ and $\lim_{n\to\infty}\|u_n - q_n\| = 0$, we find that $\lim_{n\to\infty}\|Aq_n - J_{\lambda F_2}Aq_n\| = 0$. Moreover, by Lemma 2.1 (1), one observes that

$$\begin{aligned}\|Au_n - J_{\lambda F_2}Au_n\| &\leq \|Au_n - Aq_n - (J_{\lambda F_2}Au_n - J_{\lambda F_2}Aq_n)\| + \|Aq_n - J_{\lambda F_2}Aq_n\| \\ &\leq 2\|A\|\|u_n - q_n\| + \|Aq_n - J_{\lambda F_2}Aq_n\|\,.\end{aligned}$$

This indicates that

$$\lim_{n\to\infty}\|Au_n - J_{\lambda F_2}Au_n\| = 0\,. \tag{3.10}$$

From Lemma 2.1 (1) and the definition of q_n, we get

$$\begin{aligned}\|q_n - J_{\lambda F_1}u_n\| &= \|J_{\lambda F_1}(u_n - \gamma_n A^*(I - J_{\lambda F_2})Au_n) - J_{\lambda F_1}u_n\| \\ &\leq \gamma_n\|A^*\|\|(I - J_{\lambda F_2})Au_n\|\,,\end{aligned}$$

which together with (3.10) gives that $\lim_{n\to\infty}\|q_n - J_{\lambda F_1}u_n\| = 0$. From $\lim_{n\to\infty}\|u_n - q_n\| = 0$, one obtains $\lim_{n\to\infty}\|u_n - J_{\lambda F_1}u_n\| = 0$. This combining with Lemma 2.1 (1) and $u_{n_k} \rightharpoonup p$ yields $p \in \mathrm{Fix}(J_{\lambda F_1})$. In view of the fact that A is a linear bounded operator and $u_{n_k} \rightharpoonup p$, we get $Au_{n_k} \rightharpoonup Ap$. Using (3.10) and Lemma 2.1 (1), we obtain $Ap \in \mathrm{Fix}(J_{\lambda F_2})$. Thus, we deduce that $p \in \Omega$. The proof is completed. □

Remark 3.2 It is worth noting that the proof of Lemma 3.3 does not use the definition of Armijo stepsize (3.2).

We are now in a position to prove the strong convergence result of Algorithm 3.1.

Theorem 3.1 *Suppose that Conditions (C1)–(C4) hold. Then the sequence $\{x_n\}$ formed by Algorithm 3.1 converges to $p \in \Omega$ in norm, where $p = P_\Omega \circ f(p)$.*

Proof For simplicity, we divide the proof into four claims.

Claim 1. $\{x_n\}$ is bounded. Indeed, it follows from Lemma 3.2 and $\kappa \in (0, 2)$ that

$$\|g_n - p\| \leq \|u_n - p\|, \quad \forall n \geq 1\,. \tag{3.11}$$

By the definition of u_n, we can write

$$\|u_n - p\| \le \|x_n - p\| + \sigma_n \cdot \frac{\vartheta_n}{\sigma_n}\|x_n - x_{n-1}\| . \tag{3.12}$$

According to Remark 3.1 (i), one has $\frac{\vartheta_n}{\sigma_n}\|x_n - x_{n-1}\| \to 0$. Therefore, there exists a constant $M_1 > 0$ such that

$$\frac{\vartheta_n}{\sigma_n}\|x_n - x_{n-1}\| \le M_1, \quad \forall n \ge 1 . \tag{3.13}$$

Combining (3.11), (3.12) and (3.13), we obtain

$$\|g_n - p\| \le \|u_n - p\| \le \|x_n - p\| + \sigma_n M_1, \quad \forall n \ge 1 . \tag{3.14}$$

Thus,

$$\begin{aligned}
\|x_{n+1} - p\| &\le \sigma_n\|f(x_n) - f(p)\| + \sigma_n\|f(p) - p\| + (1-\sigma_n)\|g_n - p\| \\
&\le [1 - \sigma_n(1-\rho)]\|x_n - p\| + \sigma_n(1-\rho)\frac{\|f(p) - p\| + M_1}{1-\rho} \\
&\le \max\left\{\|x_n - p\|, \frac{\|f(p) - p\| + M_1}{1-\rho}\right\} \\
&\le \cdots \le \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\| + M_1}{1-\rho}\right\}.
\end{aligned}$$

This implies that sequence $\{x_n\}$ is bounded. So, sequences $\{f(x_n)\}$, $\{u_n\}$, $\{q_n\}$ and $\{g_n\}$ are also bounded.

Claim 2.

$$(1-\sigma_n)\frac{2-\kappa}{\kappa}\|u_n - g_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \sigma_n M_4$$

for some $M_4 > 0$. Indeed, from (3.14), one sees that

$$\begin{aligned}
\|u_n - p\|^2 &\le \left(\|x_n - p\| + \sigma_n M_1\right)^2 \\
&= \|x_n - p\|^2 + \sigma_n\left(2M_1\|x_n - p\| + \sigma_n M_1^2\right) \\
&\le \|x_n - p\|^2 + \sigma_n M_2
\end{aligned} \tag{3.15}$$

for some $M_2 > 0$. Combining Lemma 3.2 and (3.15), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \sigma_n\|f(x_n) - p\|^2 + (1-\sigma_n)\|g_n - p\|^2 \\
&\le \sigma_n(\|f(x_n) - f(p)\| + \|f(p) - p\|)^2 + (1-\sigma_n)\|g_n - p\|^2 \\
&\le \sigma_n(\|x_n - p\| + \|f(p) - p\|)^2 + (1-\sigma_n)\|g_n - p\|^2 \\
&= \sigma_n\|x_n - p\|^2 + (1-\sigma_n)\|g_n - p\|^2 \\
&\quad + \sigma_n(2\|x_n - p\| \cdot \|f(p) - p\| + \|f(p) - p\|^2) \\
&\le \sigma_n\|x_n - p\|^2 + (1-\sigma_n)\|g_n - p\|^2 + \sigma_n M_3 \\
&\le \|x_n - p\|^2 - (1-\sigma_n)\frac{2-\kappa}{\kappa}\|u_n - g_n\|^2 + \sigma_n M_4 ,
\end{aligned}$$

where $M_4 := M_2 + M_3$. The desired result can be achieved by a simple conversion.

Claim 3.

$$\|x_{n+1} - p\|^2 \le (1 - (1-\rho)\sigma_n)\|x_n - p\|^2 + (1-\rho)\sigma_n \cdot \Big[\frac{3M}{1-\rho} \cdot \frac{\vartheta_n}{\sigma_n}\|x_n - x_{n-1}\| + \frac{2}{1-\rho}\langle f(p) - p, x_{n+1} - p\rangle\Big]$$

for some $M > 0$. Indeed, it follows from the definition of u_n that

$$\begin{aligned}\|u_n - p\|^2 &\le \|x_n - p\|^2 + 2\vartheta_n\|x_n - p\|\|x_n - x_{n-1}\| + \vartheta_n^2\|x_n - x_{n-1}\|^2 \\ &\le \|x_n - p\|^2 + 3M\vartheta_n\|x_n - x_{n-1}\|,\end{aligned} \tag{3.16}$$

where $M := \sup_{n\in\mathbb{N}}\{\|x_n - p\|, \vartheta\|x_n - x_{n-1}\|\} > 0$. Using (3.14) and (3.16), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\sigma_n(f(x_n) - f(p)) + (1-\sigma_n)(g_n - p) + \sigma_n(f(p) - p)\|^2 \\ &\le \|\sigma_n(f(x_n) - f(p)) + (1-\sigma_n)(g_n - p)\|^2 + 2\sigma_n\langle f(p) - p, x_{n+1} - p\rangle \\ &\le \sigma_n\|f(x_n) - f(p)\|^2 + (1-\sigma_n)\|g_n - p\|^2 + 2\sigma_n\langle f(p) - p, x_{n+1} - p\rangle \\ &\le \sigma_n\rho\|x_n - p\|^2 + (1-\sigma_n)\|u_n - p\|^2 + 2\sigma_n\langle f(p) - p, x_{n+1} - p\rangle \\ &\le (1 - (1-\rho)\sigma_n)\|x_n - p\|^2 + (1-\rho)\sigma_n \cdot \Big[\frac{3M}{1-\rho} \cdot \frac{\vartheta_n}{\sigma_n}\|x_n - x_{n-1}\| \\ &\quad + \frac{2}{1-\rho}\langle f(p) - p, x_{n+1} - p\rangle\Big].\end{aligned}$$

Claim 4. $\{\|x_n - p\|^2\}$ converges to zero. Indeed, by Lemma 2.4, it suffices to show that $\limsup_{k\to\infty}\langle f(p) - p, x_{n_k+1} - p\rangle \le 0$ for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying $\liminf_{k\to\infty}(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|) \ge 0$.

For this purpose, we assume that $\{\|x_{n_k} - p\|\}$ is a subsequence of $\{\|x_n - p\|\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|) \ge 0$. Then,

$$\begin{aligned}&\liminf_{k\to\infty}\left(\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2\right) \\ &\quad = \liminf_{k\to\infty}\left[(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|)(\|x_{n_k+1} - p\| + \|x_{n_k} - p\|)\right] \ge 0.\end{aligned}$$

From Claim 2, one sees that

$$\begin{aligned}(1-\sigma_{n_k})\frac{2-\kappa}{\kappa}\|u_{n_k} - g_{n_k}\|^2 &\le \limsup_{k\to\infty}\left[\|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2 + \sigma_{n_k}M_4\right] \\ &\le 0,\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\|g_{n_k} - u_{n_k}\| = 0. \tag{3.17}$$

This together with Lemma 3.2 finds that $\lim_{k\to\infty}\|q_{n_k} - u_{n_k}\| = 0$. Moreover, using Remark 3.1 (i) and Condition (C3), we have

$$\|x_{n_k+1} - g_{n_k}\| = \sigma_{n_k}\|g_{n_k} - f(x_{n_k})\| \to 0, \tag{3.18}$$

and

$$\|x_{n_k} - u_{n_k}\| = \sigma_{n_k} \cdot \frac{\vartheta_{n_k}}{\sigma_{n_k}}\|x_{n_k} - x_{n_k-1}\| \to 0. \tag{3.19}$$

From (3.17), (3.18) and (3.19), we conclude that

$$\|x_{n_k+1} - x_{n_k}\| \le \|x_{n_k+1} - g_{n_k}\| + \|g_{n_k} - u_{n_k}\| + \|u_{n_k} - x_{n_k}\| \to 0 . \tag{3.20}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup z$. Furthermore,

$$\limsup_{k\to\infty} \langle f(p) - p, x_{n_k} - p\rangle = \lim_{j\to\infty} \langle f(p) - p, x_{n_{k_j}} - p\rangle = \langle f(p) - p, z - p\rangle \tag{3.21}$$

We get $u_{n_k} \rightharpoonup z$ since $\|x_{n_k} - u_{n_k}\| \to 0$. This together with $\lim_{k\to\infty} \|u_{n_k} - q_{n_k}\| = 0$ and Lemma 3.3 obtains that $z \in \Omega$. From the definition of p and (3.21), we get

$$\limsup_{k\to\infty} \langle f(p) - p, x_{n_k} - p\rangle = \langle f(p) - p, z - p\rangle \le 0 . \tag{3.22}$$

Combining (3.20) and (3.22), we obtain

$$\limsup_{k\to\infty} \langle f(p) - p, x_{n_k+1} - p\rangle \le \limsup_{k\to\infty} \langle f(p) - p, x_{n_k} - p\rangle \le 0 . \tag{3.23}$$

Thus, from Remark 3.1 (i), (3.23), Claim 3 and Lemma 2.4, we conclude that $x_n \to p$. That is the desired result. □

3.2 The Algorithm 3.2

In this subsection, we propose an inertial Mann-type projection and contraction algorithm to solve (SVIP). Before proposing our iterative scheme, we first assume that the algorithm satisfies conditions (C1)–(C3) and (C5) .

(C5) Assume that the real sequence $\{\tau_n\} \subset (0, 1)$ such that $\{\tau_n\} \subset (a, b) \subset (0, 1 - \sigma_n)$ for some $a > 0, b > 0$.

The Algorithm 3.2 is of the form:

Algorithm 3.2 The inertial Mann-type projection and contraction algorithm for (SVIP)

Initialization: Set $\lambda > 0$, $\vartheta > 0$, $\zeta > 0$, $\chi \in (0, 1)$, $\delta \in (0, 1)$, $\kappa \in (0, 2)$ and let $x_0, x_1 \in \mathscr{H}$.
Iterative Steps: Calculate the next iteration point x_{n+1} as follows:

$$\begin{cases} u_n = x_n + \vartheta_n (x_n - x_{n-1}) , \\ q_n = J_{\lambda F_1}[u_n - \gamma_n A^*(I - J_{\lambda F_2}) A u_n] , \\ g_n = u_n - \kappa \mu_n c_n , \\ x_{n+1} = (1 - \sigma_n - \tau_n) u_n + \tau_n g_n , \end{cases}$$

where $\{\vartheta_n\}$, $\{\gamma_n\}$ and $\{c_n\}$ are defined in (3.1), (3.2) and (3.3), respectively.

Theorem 3.2 *Suppose that Conditions (C1)–(C3) and (C5) hold. Then the sequence $\{x_n\}$ created by Algorithm 3.2 converges to $p \in \Omega$ in norm, where $\|p\| = \min\{\|z\| : z \in \Omega\}$.*

Proof We divide this proof into four steps.

Claim 1. The sequence $\{x_n\}$ is bounded. Indeed, thanks to Lemma 3.2, we have

$$\|g_n - p\| \le \|u_n - p\| . \tag{3.24}$$

By the definition of x_{n+1}, one has

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-\sigma_n-\tau_n)(u_n-p)+\tau_n(g_n-p)-\sigma_n p\| \\ &\le \|(1-\sigma_n-\tau_n)(u_n-p)+\tau_n(g_n-p)\|+\sigma_n\|p\| .\end{aligned} \tag{3.25}$$

It follows from (3.24) that

$$\begin{aligned}&\|(1-\sigma_n-\tau_n)(u_n-p)+\tau_n(g_n-p)\|^2 \\ &\quad\le (1-\sigma_n-\tau_n)^2\|u_n-p\|^2+2(1-\sigma_n-\tau_n)\tau_n\|g_n-p\|\|u_n-p\|+\tau_n^2\|g_n-p\|^2 \\ &\quad\le (1-\sigma_n-\tau_n)^2\|u_n-p\|^2+2(1-\sigma_n-\tau_n)\tau_n\|u_n-p\|^2+\tau_n^2\|u_n-p\|^2 \\ &\quad= (1-\sigma_n)^2\|u_n-p\|^2 ,\end{aligned}$$

which yields

$$\|(1-\sigma_n-\tau_n)(u_n-p)+\tau_n(g_n-p)\| \le (1-\sigma_n)\|u_n-p\| . \tag{3.26}$$

Combining (3.14), (3.25) and (3.26), we deduce that

$$\begin{aligned}\|x_{n+1}-p\| &\le (1-\sigma_n)\|u_n-p\|+\sigma_n\|p\| \\ &\le (1-\sigma_n)\|x_n-p\|+\sigma_n(\|p\|+M_1) \\ &\le \max\{\|x_n-p\|, \|p\|+M_1\} \\ &\le \cdots \le \max\{\|x_0-p\|, \|p\|+M_1\} .\end{aligned}$$

That is, $\{x_n\}$ is bounded. So, the sequences $\{g_n\}$ and $\{u_n\}$ are also bounded.

Claim 2.

$$\tau_n\frac{2-\kappa}{\kappa}\|u_n-g_n\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\sigma_n\big(\|p\|^2+M_2\big) .$$

Indeed, using Lemma 3.2 and (3.15), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\sigma_n-\tau_n)\|u_n-p\|^2+\tau_n\|g_n-p\|^2+\sigma_n\|p\|^2 \\ &\le (1-\sigma_n-\tau_n)\|u_n-p\|^2+\tau_n\|u_n-p\|^2-\tau_n\frac{2-\kappa}{\kappa}\|u_n-g_n\|^2+\sigma_n\|p\|^2 \\ &\le \|x_n-p\|^2-\tau_n\frac{2-\kappa}{\kappa}\|u_n-g_n\|^2+\sigma_n\big(\|p\|^2+M_2\big) .\end{aligned}$$

The desired result can be achieved by a simple conversion.

Claim 3.

$$\begin{aligned}\|x_{n+1}-p\|^2 \le (1-\sigma_n)\|x_n-p\|^2+\sigma_n\Big[&2\tau_n\|u_n-g_n\|\|x_{n+1}-p\| \\ &+2\langle p, p-x_{n+1}\rangle+\frac{3M\vartheta_n}{\sigma_n}\|x_n-x_{n-1}\|\Big] .\end{aligned}$$

Setting $t_n=(1-\tau_n)u_n+\tau_n g_n$, one has

$$\|t_n-u_n\| = \tau_n\|u_n-g_n\| . \tag{3.27}$$

It follows from (3.24) that

$$\begin{aligned}\|t_n-p\| &= \|(1-\tau_n)(u_n-p)+\tau_n(g_n-p)\| \\ &\le (1-\tau_n)\|u_n-p\|+\tau_n\|u_n-p\| \\ &= \|u_n-p\| .\end{aligned} \tag{3.28}$$

From (3.16), (3.27) and (3.28), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(1-\sigma_n)(t_n-p)-\sigma_n(u_n-t_n)-\sigma_n p\|^2\\&\le (1-\sigma_n)^2\|t_n-p\|^2-2\sigma_n\langle u_n-t_n+p, x_{n+1}-p\rangle\\&\le (1-\sigma_n)\|t_n-p\|^2+2\sigma_n\|u_n-t_n\|\|x_{n+1}-p\|+2\sigma_n\langle p, p-x_{n+1}\rangle\\&\le (1-\sigma_n)\|x_n-p\|^2+\sigma_n\Big[2\tau_n\|u_n-g_n\|\|x_{n+1}-p\|\\&\quad+2\langle p, p-x_{n+1}\rangle+\frac{3M\vartheta_n}{\sigma_n}\|x_n-x_{n-1}\|\Big].\end{aligned}$$

Claim 4. The sequence $\{\|x_n-p\|^2\}$ converges to zero. We assume that $\{\|x_{n_k}-p\|\}$ is a subsequence of $\{\|x_n-p\|\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|)\ge 0$. By Claim 2 and Condition (C5), we have

$$\begin{aligned}\tau_{n_k}\frac{2-\kappa}{\kappa}\|u_{n_k}-g_{n_k}\|^2 &\le \limsup_{k\to\infty}\big[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2\big]+\limsup_{k\to\infty}\sigma_{n_k}\big(\|p\|^2+M_2\big)\\&\le 0\,,\end{aligned}$$

which indicates that

$$\lim_{k\to\infty}\|g_{n_k}-u_{n_k}\|=0\,. \tag{3.29}$$

In view of Lemma 3.2, one observes that $\lim_{k\to\infty}\|q_{n_k}-u_{n_k}\|=0$. From (3.29) and the boundedness of $\{x_n\}$, we can further obtain

$$\lim_{k\to\infty}\tau_{n_k}\|u_{n_k}-g_{n_k}\|\|x_{n_k+1}-p\|=0\,. \tag{3.30}$$

Moreover, using (3.29), Condition (C5) and Remark 3.1 (i) , we have

$$\|x_{n_k+1}-u_{n_k}\|=\sigma_{n_k}\|u_{n_k}\|+\tau_{n_k}\|g_{n_k}-u_{n_k}\|\to 0\,,$$

and

$$\|x_{n_k}-u_{n_k}\|=\sigma_{n_k}\cdot\frac{\vartheta_{n_k}}{\sigma_{n_k}}\|x_{n_k}-x_{n_k-1}\|\to 0\,.$$

Thus, we conclude that

$$\|x_{n_k+1}-x_{n_k}\|\le\|x_{n_k+1}-u_{n_k}\|+\|u_{n_k}-x_{n_k}\|\to 0\,. \tag{3.31}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}}\rightharpoonup z$. Moreover,

$$\limsup_{k\to\infty}\langle p, p-x_{n_k}\rangle=\lim_{j\to\infty}\langle p, p-x_{n_{k_j}}\rangle=\langle p, p-z\rangle\,. \tag{3.32}$$

Since $\|x_{n_k}-u_{n_k}\|\to 0$, one has $u_{n_k}\rightharpoonup z$, which, together with $\lim_{k\to\infty}\|u_{n_k}-q_{n_k}\|=0$ and Lemma 3.3, gets that $z\in\Omega$. From the definition of p and (3.32), we obtain

$$\limsup_{k\to\infty}\langle p, p-x_{n_k}\rangle=\langle p, p-z\rangle\le 0\,. \tag{3.33}$$

Combining (3.31) and (3.33), we have

$$\limsup_{k\to\infty}\langle p, p-x_{n_k+1}\rangle\le\limsup_{k\to\infty}\langle p, p-x_{n_k}\rangle\le 0\,. \tag{3.34}$$

Thus, from Remark 3.1 (i), (3.30), (3.34), Claim 3 and Lemma 2.4, we conclude that $x_n\to p$. The proof is completed. □

3.3 The Algorithm 3.3

In this subsection, an inertial Mann-type algorithm for solving (SVIP) will be given. It is worth noting that this method uses a new step size update criterion that does not require any line search process. More precisely, the approach is described as follows:

Algorithm 3.3 The self-adaptive inertial Mann-type algorithm for (SVIP)

Initialization: Set $\lambda > 0$, $\vartheta > 0$, $\varphi_n \in (0, 2)$ and let $x_0, x_1 \in \mathscr{H}$.
Iterative Steps: Calculate the next iteration point x_{n+1} as follows:

$$\begin{cases} u_n = x_n + \vartheta_n (x_n - x_{n-1}) \,, \\ g_n = J_{\lambda F_1}[u_n - \gamma_n A^*(I - J_{\lambda F_2})Au_n] \,, \\ x_{n+1} = (1 - \sigma_n - \tau_n)u_n + \tau_n g_n \,, \end{cases}$$

where $\{\vartheta_n\}$ is defined in (3.1) and the stepsize γ_n is updated by the following:

$$\gamma_n = \begin{cases} \dfrac{\varphi_n \|(I - J_{\lambda F_2})Au_n\|^2}{\|A^*(I - J_{\lambda F_2})Au_n\|^2}, & \text{if } \|A^*(I - J_{\lambda F_2})Au_n\| \neq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{3.35}$$

The following two lemmas are very important for the convergence analysis of the algorithms.

Lemma 3.4 *The sequence $\{\gamma_n\}$ formed by (3.35) is bounded.*

Proof Indeed, if $\|A^*(I - J_{\lambda F_2})Au_n\| \neq 0$, then

$$\inf \left\{ \frac{2\|(I - J_{\lambda F_2})Au_n\|^2}{\|A^*(I - J_{\lambda F_2})Au_n\|^2} - \gamma_n \right\} > 0 \,.$$

On the other hand, from the fact that A is bounded and linear, we can show that

$$\frac{\varphi_n \|(I - J_{\lambda F_2})Au_n\|^2}{\|A^*(I - J_{\lambda F_2})Au_n\|^2} \geq \frac{\varphi_n \|(I - J_{\lambda F_2})Au_n\|^2}{\|A\|^2 \|(I - J_{\lambda F_2})Au_n\|^2} = \frac{\varphi_n}{\|A\|^2} \,.$$

Therefore, $\sup \gamma_n < \infty$ and thus $\{\gamma_n\}$ is bounded. □

Lemma 3.5 *Suppose that Conditions (C1)–(C3) hold. Let the sequences $\{u_n\}$ and $\{g_n\}$ be made by Algorithm 3.3. Then*

$$\|g_n - p\|^2 \leq \|u_n - p\|^2 - \gamma_n (2 - \varphi_n) \|(I - J_{\lambda F_2})Au_n\|^2 \,.$$

Proof From Lemma 2.1 (1), Lemma 2.2 (3) and the definition of γ_n, we have

$$\begin{aligned} \|g_n - p\|^2 &\leq \|u_n - \gamma_n A^*(I - J_{\lambda F_2})Au_n - p\|^2 \\ &= \|u_n - p\|^2 + \|\gamma_n A^*(I - J_{\lambda F_2})Au_n\|^2 - 2\gamma_n \left\langle u_n - p, A^*(I - J_{\lambda F_2})Au_n \right\rangle \\ &\leq \|u_n - p\|^2 + \gamma_n^2 \|A^*(I - J_{\lambda F_2})Au_n\|^2 - 2\gamma_n \|(I - J_{\lambda F_2})Au_n\|^2 \\ &= \|u_n - p\|^2 - \gamma_n \big(2\|(I - J_{\lambda F_2})Au_n\|^2 - \gamma_n \|A^*(I - J_{\lambda F_2})Au_n\|^2\big) \\ &= \|u_n - p\|^2 - \gamma_n (2 - \varphi_n) \|(I - J_{\lambda F_2})Au_n\|^2 \,. \end{aligned}$$

The proof of the lemma is now complete. □

Theorem 3.3 *Suppose that Conditions (C1)–(C3) and (C5) hold. Then the sequence* $\{x_n\}$ *formed by Algorithm 3.3 converges to* $p \in \Omega$ *in norm, where* $\|p\| = \min\{\|z\| : z \in \Omega\}$.

Proof This proof is divided into four claims.

Claim 1. The sequence $\{x_n\}$ is bounded. Indeed, from Lemma 3.5 and $\varphi_n \in (0, 2)$, we have

$$\|g_n - p\| \le \|u_n - p\|, \quad \forall n \ge 1\,. \tag{3.36}$$

From (3.14), (3.25), (3.26) and (3.36), we get

$$\begin{aligned}\|x_{n+1} - p\| &\le (1-\sigma_n)\|u_n - p\| + \sigma_n\|p\| \\ &\le (1-\sigma_n)\|x_n - p\| + \sigma_n(\|p\| + M_1) \\ &\le \max\{\|x_n - p\|, \|p\| + M_1\} \\ &\le \cdots \le \max\{\|x_0 - p\|, \|p\| + M_1\}\,,\end{aligned}$$

where M_1 is defined in Claim 1 of Theorem 3.1. Thus, $\{x_n\}$ is bounded. Consequently, $\{u_n\}$ and $\{g_n\}$ are also bounded.

Claim 2.

$$\begin{aligned}&\tau_n\gamma_n(2-\varphi_n)\|(I - J_{\lambda F_2})Au_n\|^2 + \tau_n(1-\sigma_n-\tau_n)\|u_n - g_n\|^2 \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \sigma_n(\|p\|^2 + M_2)\,.\end{aligned}$$

Indeed, from Lemma 3.5 and (3.15), we get

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\sigma_n-\tau_n)(u_n - p) + \tau_n(g_n - p) + \sigma_n(-p)\|^2 \\ &\le (1-\sigma_n-\tau_n)\|u_n - p\|^2 + \tau_n\|g_n - p\|^2 + \sigma_n\|p\|^2 \\ &\quad - \tau_n(1-\sigma_n-\tau_n)\|u_n - g_n\|^2 \\ &\le \|x_n - p\|^2 - \tau_n\gamma_n(2-\varphi_n)\|(I - J_{\lambda F_2})Au_n\|^2 \\ &\quad + \sigma_n(\|p\|^2 + M_2) - \tau_n(1-\sigma_n-\tau_n)\|u_n - g_n\|^2\,,\end{aligned}$$

wher M_2 is defined in Claim 2 of Theorem 3.1. The desired result can be obtained by a simple conversion.

Claim 3.

$$\begin{aligned}\|x_{n+1} - p\|^2 \le (1-\sigma_n)\|x_n - p\|^2 + \sigma_n\Big[&2\tau_n\|u_n - g_n\|\|x_{n+1} - p\| \\ &+ 2\langle p, p - x_{n+1}\rangle + \frac{3M\vartheta_n}{\sigma_n}\|x_n - x_{n-1}\|\Big]\,.\end{aligned}$$

This result can be obtained by using the same facts as the Claim 3 of Theorem 3.2.

Claim 4. The sequence $\{\|x_n - p\|^2\}$ converges to zero. We assume that $\{\|x_{n_k} - p\|\}$ is a subsequence of $\{\|x_n - p\|\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|) \ge 0$. By Claim 2 and Conditions (C3) and (C5) , we have

$$\begin{aligned}&\tau_{n_k}\gamma_{n_k}(2-\varphi_{n_k})\|(I - J_{\lambda F_2})Au_{n_k}\|^2 + \tau_{n_k}(1-\sigma_{n_k}-\tau_{n_k})\|u_{n_k} - g_{n_k}\|^2 \\ &\quad\le \limsup_{k\to\infty}\left[\|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2\right] + \limsup_{k\to\infty}\sigma_{n_k}\left(\|p\|^2 + M_2\right) \le 0\,,\end{aligned}$$

which implies that $\lim_{k\to\infty} \|(I - J_{\lambda F_2})Au_{n_k}\| = 0$ and $\lim_{k\to\infty} \|g_{n_k} - u_{n_k}\| = 0$. From (3.30)–(3.34), we can show that

$$\lim_{k\to\infty} \tau_{n_k} \|u_{n_k} - g_{n_k}\| \|x_{n_k+1} - p\| = 0,$$

and

$$\limsup_{k\to\infty} \langle p, p - x_{n_k+1}\rangle \leq 0\,.$$

Combining these with Remark 3.1 (i), Claim 3 and Lemma 2.4, we deduce that $x_n \to p$. This completes the proof. □

3.4 The Algorithm 3.4

Finally, we introduce a modified version of Algorithm 3.3, which uses the viscosity-type method to ensure the strong convergence of the suggested iterative scheme. The method is stated as follows:

Algorithm 3.4 The self-adaptive inertial viscosity-type algorithm for (SVIP)

Initialization: Set $\lambda > 0$, $\vartheta > 0$, $\varphi_n \in (0, 2)$ and let $x_0, x_1 \in \mathscr{H}$.

Iterative Steps: Calculate the next iteration point x_{n+1} as follows:

$$\begin{cases} u_n = x_n + \vartheta_n(x_n - x_{n-1})\,, \\ g_n = J_{\lambda F_1}[u_n - \gamma_n A^*(I - J_{\lambda F_2})Au_n]\,, \\ x_{n+1} = \sigma_n f(x_n) + (1 - \sigma_n)g_n\,, \end{cases}$$

where $\{\vartheta_n\}$ and $\{\gamma_n\}$ are defined in (3.1) and (3.35), respectively.

Based on the proofs of Theorems 3.1 and 3.3 , we will give the convergence analysis of Algorithm 3.4 in a compact way.

Theorem 3.4 *Suppose that Conditions (C1)–(C3) and (C5) hold. Then the sequence $\{x_n\}$ created by Algorithm 3.4 converges to $p \in \Omega$ in norm, where $p = P_\Omega \circ f(p)$.*

Proof **Claim 1.** The sequence $\{x_n\}$ is bounded. Indeed, using (3.12)–(3.14) and (3.36), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\sigma_n(f(x_n) - p) + (1 - \sigma_n)(g_n - p)\| \\ &\leq \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\| + M_1}{1 - \rho}\right\}. \end{aligned}$$

This means that $\{x_n\}$ is bounded. Hence, $\{f(x_n)\}$, $\{u_n\}$ and $\{g_n\}$ are also bounded.

Claim 2.

$$(1 - \sigma_n)\gamma_n(2 - \varphi_n)\|(I - J_{\lambda F_2})Au_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \sigma_n M_4\,,$$

and

$$\begin{aligned} (1 - \sigma_n)\|g_n - u_n\|^2 \leq{}& \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \sigma_n M_4 \\ &+ 2(1 - \sigma_n)\gamma_n\|g_n - p\|\|A^*(I - J_{\lambda F_2})Au_n\|\,. \end{aligned}$$

Indeed, using (3.15) and Lemma 3.5, we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \sigma_n\|f(x_n)-p\|^2+(1-\sigma_n)\|g_n-p\|^2\\ &\le \sigma_n(\|f(x_n)-f(p)\|+\|f(p)-p\|)^2+(1-\sigma_n)\|g_n-p\|^2\\ &\le \sigma_n\|x_n-p\|^2+(1-\sigma_n)\|g_n-p\|^2\\ &\quad+\sigma_n(2\|x_n-p\|\cdot\|f(p)-p\|+\|f(p)-p\|^2)\\ &\le \sigma_n\|x_n-p\|^2+(1-\sigma_n)\|g_n-p\|^2+\sigma_n M_3\\ &\le \|x_n-p\|^2-(1-\sigma_n)\gamma_n(2-\varphi_n)\|(I-J_{\lambda F_2})Au_n\|^2+\sigma_n M_4\,,\end{aligned} \tag{3.37}$$

where $M_4 := M_2 + M_3$. The first desired result can be achieved by a simple conversion.

On the other hand, from the fact that $J_{\lambda F_1}$ is firmly nonexpansive, we get

$$\begin{aligned}2\|g_n-p\|^2 &= 2\|J_{\lambda F_1}(u_n-\gamma_n A^*(I-J_{\lambda F_2})Au_n)-J_{\lambda F_1}(p)\|^2\\ &\le 2\langle g_n-p, u_n-\gamma_n A^*(I-J_{\lambda F_2})Au_n-p\rangle\\ &= \|g_n-p\|^2+\|u_n-\gamma_n A^*(I-J_{\lambda F_2})Au_n-p\|^2\\ &\quad-\|g_n-p-(u_n-\gamma_n A^*(I-J_{\lambda F_2})Au_n-p)\|^2\\ &= \|g_n-p\|^2+\|u_n-p\|^2+\gamma_n^2\|A^*(I-J_{\lambda F_2})Au_n\|^2\\ &\quad-2\langle u_n-p,\gamma_n A^*(I-J_{\lambda F_2})Au_n\rangle-\|g_n-u_n\|^2\\ &\quad-\gamma_n^2\|A^*(I-J_{\lambda F_2})Au_n\|^2-2\langle g_n-u_n,\gamma_n A^*(I-J_{\lambda F_2})Au_n\rangle\\ &= \|g_n-p\|^2+\|u_n-p\|^2-\|g_n-u_n\|^2\\ &\quad+2\langle g_n-p,\gamma_n A^*(J_{\lambda F_2}-I)Au_n\rangle\,,\end{aligned}$$

which implies that

$$\|g_n-p\|^2\le\|u_n-p\|^2-\|g_n-u_n\|^2+2\gamma_n\|g_n-p\|\|A^*(I-J_{\lambda F_2})Au_n\|\,.$$

This together with (3.15) and (3.37) obtains

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \sigma_n\|x_n-p\|^2+(1-\sigma_n)\|g_n-p\|^2+\sigma_n M_3\\ &\le \sigma_n\|x_n-p\|^2+(1-\sigma_n)\|u_n-p\|^2+\sigma_n M_3\\ &\quad-(1-\sigma_n)\big(\|g_n-u_n\|^2-2\gamma_n\|g_n-p\|\|A^*(I-J_{\lambda F_2})Au_n\|\big)\\ &\le \|x_n-p\|^2-(1-\sigma_n)\big(\|g_n-u_n\|^2-2\gamma_n\|g_n-p\|\|A^*(I-J_{\lambda F_2})Au_n\|\big)+\sigma_n M_4\,.\end{aligned}$$

By a simple transformation, we get the second desired result.

Claim 3.

$$\begin{aligned}\|x_{n+1}-p\|^2 \le (1-(1-\rho)\sigma_n)\|x_n-p\|^2+(1-\rho)\sigma_n\cdot\Big[\frac{3M}{1-\rho}\cdot\frac{\vartheta_n}{\sigma_n}\|x_n-x_{n-1}\|\\ +\frac{2}{1-\rho}\langle f(p)-p, x_{n+1}-p\rangle\Big],\end{aligned}$$

This result can be obtained by using the same facts as the Claim 3 of Theorem 3.1.

Claim 4. $\{\|x_n - p\|^2\}$ converges to zero. We assume that $\left\{\|x_{n_k} - p\|\right\}$ is a subsequence of $\{\|x_n - p\|\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|) \geq 0$. By Claim 2 and Condition (C3), we have

$$(1-\sigma_{n_k})\gamma_{n_k}(2-\varphi_{n_k})\|(I - J_{\lambda F_2})Au_{n_k}\|^2 \leq \limsup_{k\to\infty}\left[\|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2 + \sigma_{n_k}M_4\right] \leq 0\,,$$

which implies that $\lim_{k\to\infty}\|(I - J_{\lambda F_2})Au_{n_k}\| = 0$. This together with Claim 2 yields that $\lim_{k\to\infty}\|g_{n_k} - u_{n_k}\| = 0$. From (3.18)–(3.23), we observe that

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_k+1} - p\rangle \leq 0\,.$$

This together with Remark 3.1 (i), Claim 3 and Lemma 2.4 concludes that $x_n \to p$. The proof of the theorem is now complete. □

Remark 3.3 We note here that the proposed algorithms directly improve some known results in the literature. The details are as follows:

(i) Our presented methods have strong convergence in real Hilbert spaces, which is more preferable than the weak convergence results of Byrne et al. [7], Chuang [17], Majee and Nahak [18] and Kesornprom and Cholamjiak [28]. Moreover, our Algorithms 3.1 and 3.4 use the viscosity-type method to ensure strong convergence, which makes them faster than the Halpern-type methods in the literature [7,8,34].

(ii) The selection of the step size in the algorithms provided by [7–10,12,13,17,18,24–26] requires the prior information of the operator (matrix) norm, while our algorithms can adaptively update the step size of each iteration. On the one hand, it is not easy to estimate the operator (matrix) norm of the bounded linear operator A in practical applications. On the other hand, it should be pointed out that Armijo-type search methods need to evaluate the value of the iterative sequences $\{u_n, q_n\}$ at operator A and the resolvent mapping of F_2 multiple times in each iteration. The proposed Algorithms 3.3 and 3.4 use a method that does not involve any line search process. The method only needs to use known information for a simple calculation in each iteration to complete the step size update. Therefore, our self-adaptive iterative schemes (especially for Algorithms 3.3 and 3.4) are more preferable than the fixed-step methods and the Armijo-type methods [28].

(iii) In [24, Algorithm 3.3], Thong et al. [24] calculated g_n by $g_n = u_n - \mu_n c_n$, however, our Algorithms 3.1 and 3.2 calculate g_n via $g_n = u_n - \kappa\mu_n c_n$, where $\kappa \in (0, 2)$. Obviously, our two methods for calculating g_n are preferable to Thong et al. [24]. Furthermore, Algorithm 3.3 and Anh et al.'s Algorithm [26, Algorithm 4] update x_{n+1} differently. To be more precise, we calculate $x_{n+1} = (1-\sigma_n-\tau_n)u_n + \tau_n g_n$, while Anh et al. calculated $x_{n+1} = (1 - \sigma_n - \tau_n)x_n + \tau_n g_n$. Numerical experiments show that our iterative scheme is more efficient than Anh et al.'s algorithm (cf. Sect. 5).

4 Applications

In this section, we apply the proposed algorithms 3.1–3.4 to split feasibility problems and split minimization problems.

4.1 Application to Split Feasibility Problems

Recall that the split feasibility problem (SFP) introduced by Censor and Elfving [2] is described as follows:

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q\,, \tag{SFP}$$

where C and Q are closed convex subsets of real Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, and $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator with adjoint operator A^*. We shall denote Ω the solution set of (SFP). Based on Algorithm 3.1, we obtain the following result.

Corollary 4.1 *Let $\mathscr{H}_1$, $\mathscr{H}_2$, C, Q, A, A^* and Ω be the same as the above statement. Suppose that $\Omega \neq \emptyset$, $\vartheta > 0$, $\zeta > 0$, $\chi \in (0, 1)$, $\delta \in (0, 1)$, $\kappa \in (0, 2)$, and Conditions (C3) and (C4) hold. Let $x_0, x_1 \in \mathscr{H}$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = x_n + \vartheta_n(x_n - x_{n-1})\,, \\ q_n = P_C\left[u_n - \gamma_n A^*\left(I - P_Q\right)Au_n\right]\,, \\ g_n = u_n - \kappa\mu_n c_n\,, \\ x_{n+1} = \sigma_n f(x_n) + (1 - \sigma_n)g_n\,, \end{cases}$$

where ϑ_n is defined in (3.1), c_n and μ_n are defined as follows:

$$c_n = u_n - q_n - \gamma_n[A^*(I - P_Q)Au_n - A^*(I - P_Q)Aq_n]\,,\ \mu_n = \frac{\langle u_n - q_n, c_n\rangle}{\|c_n\|^2}\,,$$

and $\gamma_n = \zeta\chi^{r_n}$ and r_n is the smallest nonnegative integer such that

$$\gamma_n\|A^*(I - P_Q)Au_n - A^*(I - P_Q)Aq_n\| \le \delta\|u_n - q_n\|\,.$$

Then the iterative sequence $\{x_n\}$ provided above converges to $p \in \Omega$ in norm.

In Algorithms 3.2–3.4, if $J_{\lambda F_1} = P_C$ and $J_{\lambda F_2} = P_Q$, we can also obtain some sub-results on the split feasibility problem. We omit them here.

4.2 Application to Split Minimization Problems

In this subsection, we explore the solution of the following split minimization problem (SMP):

$$\text{find } x^* \in \mathscr{H}_1 \ \text{ such that } \ x^* \in \operatorname{argmin}_{x\in\mathscr{H}_1} f(x) \text{ and } Ax^* \in \operatorname{argmin}_{y\in\mathscr{H}_2} g(y)\,, \tag{SMP}$$

where $\mathscr{H}_1$ and $\mathscr{H}_2$ represent two real Hilbert spaces, $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a linear bounded operator with its adjoint A^*, and convex functions $f : \mathscr{H}_1 \to \mathbb{R}$ and $g : \mathscr{H}_2 \to \mathbb{R}$ are proper lower semicontinuous. For convenience, we also use Ω to represent the solution set of (SMP) and assume that $\Omega \neq \emptyset$. Let $\text{prox}_{\lambda f}$ represent the proximal mapping of a proper convex and lower semicontinuous function $f : \mathscr{H}_1 \to \mathbb{R}$ with a parameter $\lambda > 0$, which is defined as follows:

$$\text{prox}_{\lambda f}(x) := \operatorname{argmin}_{y\in\mathscr{H}_1}\left\{\lambda f(y) + \frac{1}{2}\|y - x\|^2\right\}.$$

It is well known that $\text{prox}_{\lambda f}(x) = (I+\lambda\partial f)^{-1}(x) = J_{\lambda\partial f}(x)$, where ∂f is the subdifferential of f defined by

$$\partial f(x) := \{z \in \mathscr{H} : f(y) - f(x) \ge \langle z, y - x\rangle,\ \forall y \in \mathscr{H}\}, \quad \forall x \in \text{Dom}(f)\,.$$

It is known that ∂f is maximal monotone and $\text{prox}_{\lambda f}$ is firmly nonexpansive. Thus, the following corollary follows directly from Algorithm 3.3.

Corollary 4.2 *Let $\mathscr{H}_1$, $\mathscr{H}_2$, f, g, A, A^* and Ω be the same as the above statement. Assume that $\Omega \neq \emptyset$, $\lambda > 0$, $\vartheta > 0$, $\varphi_n \in (0,2)$, and Conditions (C3) and (C5) hold. Let $x_0, x_1 \in \mathscr{H}$ and $\{x_n\}$ be a sequence created by*

$$\begin{cases} u_n = x_n + \vartheta_n(x_n - x_{n-1})\,, \\ g_n = \text{prox}_{\lambda f}[u_n - \gamma_n A^*(I - \text{prox}_{\lambda g})Au_n]\,, \\ x_{n+1} = (1 - \sigma_n - \tau_n)u_n + \tau_n g_n\,, \end{cases}$$

where ϑ_n is defined in (3.1) and the stepsize γ_n is updated by the following:

$$\gamma_n = \begin{cases} \dfrac{\varphi_n \|(I - \text{prox}_{\lambda g})Au_n\|^2}{\|A^*(I - \text{prox}_{\lambda g})Au_n\|^2}, & \text{if } \|A^*(I - \text{prox}_{\lambda g})Au_n\| \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then the iterative sequence $\{x_n\}$ constructed above converges to $p \in \Omega$ in norm.

In Algorithms 3.1, 3.2 and 3.4 , if $J_{\lambda F_1} = \text{prox}_{\lambda f}$ and $J_{\lambda F_2} = \text{prox}_{\lambda g}$, we can also obtain some sub-results on the split minimization problem. We omit them here.

5 Numerical Experiments

In this section, we provide some numerical examples occurring in finite- and infinite- dimensional spaces to show the advantages of our algorithms, and compare them with some known strongly convergent algorithms, including Byrne et al.'s Algorithm 4.4 (shortly, BCGR Alg. 4.4) [7], Kazmi and Rizvi's Algorithm (3.1) (KR Alg. (3.1)) [8], Thong et al.'s Algorithm 3.3 (TDC Alg. 3.3) [24], Long et al.'s Algorithm (49) (LTD Alg. (49)) [25], Anh et al.'s Algorithm (4) (ATD Alg. 4) [26] and Suantai et al.'s Algorithm 3 (SKC Alg. 3) [34]. All the programs are implemented in MATLAB 2018a on a Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz computer with RAM 8.00 GB. Before starting our numerical experiments, we first review the strongly convergent algorithms that need to be compared. These iterative schemes and their convergence conditions are described in Table 1.

In the following numerical experiments, the parameters of all algorithms are set as follows:

- In all algorithm, we set $\lambda = 1$, $\sigma_n = 1/(n+1)$, $\tau_n = 0.5(1 - \sigma_n)$ and $f(x) = 0.5x$.
- In BCGR Alg. 4.4, KR Alg. (3.1), TDC Alg. 3.3, LTD Alg. (49) and ATD Alg. 4, we choose stepsize $\gamma = 0.5/\|A\|^2$.
- In TDC Alg. 3.3, LTD Alg. (49) and ATD Alg. 4, we update the inertia parameter ϑ_n through (3.1). In these three algorithms and the offered algorithms 3.1–3.4, we take $\varpi_n = 1/(n+1)^2$ and $\vartheta = 0.5$.
- In our Algorithms 3.1 and 3.2 , we adopt $\zeta = 2$, $\chi = 0.5$, $\delta = 0.5$, $\kappa = 1$. Set $\varphi_n = 1.5$ in our Algorithms 3.3 and 3.4 . In SKC Alg. 3, we take $u = x_0$, $\varphi_n = 3$ and $\iota_n = 1/n^3$.

Example 5.1 Assume that $A, A_1, A_2 : \mathbb{R}^m \to \mathbb{R}^m$ are created from a normal distribution with mean zero and unit variance. Let $F_1 : \mathbb{R}^m \to \mathbb{R}^m$ and $F_2 : \mathbb{R}^m \to \mathbb{R}^m$ be defined by $F_1(x) = A_1^* A_1 x$ and $F_2(y) = A_2^* A_2 y$, respectively. Consider the problem of finding a point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)^\mathsf{T} \in \mathbb{R}^m$ such that $F_1(\bar{x}) = (0, \ldots, 0)^\mathsf{T}$ and $F_2(A\bar{x}) = (0, \ldots, 0)^\mathsf{T}$. It is easy to see that the minimum norm solution of the problem mentioned above is $x^* = (0, \ldots, 0)^\mathsf{T}$.

Table 1 Strongly convergent algorithms and their convergence conditions

Strongly convergent algorithms	Convergence conditions ($L = \|A\|^2$)
BCGR Alg. 4.4 $\begin{cases} g_n = J_{\lambda F_1}[x_n - \gamma A^*(I - J_{\lambda F_2})Ax_n], \\ x_{n+1} = \sigma_n x_0 + (1-\sigma_n) g_n, \end{cases}$	$\lambda > 0,\ \gamma \in (0, 2/L),\ \{\sigma_n\} \subset (0,1),$ $\lim_{n\to\infty} \sigma_n = 0,\ \sum_{n=1}^{\infty} \sigma_n = \infty.$
KR Alg. (3.1) $\begin{cases} g_n = J_{\lambda F_1}[x_n - \gamma A^*(I - J_{\lambda F_2})Ax_n], \\ x_{n+1} = \sigma_n f(x_n) + (1-\sigma_n) g_n, \end{cases}$	$\lambda > 0,\ \gamma \in (0, 1/L),\ \{\sigma_n\} \subset (0,1),\ \lim_{n\to\infty} \sigma_n = 0,$ $\sum_{n=1}^{\infty} \sigma_n = \infty,\ \sum_{n=1}^{\infty} \left|\sigma_n - \sigma_{n-1}\right| < \infty.$
TDC Alg. 3.3 $\begin{cases} u_n = x_n + \vartheta_n(x_n - x_{n-1}), \\ q_n = J_{\lambda F_1}[u_n - \gamma A^*(I - J_{\lambda F_2})Au_n], \\ g_n = u_n - \mu_n c_n, \\ x_{n+1} = \sigma_n f(x_n) + (1-\sigma_n) g_n, \end{cases}$	$\lambda > 0,\ \gamma \in [a,b] \in (0, 1/L),\ \{\vartheta_n\}$ and $\{c_n\}$ are defined in (3.1) and (3.3), resp., Conditions (C3) and (C4) hold.
LTD Alg. (49) $\begin{cases} u_n = x_n + \vartheta_n(x_n - x_{n-1}), \\ g_n = J_{\lambda F_1}[u_n - \gamma A^*(I - J_{\lambda F_2})Au_n], \\ x_{n+1} = \sigma_n f(x_n) + (1-\sigma_n) g_n, \end{cases}$	$\lambda > 0,\ \gamma \in [a,b] \in (0, 1/L),\ \{\vartheta_n\} \in [0, \vartheta],$ $\{\sigma_n\} \subset (0,1),\ \lim_{n\to\infty} \sigma_n = 0,\ \sum_{n=1}^{\infty} \sigma_n = \infty,$ $\lim_{n\to\infty} \vartheta_n/\sigma_n \cdot \left\|x_n - x_{n-1}\right\| = 0,$ (C4) holds.
ATD Alg. 4 $\begin{cases} u_n = x_n + \vartheta_n(x_n - x_{n-1}), \\ g_n = J_{\lambda F_1}[u_n - \gamma A^*(I - J_{\lambda F_2})Au_n], \\ x_{n+1} = (1 - \sigma_n - \tau_n) x_n + \tau_n g_n, \end{cases}$	$\lambda > 0,\ \gamma \in [c,d] \in (0, 1/L),\ \{\vartheta_n\} \in [0, \vartheta],$ $\{\sigma_n\} \subset (0,1),\ \lim_{n\to\infty} \sigma_n = 0,\ \sum_{n=1}^{\infty} \sigma_n = \infty,$ $\lim_{n\to\infty} \vartheta_n/\sigma_n \cdot \left\|x_n - x_{n-1}\right\| = 0,$ (C5) holds.
SKC Alg. 3 $\begin{cases} g_n = J_{\lambda F_1}[x_n - \gamma_n A^*(I - J_{\lambda F_2})Ax_n], \\ x_{n+1} = \sigma_n u + (1-\sigma_n) g_n, \end{cases}$	$u \in \mathscr{H}_1,\ \lambda > 0,\ \gamma \in (0, 2/L),\ \{\sigma_n\} \subset (0,1),$ $\lim_{n\to\infty} \sigma_n = 0,\ \sum_{n=1}^{\infty} \sigma_n = \infty,\ \inf \varphi_n (4 - \varphi_n) > 0,$ $\gamma_n = \frac{0.5\varphi_n \|(I - J_{\lambda F_2})Ax_n\|^2}{\|A^*(I - J_{\lambda F_2})Ax_n\|^2 + \iota_n},\ \lim_{n\to\infty} \iota_n = 0.$

Table 2 The number of termination iterations and execution time of all algorithms with different stopping criteria ($m = 100$) in Example 5.1

Algotithms	$\epsilon_n = 10^{-4}$		$\epsilon_n = 10^{-5}$		$\epsilon_n = 10^{-6}$		$\epsilon_n = 10^{-7}$	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
Our Alg. 3.1	23	0.0345	32	0.0365	36	0.0336	39	0.0407
Our Alg. 3.2	22	0.0342	31	0.0353	39	0.0363	49	0.0543
Our Alg. 3.3	17	0.0185	19	0.0172	25	0.0170	29	0.0218
Our Alg. 3.4	11	0.0147	15	0.0135	18	0.0125	21	0.0142
BCGR Alg. 4.4	299	0.4177	299	0.4251	299	0.4180	299	0.4335
KR Alg. (3.1)	80	0.1163	103	0.1449	126	0.1724	149	0.2090
TDC Alg. 3.3	39	0.0673	47	0.0718	55	0.0747	63	0.0879
LTD Alg. (49)	41	0.0659	49	0.0734	57	0.0764	66	0.0909
ATD Alg. 4	71	0.1075	100	0.1471	131	0.1830	164	0.2346
SKC Alg. 3	299	0.2086	299	0.2015	299	0.2060	299	0.2112

We use $E_n = \|x_n - x^*\|$ to measure the iteration error of all the algorithms. The stopping condition is either $E_n < \epsilon$, or maximum number of iterations which is set to 299. First, choosing $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$. We test the convergence behavior of all algorithms under different stopping conditions. The numerical results are shown in Table 2 and Fig. 1. Second, Table 3 and Fig. 2 describe the numerical behavior of all algorithms in different dimensions with the same stopping criterion $\epsilon = 10^{-7}$.

Remark 5.1 From the numerical results of Example 5.1, we have the following observations:

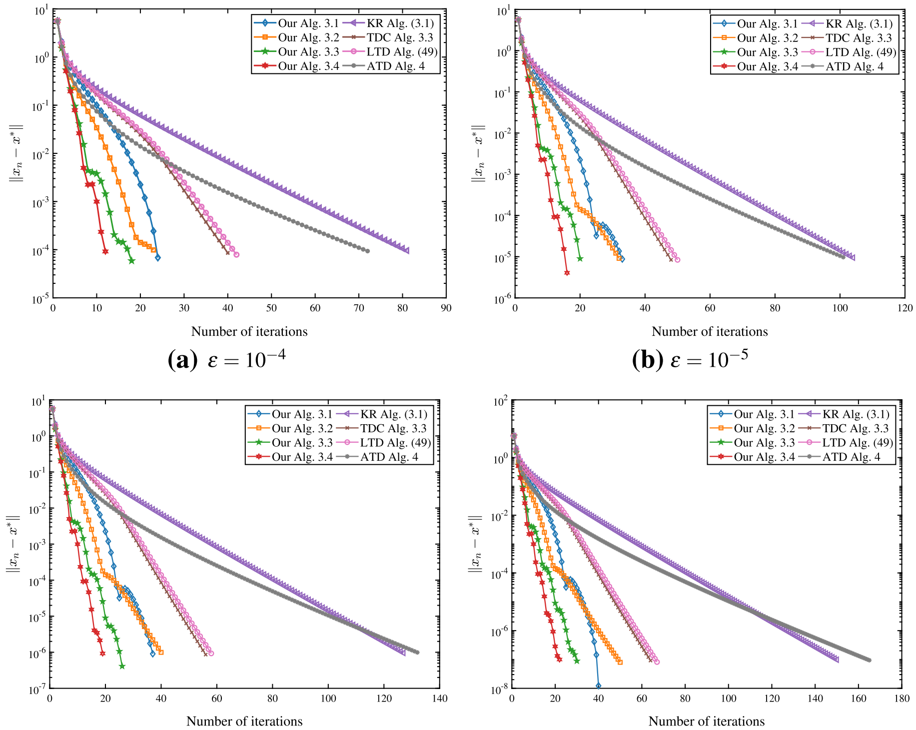
(1) The four iterative schemes proposed in this paper are efficient and easy to implement. The most important thing is that they converge quickly.
(2) Our offered methods converge faster than some known algorithms in the literature in terms of the number of iterations and execution time, and these observations have no significant relationship with the dimensions of the problem and the selection of initial values (cf. Table 2, Table 3, Figs. 1, 2).

Example 5.2 Assume that $\mathscr{H}_1$ and $\mathscr{H}_2$ are real Hilbert spaces, and $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator with its adjoint A^*. Let C and Q be nonempty closed and convex subsets of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. We consider the split feasibility problem (SFP) in infinite-dimensional Hilbert spaces, which reads as

$$\text{find } x^* \in C \quad \text{such that} \quad Ax^* \in Q\,.$$

For any $x, y \in L^2([0, 1])$, we consider $\mathscr{H}_1 = \mathscr{H}_2 = L^2([0, 1])$ embedded with the inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)\,\mathrm{d}t$ and the induced norm $\|x\| := \left(\int_0^1 |x(t)|^2\,\mathrm{d}t\right)^{1/2}$. Consider the following nonempty closed and convex subsets C and Q in $L^2([0, 1])$:

$$C = \left\{x \in L_2([0, 1]) \mid \int_0^1 x(t)\,\mathrm{d}t \leq 1\right\},$$
$$Q = \left\{x \in L_2([0, 1]) \mid \int_0^1 |x(t) - \sin(t)|^2\,\mathrm{d}t \leq 16\right\}.$$



(a) $\varepsilon = 10^{-4}$ **(b)** $\varepsilon = 10^{-5}$

(c) $\varepsilon = 10^{-6}$ **(d)** $\varepsilon = 10^{-7}$

Fig. 1 Numerical behavior of all algorithms with different stopping criteria in Example 5.1

Let $A : L^2([0,1]) \to L^2([0,1])$ be the Volterra integration operator, which is given by $(Ax)(t) = \int_0^t x(s)\,\mathrm{d}s$, $\forall t \in [0,1]$, $x \in \mathscr{H}$. Then A is a bounded linear operator (see [39, Exercise 20.16]) and its operator norm is $\|A\| = \frac{2}{\pi}$. Moreover, the adjoint A^* of A is defined by $(A^*x)(t) = \int_t^1 x(s)\,\mathrm{d}s$. Note that $x(t) = 0$ is a solution of (SFP) and thus the solution set of the problem is nonempty. On the other hand, it is known that projections on sets C and Q have display formulas, that is,

$$P_C(x) = \begin{cases} 1 - a + x\,, & a > 1\,; \\ x\,, & a \le 1\,. \end{cases} \quad \text{and } P_Q(x) = \begin{cases} \sin(\cdot) + \frac{4(x - \sin(\cdot))}{\sqrt{b}}\,, & b > 16\,; \\ x\,, & b \le 16\,, \end{cases}$$

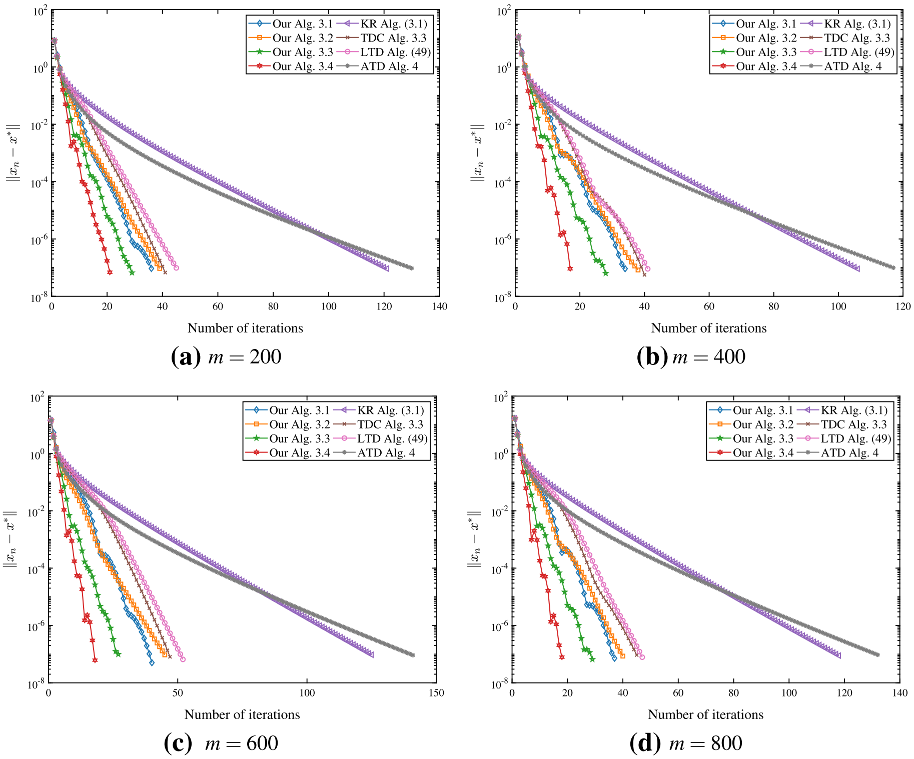
where $a := \int_0^1 x(t)\,\mathrm{d}t$ and $b := \int_0^1 |x(t) - \sin(t)|^2\,\mathrm{d}t$.

We use symbolic computation in MATLAB to implement these algorithms for generating the sequences of iterates and use $E_n = \|(I - P_C)x_n\|^2 + \left\|A^*(I - P_Q)Ax_n\right\|^2 < 10^{-5}$ for stopping criterion. We do not report the numerical results of BCGR Alg. 4.4 and SKC Alg. 3 here because they converge slowly. Table 4 and Fig. 3 show the numerical behavior of all algorithms (except BCGR Alg. 4.4 and SKC Alg. 3) with four different initial values $x_0 = x_1$.

Remark 5.2 It can be seen from Table 4 and Fig. 3 that the proposed approaches are easy to implement and efficient. In addition, our suggested methods (especially Alg. 3.3 and Alg. 3.4) require fewer iterations than some algorithms in the literature to achieve the same

Table 3 The number of termination iterations and execution time of all algorithms with different dimensions ($\epsilon_n = 10^{-7}$) in Example 5.1

Algotithms	$m = 200$		$m = 400$		$m = 600$		$m = 800$	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
Our Alg. 3.1	35	0.1523	33	0.5510	39	1.1640	36	2.4639
Our Alg. 3.2	38	0.1603	37	0.6209	44	1.3438	39	2.7218
Our Alg. 3.3	28	0.0881	27	0.3723	26	0.6524	28	1.2189
Our Alg. 3.4	20	0.0689	16	0.2252	17	0.4076	17	0.7902
BCGR Alg. 4.4	299	1.8065	299	8.7384	299	17.9115	299	37.5884
KR Alg. (3.1)	120	0.7502	105	3.1241	124	7.4629	117	15.2309
TDC Alg. 3.3	40	0.2644	39	1.1668	46	2.7830	44	5.7761
LTD Alg. (49)	44	0.2896	40	1.1718	51	3.3261	46	6.0827
ATD Alg. 4	129	0.8595	116	3.4534	140	8.7756	131	17.7027
SKC Alg. 3	299	0.9027	299	4.2046	299	6.6302	299	13.7081

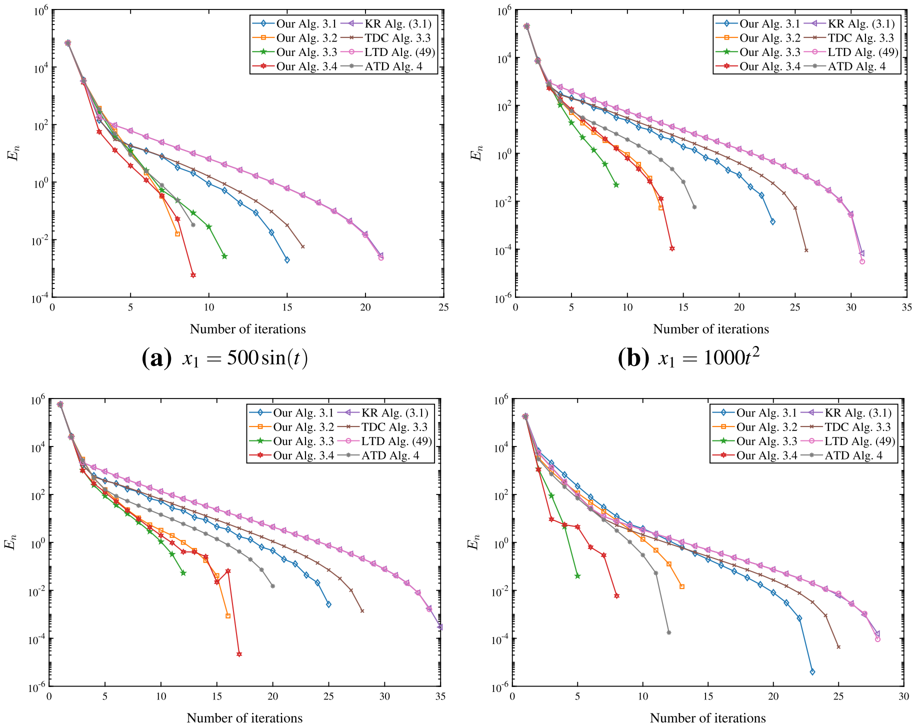


(a) $m = 200$ **(b)** $m = 400$ **(c)** $m = 600$ **(d)** $m = 800$

Fig. 2 Numerical behavior of all algorithms with different dimensions in Example 5.1

Table 4 The number of termination iterations and execution time of all algorithms with different initial values ($x_0 = x_1$) in Example 5.2

Algotithms	$x_1 = 500\sin(t)$		$x_1 = 1000t^2$		$x_1 = 500(t^3 + 2t)$		$x_1 = 300\log(t)$	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
Our Alg. 3.1	15	37.529	23	29.429	25	69.7555	22	144.176
Our Alg. 3.2	8	14.119	13	15.578	16	42.096	13	75.2286
Our Alg. 3.3	11	7.4645	9	4.0796	12	10.9242	5	3.476
Our Alg. 3.4	9	7.3766	14	7.4795	17	16.4684	8	6.5769
KR Alg. (3.1)	21	4.5369	31	5.3176	35	6.5202	28	5.7873
TDC Alg. 3.3	16	26.817	26	24.806	28	51.2577	25	160.579
LTD Alg. (49)	21	18.055	31	15.019	34	31.9965	28	56.5573
ATD Alg. 4	9	6.7382	16	7.4592	20	16.6949	12	12.249



(a) $x_1 = 500\sin(t)$ **(b)** $x_1 = 1000t^2$ **(c)** $x_1 = 500(t^3 + 2t)$ **(d)** $x_1 = 300\log(t)$

Fig. 3 Numerical behavior of all algorithms with different initial values in Example 5.2

error accuracy, and these results are independent of the selection of initial values. It is worth noting that our Algorithms 3.1 and 3.2 enjoy fewer iterations while accompanied by more execution time (because the Armijo-type line search criterion (3.2) takes more time to find a suitable stepsize). Moreover, it should be pointed out that the operator norm $\|A\|$ of this

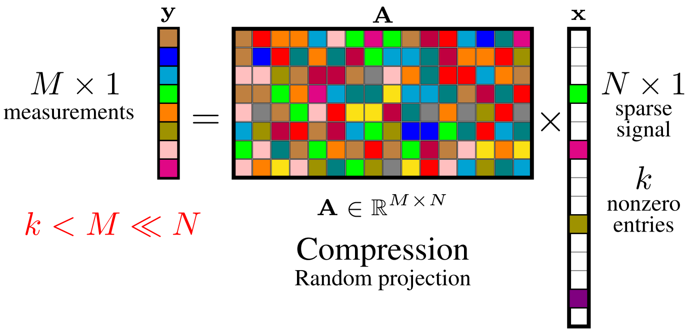


Fig. 4 Structure of compressive sensing matrices

Table 5 The numerical results of all algorithms for solving (LASSO) in case $M = 256$, $N = 512$ and $k = 10$

Measurement result	Our algotithms			
	Our Alg. 3.1	Our Alg. 3.2	Our Alg. 3.3	Our Alg. 3.4
MSE ($\times 10^{-4}$)	0.1786	0.2084	0.1838	0.1730
Time (s)	0.4887	0.5200	0.1202	0.1184
Measurement result	Comparison algotithms			
	KR Alg. (3.1)	TDC Alg. 3.3	LTD Alg. (49)	ATD Alg. 4
MSE ($\times 10^{-4}$)	0.1801	0.1786	0.1801	0.2153
Time (s)	1.6703	0.1681	1.6658	1.6427

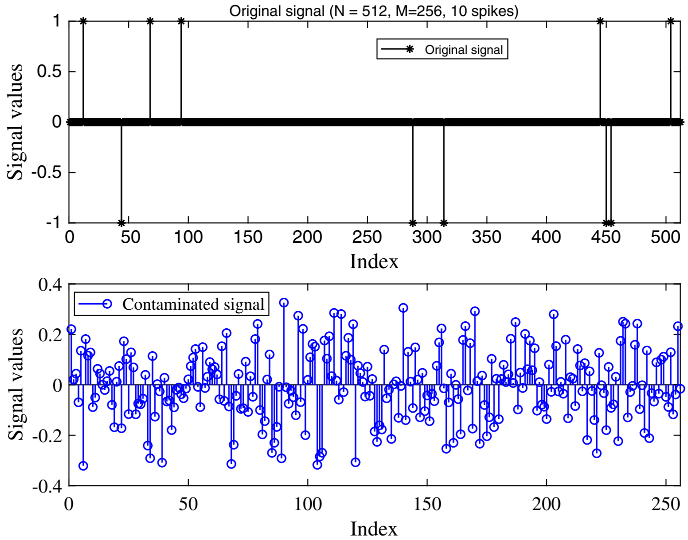
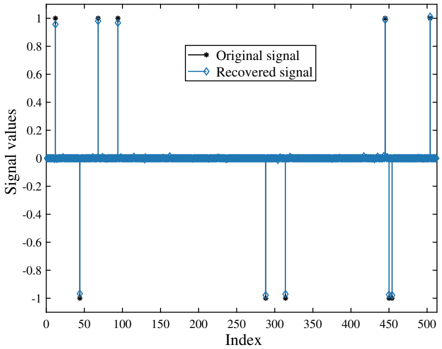
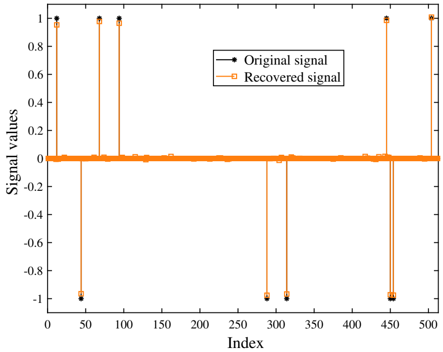


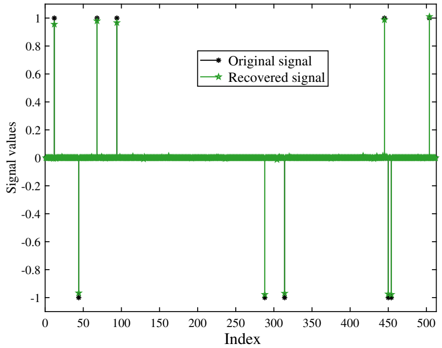
Fig. 5 Original signal and contaminated signal



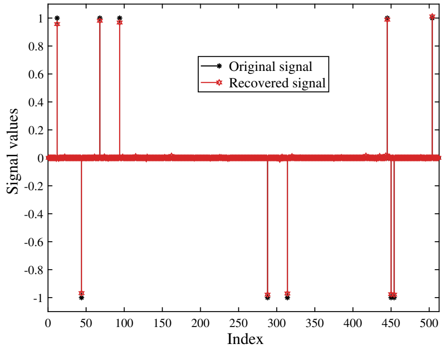
(a) Recovered by our Alg. 3.1



(b) Recovered by our Alg. 3.2



(c) Recovered by our Alg. 3.3



(d) Recovered by our Alg. 3.4

Fig. 6 The original signal and the signal recovered by our algorithms

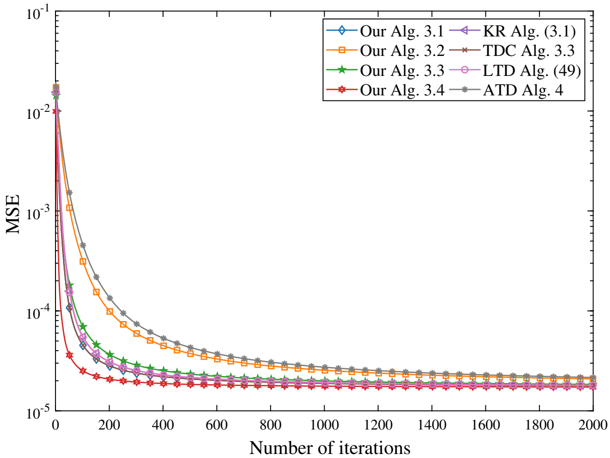
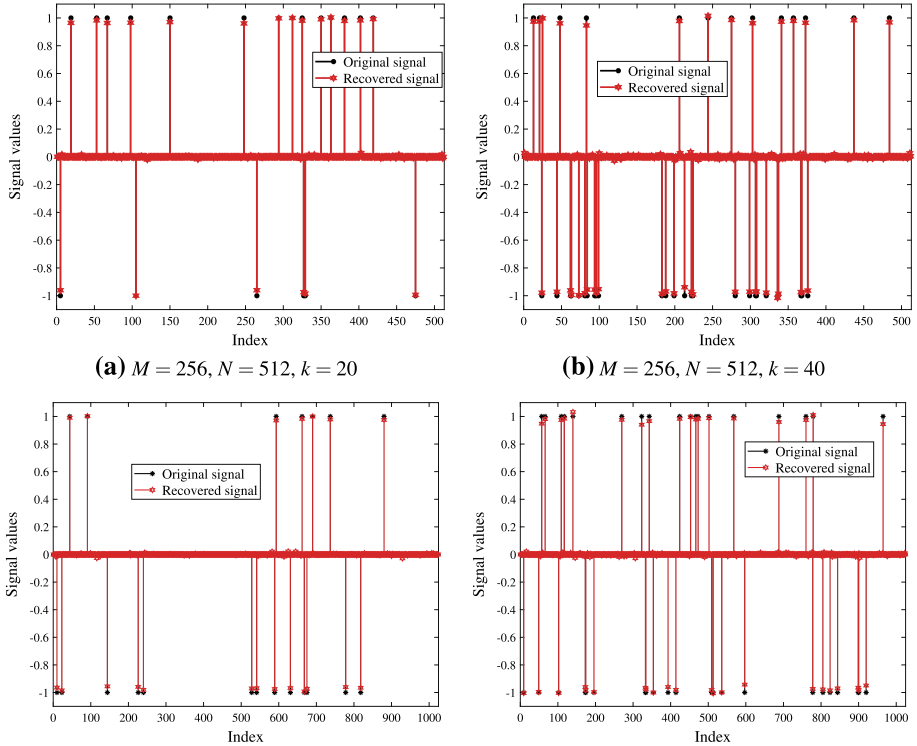


Fig. 7 The discrepancy of mean squared error (MSE) of all algorithms

Table 6 The numerical results of all algorithms for solving (LASSO) with different situations

Algotithms	$M=256,\ N=512\,k=20$		$M=256,\ N=512\,k=40$		$M=512,\ N=1024\,k=20$		$M=512,\ N=1024\,k=40$	
	MSE ($\times 10^{-4}$)	Time (s)	MSE ($\times 10^{-3}$)	Time (s)	MSE ($\times 10^{-4}$)	Time (s)	MSE ($\times 10^{-4}$)	Time (s)
Our Alg. 3.1	0.3317	0.4874	0.1003	0.4693	0.2082	1.4892	0.4342	1.5198
Our Alg. 3.2	0.384	0.4876	0.1248	0.4879	0.2411	1.4459	0.5001	1.5098
Our Alg. 3.3	0.3417	0.1225	0.1064	0.1485	0.214	0.3615	0.4481	0.3817
Our Alg. 3.4	0.3221	0.1157	0.0966	0.1156	0.2021	0.3207	0.4224	0.368
KR Alg. (3.1)	0.3355	1.5696	0.1029	1.5717	0.2098	8.2411	0.4394	7.7349
TDC Alg. 3.3	0.3317	0.184	0.1003	0.1795	0.2082	0.8648	0.4342	0.597
LTD Alg. (49)	0.3355	1.5704	0.1029	1.5271	0.2098	8.0863	0.4393	7.6975
ATD Alg. 4	0.402	1.5776	0.1416	1.5861	0.249	7.6964	0.5242	7.7073



(a) $M = 256, N = 512, k = 20$ **(b)** $M = 256, N = 512, k = 40$

(c) $M = 512, N = 1024, k = 20$ **(d)** $M = 512, N = 1024, k = 40$

Fig. 8 The original signal and the signal recovered by our Algorithm 3.4

problem is not easy to obtain, which means that the algorithms of fixed step size will fail. However, the self-adaptive algorithms presented in this paper can work well.

Example 5.3 Compressed sensing is an effective method to recover a clean signal from a polluted signal. This requires us to solve the following underdetermined system problems:

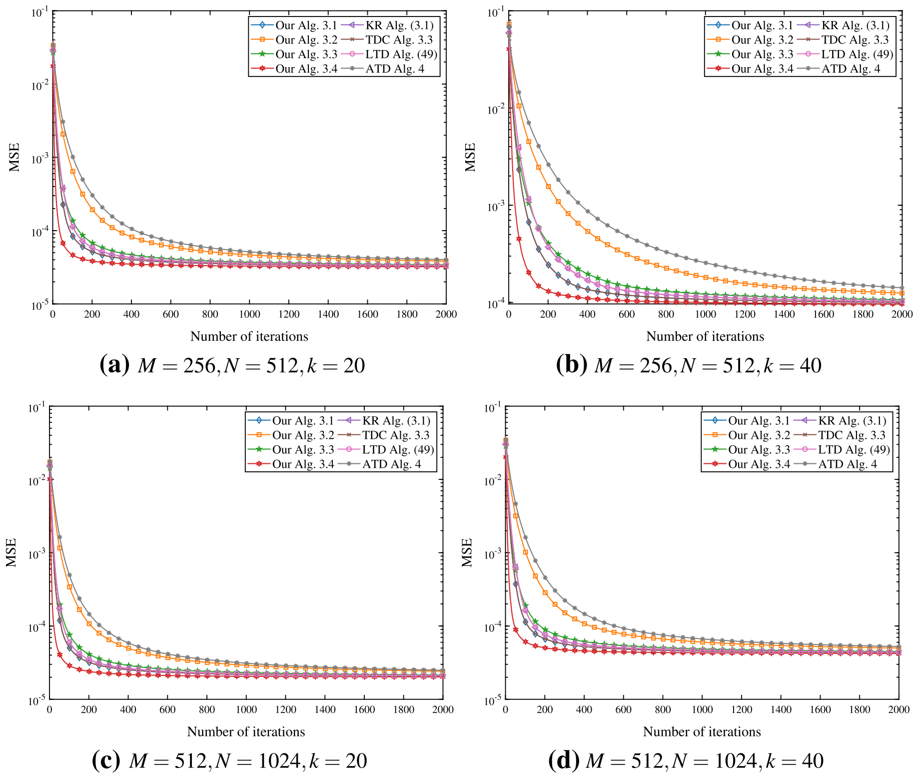
$$\mathbf{y} = \mathbf{A}\mathbf{x} + \epsilon\,,$$

where $\mathbf{y} \in \mathbb{R}^M$ is the observed noise data, $\mathbf{A} : \mathbb{R}^{M \times N}$ is a bounded linear observation operator, $\mathbf{x} \in \mathbb{R}^N$ with k $(k \ll N)$ non-zero elements is the original and clean data that needs to be restored, and ϵ is the noise observation encountered during data transmission. An important consideration of this problem is that the signal $\mathbf{x}$ is sparse, that is, the number of non-zero elements in the signal $\mathbf{x}$ is much smaller than the dimension of the signal $\mathbf{x}$. Figure 4 visually shows the matrix structure expression of compressed sensing.

A successful model used to solve the above problem can be translated into the following convex constraint minimization problem:

$$\min_{\mathbf{x} \in \mathbb{R}^N} \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2 \quad \text{subject to} \quad \|\mathbf{x}\|_1 \leq t\,, \tag{LASSO}$$

where t is a positive constant. It should be pointed out that this problem is related to the least absolute shrinkage and selection operator (LASSO) problem. Note that the (LASSO) problem



(a) $M = 256, N = 512, k = 20$ **(b)** $M = 256, N = 512, k = 40$

(c) $M = 512, N = 1024, k = 20$ **(d)** $M = 512, N = 1024, k = 40$

Fig. 9 The discrepancy of mean squared error (MSE) of all algorithms

described above can be regarded as a special case of (SFP) when $C = \left\{\mathbf{x} \in \mathbb{R}^N : \|\mathbf{x}\|_1 \leq t\right\}$ and $Q = \{\mathbf{y}\}$. In this situation, we can use the projection formulas described in Sect. 2 to calculate P_C and P_Q.

We now consider using the proposed iterative schemes to solve (LASSO) and compare them with some known algorithms in the literature. In our numerical experiments, the matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$ is created from a standard normal distribution with zero mean and unit variance and then orthonormalizing the rows. The clean signal $\mathbf{x} \in \mathbb{R}^N$ contains k ($k \ll N$) randomly generated ± 1 spikes. The observation $\mathbf{y}$ is formed by $\mathbf{y} = \mathbf{A}\mathbf{x} + \epsilon$ with white Gaussian noise ϵ of variance 10^{-4}. The recovery process starts with the initial signals $\mathbf{x}_0 = \mathbf{x}_1 = \mathbf{0}$ and ends after 2000 iterations. We use the mean squared error $\text{MSE} = (1/N)\,\|\mathbf{x}^* - \mathbf{x}\|^2$ ($\mathbf{x}^*$ is an estimated signal of $\mathbf{x}$) to measure the restoration accuracy of all algorithms. In our first test, we set $M = 256$, $N = 512$ and $k = 10$. The numerical results are shown in Table 5 and Figs. 5, 6 and 7. Figure 5 displays the original signal and the contaminated signal. The recovery results of the suggested algorithms are shown in Fig. 6. Table 5 presents the numerical results of all algorithms, including the mean squared error (MSE) of the restored signal and the original signal, and the execution time required for the iterative process. Figure 7 gives the numerical behavior of the MSE of all algorithms in the iteration process.

Next, in order to demonstrate the robustness of the proposed algorithms, we conduct signal recovery tests with different dimensions and different sparsity. The numerical results are reported in Table 6, Figs. 8 and 9.

Remark 5.3 As can be seen from the numerical results of Example 5.3, the proposed algorithms can be applied to signal processing problems in compressed sensing, and they can work well (see Figs. 6, 8). Under the same number of iterations, the presented algorithms have smaller mean squared error and cpu time than the compared algorithms (cf. Tables 5, 6), which implies that our proposed algorithms perform better and converge faster in the signal recovery tests (cf. Figs. 7, 9). Furthermore, as shown in the previous two examples, our algorithms are still robust in this example, because the dimension and sparsity of the signal have no significant influence on our results.

6 The Conclusion

In this paper, we presented four inertial algorithms for finding the solution to the split variational inclusion problem in real Hilbert spaces. Our approaches can adaptively update the iteration step size without knowing the prior information of the operator norm. Under some suitable conditions, we established the strong convergence theorems of the suggested algorithms. The applications of our results in split feasibility problems and split minimization problems were given. Finally, we demonstrated the computational efficiency of the offered algorithms compared with other ones through numerical experiments in finite- and infinite-dimensional spaces as well as signal recovery problems. The algorithms obtained in this paper improved and extended some known results in the literature.

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Compliance with ethical standards

Conflict of interest The authors declare that there have no conflict of interest.

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