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ORIGINAL RESEARCH

Viscosity-type inertial extragradient algorithms for solving variational inequality problems and fixed point problems

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Abstract

The paper presents two inertial viscosity-type extragradient algorithms for finding a common solution of the variational inequality problem involving a monotone and Lipschitz continuous operator and of the fixed point problem with a demicontractive mapping in real Hilbert spaces. Our algorithms use a simple step size rule which is generated by some calculations at each iteration. Two strong convergence theorems are obtained without the prior knowledge of the Lipschitz constant of the operator. The numerical behaviors of the proposed algorithms in some numerical experiments are reported and compared with previously known ones.

Keywords Variational inequality · Fixed point problem · Subgradient extragradient method · Tseng's extragradient method · Inertial method

Mathematics Subject Classification. 47H05 · 47H09 · 49J15 · 47J20 · 65K15

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space $\mathscr{H}$ with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. The main purpose of this paper is for finding a common solution of variational inequality problems and fixed point problems in real Hilbert spaces. The motivation for studying such problems is that it is possible to apply them to mathematical models whose constraints can be expressed as

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variational inequalities and/or fixed point problems. This situation occurs especially in practical problems, such as signal processing, composite minimization problems, optimal control problems and image restoration; see, e.g., [1–5]. Let us recall the involved problems. The variational inequality problem (in short, VI) for operator $A : \mathscr{H} \to \mathscr{H}$ on C is to find a point $p \in C$ such that

$$\langle Ap, x - p\rangle \geq 0, \quad \forall x \in C . \tag{VI}$$

Let $\mathrm{VI}(C, A)$ be the solution set of (VI). Variational inequalities theory arises in various models for a large number of mathematical, engineering, physical and other problems. In recent years, considerable interest has been shown in developing efficient and implementable numerical methods of variational inequalities, see, e.g., [6–11] and the references therein. There have been two general methods to study the monotone variational inequality problem under mild conditions: regularized methods and projection-type methods. In this paper, we focus on projection-based methods. The simplest projection method to solve (VI) is the projected gradient method:

$$x_{n+1} = P_C\left(x_n - \tau_n A x_n\right) .$$

Note that only one projection onto the feasible set is performed. However, the convergence of the projected gradient method requires a slightly strong hypothesis that the operator is strongly monotone or inverse strongly monotone. To avoid this strong hypothesis, Korpelevich [12] introduced the extragradient method to solve saddle point problems in Euclidean spaces. Indeed, the extragradient method is of the form:

$$\begin{cases} y_n = P_C\left(x_n - \tau A x_n\right) , \\ x_{n+1} = P_C\left(x_n - \tau A y_n\right) , \end{cases} \tag{1.1}$$

where operator A is monotone and L-Lipschitz continuous, P_C is denoted by the metric projection from $\mathscr{H}$ onto C and $\tau \in (0, 1/L)$. It is know that the sequence $\{x_n\}$ generated by the process (1.1) converges to an element in $\mathrm{VI}(C, A)$ when the solution set $\mathrm{VI}(C, A)$ is nonempty.

Note that calculating the projection onto a closed convex set C is equivalent to finding the solution of the minimum distance problem. We point out that the extragradient method requires to calculate two projections onto C in each iteration. This may require a prohibitive amount of computation time when C is a general closed and convex set. To overcome this shortcoming, Censor, Gibali and Reich [13] obtained the subgradient extragradient method by modifying the extragradient algorithm. The purpose of this modification is to replace the second projection on C with a projection onto a half-space. It is worth noting that the projection onto a half-space can be calculated explicitly. Indeed, they obtained the following algorithm:

$$\begin{cases} y_n = P_C\left(x_n - \tau A x_n\right) , \\ T_n = \{x \in \mathscr{H} \mid \langle x_n - \tau A x_n - y_n, x - y_n\rangle \leq 0\} , \\ x_{n+1} = P_{T_n}\left(x_n - \tau A y_n\right) , \end{cases} \tag{1.2}$$

where $\tau \in (0, 1/L)$ and the operator A is monotone and L-Lipschitz continuous. If the solution set $\mathrm{VI}(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.2) converges weakly to a solution of (VI). The second method is the Tseng's extragradient method proposed by Tseng in [14]:

$$\begin{cases} y_n = P_C\left(x_n - \tau A x_n\right) , \\ x_{n+1} = y_n - \tau\left(A y_n - A x_n\right) , \end{cases} \tag{1.3}$$

where $\tau \in (0, 1/L)$ and the operator A is monotone and L-Lipschitz continuous. The subgradient extragradient method and the Tseng's extragradient method only require to calculate one projection onto the feasible set in each iteration.

Let $U : C \to C$ be a nonlinear mapping. A point $x \in \mathscr{H}$ is called a fixed point of mapping U if $Ux = x$. The set of fixed points of U is denoted by $\mathrm{Fix}(U)$. The fixed point problem (shortly, FPP) is defined as follows:

$$\text{find} \quad p \in C \quad \text{such that} \quad Up = p\,. \tag{FPP}$$

Our focus in this paper is to find a common solution of (VI) and (FPP). That is, we find a point p such that

$$p \in \mathrm{Fix}(U) \cap \mathrm{VI}(C, A)\,. \tag{VIFPP}$$

There are many numerical algorithms have been proposed for solving (VIFPP) in infinite-dimensional spaces, see, e.g., [15–18] and the references therein. Takahashi and Toyoda [19] proposed an iterative method for finding a solution of (VIFPP) as follows:

$$x_{n+1} = \left(1 - \alpha_n\right) x_n + \alpha_n U P_C\left(x_n - \tau_n A x_n\right) , \tag{1.4}$$

where mapping $A : C \to \mathscr{H}$ is λ-inverse strongly monotone and mapping $U : C \to C$ is nonexpansive. They proved that the sequence $\{x_n\}$ generated by (1.4) converges weakly to a solution of (VIFPP) under some conditions. Recently, under the assumptions that mapping A is Lipschitz continuous monotone and mapping U is nonexpansive, Censor, Gibali and Reich [20] proved that the iterative scheme generated by their algorithm (1.5) converges weakly to a solution of (VIFPP). Their algorithm is described as follows:

$$\begin{cases} y_n = P_C\left(x_n - \tau A x_n\right) , \\ T_n = \{x \in \mathscr{H} \mid \langle x_n - \tau A x_n - y_n, x - y_n\rangle \leq 0\} , \\ x_{n+1} = \alpha_n x_n + \left(1 - \alpha_n\right) U P_{T_n}\left(x_n - \tau A y_n\right) . \end{cases} \tag{1.5}$$

We note that norm convergence is generally desirable than weak convergence in infinite-dimensional spaces. Therefore, a natural question is how to design an algorithm that provides strong convergence to solve the (VIFPP) in the sense of infinite-dimensional Hilbert spaces, when mapping A is only monotone and Lipschitz continuous. Kraikaew and Saejung [21] combined the subgradient extragradient method and the Halpern method, and proposed an algorithm which is called the Halpern subgradient extragradient method (shortly, HSEGM) for solving (VIFPP). Their algorithm is of the form:

$$\begin{cases} y_n = P_C\left(x_n - \tau A x_n\right), \\ T_n = \{x \in \mathscr{H} \mid \langle x_n - \tau A x_n - y_n, x - y_n\rangle \le 0\}, \\ z_n = \alpha_n x_0 + (1-\alpha_n) P_{T_n}\left(x_n - \tau A y_n\right), \\ x_{n+1} = \beta_n x_n + (1-\beta_n) U z_n, \end{cases} \tag{HSEGM}$$

where $\tau \in (0, 1/L)$, $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $\{\beta_n\} \subset [a,b] \subset (0,1)$, A is a monotone and L-Lipschitz continuous mapping and $U : \mathscr{H} \to \mathscr{H}$ is a quasi-nonexpansive mapping. They proved that the sequence $\{x_n\}$ generated by the iterative scheme (HSEGM) converges strongly to $P_{\mathrm{VI}(C,A)\cap \mathrm{Fix}(U)}(x_0)$.

In this paper, we focus on U is a demicontractive mapping, which covers the quasi-nonexpansive mappings. Recently, Thong and Hieu [22] combined the subgradient extragradient method and the Mann-like method, and proposed a modified subgradient extragradient algorithm to find common solution elements of the variational inequality problem solution set and the fixed point set of a demicontractive mapping. Indeed, their algorithm is as follows:

$$\begin{cases} y_n = P_C\left(x_n - \tau A x_n\right), \\ T_n = \{x \in \mathscr{H} \mid \langle x_n - \tau A x_n - y_n, x - y_n\rangle \le 0\}, \\ z_n = P_{T_n}\left(x_n - \tau A y_n\right), \\ x_{n+1} = (1-\alpha_n-\beta_n) z_n + \beta_n U z_n, \end{cases} \tag{MSEGM}$$

where $\tau \in (0, 1/L)$, $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$, $\beta_n \in (a,b) \subset (0, (1-\lambda)(1-\alpha_n))$ for some $a > 0, b > 0$, $A : \mathscr{H} \to \mathscr{H}$ is a monotone and L-Lipschitz continuous mapping and $U : \mathscr{H} \to \mathscr{H}$ is a λ-demicontractive mapping such that $(I-U)$ is demiclosed at zero. Assume that $\mathrm{VI}(C,A) \cap \mathrm{Fix}(U) \ne \emptyset$, they proved that the iterative sequence $\{x_n\}$ generated by (MSEGM) converges strongly to an element $p \in \mathrm{VI}(C,A) \cap \mathrm{Fix}(U)$, where $\|p\| = \min\{\|z\| : z \in \mathrm{VI}(C,A) \cap \mathrm{Fix}(U)\}$.

Note that the algorithms (HSEGM) and (MSEGM) need to know the prior knowledge of the Lipschitz constant of the mapping A. However, in many cases, we cannot obtain the prior knowledge of the operator A in advance. Recently, Thong and Hieu [23] introduced two extragradient-viscosity algorithms for solving (VIFPP). They used a simple rule to automatically update the step size and thus the Lipschitz constant of the mapping is not required. Indeed, their algorithms are described as follows:

$$\begin{cases} y_n = P_C\left(x_n - \tau_n A x_n\right), \\ T_n = \{x \in \mathscr{H} \mid \langle x_n - \tau_n A x_n - y_n, x - y_n\rangle \le 0\}, \\ z_n = P_{T_n}\left(x_n - \tau_n A y_n\right), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\left[(1-\beta_n) z_n + \beta_n U z_n\right], \end{cases} \tag{VSEGM}$$

and

$$\begin{cases} y_n = P_C\left(x_n - \tau_n A x_n\right), \\ z_n = y_n - \tau_n\left(A y_n - A x_n\right), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\left[(1-\beta_n) z_n + \beta_n U z_n\right], \end{cases} \tag{VTEGM}$$

where Algorithms (VSEGM) and (VTEGM) update the step size $\{\tau_n\}$ by the following rule:

$$\tau_{n+1} = \begin{cases} \min\left\{\frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \tau_n\right\}, & \text{if } Ax_n - Ay_n \neq 0; \\ \tau_n, & \text{otherwise}, \end{cases}$$

where $\tau_0 > 0, \mu \in (0,1), \alpha_n \subset (0,1), \beta_n \subset (a, 1-\lambda) \subset (0,1)$, A is a monotone and Lipschitz continuous mapping, U is a λ-demicontractive mapping and f is a contraction mapping. Under some mild assumptions, the sequences generated by (VSEGM) and (VTEGM) converge strongly to $q \in \operatorname{Fix}(U) \cap \operatorname{VI}(C, A)$, where $q = P_{\operatorname{Fix}(U)\cap\operatorname{VI}(C,A)}(f(q))$.

In recent years, the development of fast iterative algorithms has attracted enormous interest, especially for inertial technology, which is based on discrete versions of a second-order dissipative dynamic system (see [24,25] for more details). Many researchers have constructed various fast iterative algorithms by using inertial technology, see, e.g., [26–31] and the references therein. One of the common features of these algorithms is that the next iteration depends on the combination of the previous two iterations. Note that this minor change greatly improves the performance of the algorithm.

Motivated and inspired by the above work, in this paper, we introduce two inertial extragradient algorithms for finding a common element of the solution set of the monotone variational inequality problem and the fixed point set of a demicontractive mapping in real Hilbert spaces. We provide a choice of step size rule which allows the algorithms to work without the previously known information of the Lipschitz constant of the mapping. Under suitable conditions, we prove that the sequences generated by the suggested algorithms converge strongly to a solution of variational inequality problems and fixed point problems. Some numerical experiments are presented to support the theoretical results. Our numerical results show that the new algorithms have a better convergence speed than the existing ones [21–23].

The remainder of this paper is organized as follows. In Section 2, we recall some preliminary results and lemmas for further use. Section 3 analyzes the convergence of the proposed algorithms. In Section 4, some numerical examples are provided to illustrate the numerical behavior of the proposed algorithms and compare them with other ones. Finally, we give a brief summary of the paper in Section 5, the last section.

2 Preliminaries

Let C be a nonempty closed and convex subset of a real Hilbert space $\mathscr{H}$. The weak convergence and strong convergence of $\{x_n\}$ to x are represented by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. For each $x, y \in \mathscr{H}$ and $\alpha \in \mathbb{R}$, we have the following facts:

1. $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;
2. $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;
3. $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$.

For every point $x \in \mathscr{H}$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that $P_C(x) := \operatorname{argmin}\{\|x-y\|,\ y \in C\}$. P_C is called the metric projection of $\mathscr{H}$ onto C. It is known that P_C is nonexpansive and P_C has the following basic properties:

- $\langle x - P_C(x), y - P_C(x)\rangle \leq 0,\ \forall y \in C$;
- $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle,\ \forall y \in \mathscr{H}$.

Definition 2.1 ([32]) Assume that $T : \mathscr{H} \to \mathscr{H}$ is a nonlinear operator with $\operatorname{Fix}(T) \neq \emptyset$. Then, $I - T$ is said to be demiclosed at zero if for any $\{x_n\}$ in $\mathscr{H}$, the following implication holds:

$$x_n \rightharpoonup x \text{ and } (I-T)x_n \to 0 \Longrightarrow x \in \operatorname{Fix}(T)\,.$$

Definition 2.2 Recall that a mapping $T : \mathscr{H} \to \mathscr{H}$ is said to be:

- L-Lipschitz continuous with $L > 0$ if

$$\|Tx - Ty\| \leq L\|x-y\|, \quad \forall x, y \in \mathscr{H}\,.$$

- η-strongly monotone if there exists $\eta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \eta\|x-y\|, \quad \forall x, y \in \mathscr{H}\,.$$

- η-inverse strongly monotone if there exists $\eta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \eta\|Tx-Ty\|^2, \quad \forall x, y \in \mathscr{H}\,.$$

- monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in \mathscr{H}\,.$$

- quasi-nonexpansive if

$$\|Tx - z\| \leq \|x - z\|, \quad \forall z \in \operatorname{Fix}(T), x \in \mathscr{H}\,.$$

- λ-strictly pseudocontractive with $0 \leq \lambda < 1$ if

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \lambda\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in \mathscr{H}\,.$$

- β-demicontractive with $0 \leq \beta < 1$ if

$$\|Tx - z\|^2 \leq \|x - z\|^2 + \beta\|(I - T)x\|^2, \quad \forall z \in \mathrm{Fix}(T), x \in \mathscr{H},$$

or equivalently

$$\langle Tx - x, x - z\rangle \leq \frac{\beta - 1}{2}\|x - Tx\|^2, \quad \forall z \in \mathrm{Fix}(T), x \in \mathscr{H}. \tag{2.1}$$

Remark 2.1 From the above definitions, we have the facts that:

- The class of demicontractive mappings includes the class of quasi-nonexpansive mappings.
- Every strictly pseudocontractive mapping with a nonempty fixed point set is demicontractive.

We need the following lemmas to prove the convergence of our algorithms.

Lemma 2.1 *([21]) Let $A : \mathscr{H} \to \mathscr{H}$ be a monotone and L-Lipschitz continuous mapping on C. Let $S = P_C(I - \mu A)$, where $\mu > 0$. If $\{x_n\}$ is a sequence in $\mathscr{H}$ satisfying $x_n \rightharpoonup q$ and $x_n - Sx_n \to 0$, then $q \in \mathrm{VI}(C, A) = \mathrm{Fix}(S)$.*

Lemma 2.2 *([33]) Let $\{p_n\}$ be a positive sequence, $\{q_n\}$ be a sequence of real numbers, and $\{\sigma_n\}$ be a sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \sigma_n = \infty$. Suppose that*

$$p_{n+1} \leq \sigma_n q_n + (1 - \sigma_n)\, p_n, \quad \forall n \geq 1.$$

If $\limsup_{k\to\infty} q_{n_k} \leq 0$ for every subsequence $\{p_{n_k}\}$ of $\{p_n\}$ satisfying $\liminf_{k\to\infty} \left(p_{n_k+1} - p_{n_k}\right) \geq 0$, then $\lim_{n\to\infty} p_n = 0$.

Lemma 2.3 *Let $U : \mathscr{H} \to \mathscr{H}$ be a β-demicontractive with $\mathrm{Fix}(U) \neq \emptyset$ and let $U_\lambda = (1 - \lambda)I + \lambda U$, where $\lambda \in (0, 1 - \beta)$. Then:*

(1) $\mathrm{Fix}(U) = \mathrm{Fix}(U_\lambda)$;
(2) $\|U_\lambda x - z\|^2 \leq \|x - z\|^2 - \lambda(1 - \beta - \lambda)\,\|(I - U)\,x\|^2, \quad \forall x \in \mathscr{H}, z \in \mathrm{Fix}(U)$;
(3) $\mathrm{Fix}(U)$ *is a closed convex subset of* $\mathscr{H}$.

Proof (1) It is obvious.
(2) From the definition of U_λ and (2.1), we have

$$\begin{aligned}
\|U_\lambda x - z\|^2 &= \|(1 - \lambda)x + \lambda Ux - z\|^2 \\
&= \|x - z\|^2 + 2\lambda\langle x - z, Ux - x\rangle + \lambda^2\|Ux - x\|^2 \\
&\leq \|x - z\|^2 + \lambda(\beta - 1)\|Ux - x\|^2 + \lambda^2\|Ux - x\|^2 \\
&= \|x - z\|^2 - \lambda(1 - \beta - \lambda)\,\|(I - U)\,x\|^2 .
\end{aligned}$$

(3) It is a consequence of Proposition 1 in [34].

□

3 Main results

In this section, we introduce two new inertial extragradient algorithms for solving variational inequality problems and fixed point problems and analyze their convergence. First, we assume that our proposed algorithms satisfy the following conditions.

(C1) The solution set $\operatorname{Fix}(U) \cap \operatorname{VI}(C, A) \neq \emptyset$.
(C2) The mapping $A : \mathscr{H} \to \mathscr{H}$ is monotone and L-Lipschitz continuous.
(C3) The mapping $U : \mathscr{H} \to \mathscr{H}$ is λ-demicontractive such that $(I - U)$ is demiclosed at zero.
(C4) The mapping $f : \mathscr{H} \to \mathscr{H}$ is ρ-contraction with constant $\rho \in [0, 1)$.
(C5) Let $\{\epsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, where $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{\beta_n\}$ be a real sequence in $(0, 1)$ such that $\{\beta_n\} \subset (a, 1 - \lambda)$ for some $a > 0$.

3.1 The viscosity-type inertial subgradient extragradient algorithm

Now, we introduce a viscosity-type inertial subgradient extragradient algorithm for solving variational inequality problems and fixed point problems. The Algorithm 3.1 is of the form.

Algorithm 3.1 The viscosity-type inertial subgradient extragradient algorithm

Initialization: Take $\theta > 0$, $\tau_1 > 0$, $\mu \in (0, 1)$. Let $x_0, x_1 \in \mathscr{H}$ be arbitrary.
Iterative Steps: Calculate x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$. Set $w_n = x_n + \theta_n \left(x_n - x_{n-1}\right)$, where

$$\theta_n = \begin{cases} \min\left\{\dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}, \theta\right\}, & \text{if } x_n \neq x_{n-1}\,; \\ \theta, & \text{otherwise}\,. \end{cases} \tag{3.1}$$

Step 2. Compute $y_n = P_C\left(w_n - \tau_n A w_n\right)$.
Step 3. Compute $z_n = P_{T_n}\left(w_n - \tau_n A y_n\right)$, where the half-space T_n is defined by

$$T_n := \{x \in \mathscr{H} \mid \langle w_n - \tau_n A w_n - y_n, x - y_n \rangle \leq 0\}\,.$$

Step 4. Compute $x_{n+1} = \alpha_n f\left(x_n\right) + \left(1 - \alpha_n\right)\left[\left(1 - \beta_n\right) z_n + \beta_n U z_n\right]$, and update

$$\tau_{n+1} = \begin{cases} \min\left\{\dfrac{\mu \|w_n - y_n\|}{\|A w_n - A y_n\|}, \tau_n\right\}, & \text{if } A w_n - A y_n \neq 0\,; \\ \tau_n, & \text{otherwise}\,. \end{cases} \tag{3.2}$$

Set $n := n + 1$ and go to **Step 1**.

Remark 3.1 It follows from (3.1) that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0\,.$$

Indeed, we have $\theta_n \|x_n - x_{n-1}\| \leq \epsilon_n$ for all $n \geq 1$, which together with $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$ implies that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0\,.$$

The following lemmas are quite helpful to analyze the convergence of the algorithm.

Lemma 3.1 *The sequence $\{\tau_n\}$ generated by (3.2) is a nonincreasing sequence and*

$$\lim_{n\to\infty} \tau_n = \tau \geq \min\left\{\tau_1, \frac{\mu}{L}\right\}.$$

Proof It follows from (3.2) that $\tau_{n+1} \leq \tau_n$ for all $n \in \mathbb{N}$. Hence, $\{\tau_n\}$ is nonincreasing. On the other hand, we get $\|Aw_n - Ay_n\| \leq L\|w_n - y_n\|$ since A is L-Lipschitz continuous. Thus,

$$\mu \frac{\|w_n - y_n\|}{\|Aw_n - Ay_n\|} \geq \frac{\mu}{L}, \quad \text{if } \; Aw_n \neq Ay_n\,,$$

which together with (3.2) implies that

$$\tau_n \geq \min\left\{\tau_1, \frac{\mu}{L}\right\}.$$

Therefore, $\lim_{n\to\infty} \tau_n = \tau \geq \min\left\{\tau_1, \frac{\mu}{L}\right\}$ since the sequence $\{\tau_n\}$ is nonincreasing and lower bounded. □

Lemma 3.2 *([35]) Assume that Condition (C2) holds. Let $\{z_n\}$ be a sequence generated by Algorithm 3.1. Then*

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|y_n - w_n\|^2 - \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|z_n - y_n\|^2 \tag{3.3}$$

for all $p \in \mathrm{VI}(C, A)$.

Theorem 3.1 *Assume that Conditions (C1)–(C5) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges to $q \in \mathrm{Fix}(U) \cap \mathrm{VI}(C, A)$ in norm, where $q = P_{\mathrm{Fix}(U)\cap\mathrm{VI}(C,A)}(f(q))$.*

Proof Note that $\mathrm{VI}(C, A)$ is a closed convex subset, and $\mathrm{Fix}(U)$ is also a closed convex subset by Lemma 2.3. Hence, the mapping $P_{\mathrm{Fix}(U)\cap\mathrm{VI}(C,A)}(f) : \mathscr{H} \to \mathscr{H}$ is a contraction. From the Banach contraction principle, there exists a unique point $q \in \mathscr{H}$ such that $q = P_{\mathrm{Fix}(U)\cap\mathrm{VI}(C,A)}(f(q))$. In particular, $q \in \mathrm{Fix}(U) \cap \mathrm{VI}(C, A)$ and

$$\langle f(q) - q, z - q\rangle \leq 0, \quad \forall z \in \mathrm{Fix}(U) \cap \mathrm{VI}(C, A)\,.$$

We divide the proof into four parts.

Claim 1. The sequence $\{x_n\}$ is bounded. Indeed, put $t_n = (1-\beta_n) z_n + \beta_n U z_n$. From Lemma 2.3 (ii), we have

$$\|t_n - q\|^2 \leq \|z_n - q\|^2 - \beta_n (1-\lambda-\beta_n) \|U z_n - z_n\|^2 .$$

In view of Lemma 3.2 and $\{\beta_n\} \subset (a, 1-\lambda)$, we obtain

$$\begin{aligned} \|t_n - q\|^2 \leq \|w_n - q\|^2 - \left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right) \|y_n - w_n\|^2 - \left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right) \|z_n - y_n\|^2 \\ - \beta_n (1-\lambda-\beta_n) \|U z_n - z_n\|^2 . \end{aligned} \tag{3.4}$$

According to Lemma 3.1, we get

$$\lim_{n\to\infty} \left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right) = 1-\mu > 0 ,$$

which implies that there exists $n_0 \in \mathbb{N}$ such that $(1-\mu\frac{\tau_n}{\tau_{n+1}}) > 0,\ \forall n \geq n_0$. By (3.4), one has

$$\|t_n - q\| \leq \|w_n - q\| , \quad \forall n \geq n_0 . \tag{3.5}$$

From the definition of w_n, we can write

$$\begin{aligned} \|w_n - q\| &= \|x_n + \theta_n (x_n - x_{n-1}) - q\| \\ &\leq \|x_n - q\| + \alpha_n \cdot \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| . \end{aligned} \tag{3.6}$$

By Remark 3.1, we have $\frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \to 0$ as $n \to \infty$. Thus, there exists a constant $M_1 > 0$ such that

$$\frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \leq M_1, \quad \forall n \geq 1 . \tag{3.7}$$

Combining (3.5), (3.6) and (3.7), we find

$$\|t_n - q\| \leq \|w_n - q\| \leq \|x_n - q\| + \alpha_n M_1, \quad \forall n \geq n_0 . \tag{3.8}$$

Using (3.8), we obtain

$$\begin{aligned}\|x_{n+1}-q\| &= \|\alpha_n f(x_n)+(1-\alpha_n)t_n-q\| \\ &\le \alpha_n\|f(x_n)-f(q)\|+\alpha_n\|f(q)-q\|+(1-\alpha_n)\|t_n-q\| \\ &\le \alpha_n\rho\|x_n-q\|+\alpha_n\|f(q)-q\|+(1-\alpha_n)\|w_n-q\| \\ &\le [1-\alpha_n(1-\rho)]\|x_n-q\|+\alpha_n(1-\rho)\frac{\|f(q)-q\|+M_1}{1-\rho} \\ &\le \max\left\{\|x_n-q\|,\frac{\|f(q)-q\|+M_1}{1-\rho}\right\} \\ &\le \cdots \le \max\left\{\left\|x_{n_0}-q\right\|,\frac{\|f(q)-q\|+M_1}{1-\rho}\right\},\ \forall n\ge n_0,\end{aligned}$$

which implies that the sequence $\{x_n\}$ is bounded. So the sequences $\{w_n\}$, $\{f(x_n)\}$, $\{y_n\}$ and $\{z_n\}$ are also bounded.

Claim 2.

$$\begin{aligned}&(1-\alpha_n)\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|y_n-w_n\|^2+(1-\alpha_n)\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|z_n-y_n\|^2 \\ &\quad+(1-\alpha_n)\beta_n(1-\lambda-\beta_n)\|Uz_n-z_n\|^2 \\ &\le \|x_n-q\|^2-\|x_{n+1}-q\|^2+\alpha_n\|f(x_n)-q\|^2+\alpha_n M_2\end{aligned}$$

for some $M_2>0$. Indeed, it follows from (3.8) that

$$\begin{aligned}\|w_n-q\|^2 &\le (\|x_n-q\|+\alpha_n M_1)^2 \\ &= \|x_n-q\|^2+\alpha_n\left(2M_1\|x_n-q\|+\alpha_n M_1^2\right) \\ &\le \|x_n-q\|^2+\alpha_n M_2\end{aligned} \tag{3.9}$$

for some $M_2>0$. Using (3.4) and (3.9), we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|\alpha_n(f(x_n)-q)+(1-\alpha_n)(t_n-q)\|^2 \\ &\le \alpha_n\|f(x_n)-q\|^2+(1-\alpha_n)\|t_n-q\|^2 \\ &\le \alpha_n\|f(x_n)-q\|^2+\|x_n-q\|^2 \\ &\quad+\alpha_n M_2-(1-\alpha_n)\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|y_n-w_n\|^2 \\ &\quad-(1-\alpha_n)\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|z_n-y_n\|^2 \\ &\quad-(1-\alpha_n)\beta_n(1-\lambda-\beta_n)\|Uz_n-z_n\|^2.\end{aligned}$$

The desired result can be obtained by a simple deformation.

Claim 3.

$$\|x_{n+1}-q\|^2 \le (1-(1-\rho)\alpha_n)\|x_n-q\|^2 + (1-\rho)\alpha_n\Big[\frac{2}{1-\rho}\langle f(q)-q, x_{n+1}-q\rangle + \frac{3M\theta_n}{(1-\rho)\alpha_n}\|x_n-x_{n-1}\|\Big], \quad \forall n \ge n_0 .$$

Indeed, by the definition of w_n, one obtains

$$\begin{aligned}\|w_n-q\|^2 &= \|x_n+\theta_n(x_n-x_{n-1})-q\|^2 \\ &= \|x_n-q\|^2 + 2\theta_n\langle x_n-q, x_n-x_{n-1}\rangle + \theta_n^2\|x_n-x_{n-1}\|^2 \\ &\le \|x_n-q\|^2 + 3M\theta_n\|x_n-x_{n-1}\| ,\end{aligned} \tag{3.10}$$

where $M := \sup_{n\in\mathbb{N}}\{\|x_n-q\|, \theta\|x_n-x_{n-1}\|\} > 0$. Using (3.8) and (3.10), we get

$$\begin{aligned}&\|x_{n+1}-q\|^2 = \|\alpha_n f(x_n) + (1-\alpha_n)t_n - q\|^2 \\ &= \|\alpha_n(f(x_n)-f(q)) + (1-\alpha_n)(t_n-q) + \alpha_n(f(q)-q)\|^2 \\ &\le \|\alpha_n(f(x_n)-f(q)) + (1-\alpha_n)(t_n-q)\|^2 + 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle \\ &\le \alpha_n\|f(x_n)-f(q)\|^2 + (1-\alpha_n)\|t_n-q\|^2 + 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle \\ &\le \alpha_n\rho\|x_n-q\|^2 + (1-\alpha_n)\|x_n-q\|^2 + 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle + 3M\theta_n\|x_n-x_{n-1}\| \\ &= (1-(1-\rho)\alpha_n)\|x_n-q\|^2 + (1-\rho)\alpha_n\Big[\frac{2}{1-\rho}\langle f(q)-q, x_{n+1}-q\rangle \\ &\quad + \frac{3M\theta_n}{(1-\rho)\alpha_n}\|x_n-x_{n-1}\|\Big], \quad \forall n \ge n_0 .\end{aligned} \tag{3.11}$$

Claim 4. The sequence $\{\|x_n-q\|^2\}$ converges to zero. Indeed, from Lemma 2.2 and Remark 3.1, it suffices to show that $\limsup_{k\to\infty}\langle f(q)-q, x_{n_k+1}-q\rangle \le 0$ for every subsequence $\{\|x_{n_k}-q\|\}$ of $\{\|x_n-q\|\}$ satisfying

$$\liminf_{k\to\infty}\left(\|x_{n_k+1}-q\| - \|x_{n_k}-q\|\right) \ge 0 . \tag{3.12}$$

For this purpose, we assume that $\{\|x_{n_k}-q\|\}$ is a subsequence of $\{\|x_n-q\|\}$ such that (3.12) holds. Then

$$\begin{aligned}&\liminf_{k\to\infty}\left(\|x_{n_k+1}-q\|^2 - \|x_{n_k}-q\|^2\right) \\ &= \liminf_{k\to\infty}\left[\left(\|x_{n_k+1}-q\| - \|x_{n_k}-q\|\right)\left(\|x_{n_k+1}-q\| + \|x_{n_k}-q\|\right)\right] \ge 0 .\end{aligned}$$

It follows from Claim 2 and Condition (C5) that

$$\begin{aligned}
&\limsup_{k\to\infty}\Big\{(1-\alpha_{n_k})\Big(1-\mu\frac{\tau_{n_k}}{\tau_{n_k+1}}\Big)\left\|y_{n_k}-w_{n_k}\right\|^2\\
&\quad+(1-\alpha_{n_k})\Big(1-\mu\frac{\tau_{n_k}}{\tau_{n_k+1}}\Big)\left\|z_{n_k}-y_{n_k}\right\|^2\\
&\quad+(1-\alpha_{n_k})\beta_{n_k}\left(1-\lambda-\beta_{n_k}\right)\left\|Uz_{n_k}-z_{n_k}\right\|^2\Big\}\\
&\quad\le\limsup_{k\to\infty}\Big[\left\|x_{n_k}-q\right\|^2-\|x_{n_k+1}-q\|^2+\alpha_{n_k}\left\|f\left(x_{n_k}\right)-q\right\|^2+\alpha_{n_k}M_2\Big]\\
&\quad=-\liminf_{k\to\infty}\Big[\|x_{n_k+1}-q\|^2-\left\|x_{n_k}-q\right\|^2\Big]\le 0\,,
\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\left\|w_{n_k}-y_{n_k}\right\|=0,\ \text{and}\ \lim_{k\to\infty}\left\|z_{n_k}-Uz_{n_k}\right\|=0,\ \text{and}\ \lim_{k\to\infty}\left\|z_{n_k}-y_{n_k}\right\|=0\,. \tag{3.13}$$

Therefore, we get $\lim_{k\to\infty}\left\|z_{n_k}-w_{n_k}\right\|=0$. According to the definition of w_n, one has

$$\left\|x_{n_k}-w_{n_k}\right\|=\theta_{n_k}\left\|x_{n_k}-x_{n_k-1}\right\|=\alpha_{n_k}\cdot\frac{\theta_{n_k}}{\alpha_{n_k}}\left\|x_{n_k}-x_{n_k-1}\right\|\to 0,\ \ \text{as } k\to\infty\,. \tag{3.14}$$

This together with $\lim_{k\to\infty}\left\|z_{n_k}-w_{n_k}\right\|=0$ yields that

$$\lim_{k\to\infty}\left\|z_{n_k}-x_{n_k}\right\|=0\,. \tag{3.15}$$

From $t_{n_k}=\left(1-\beta_{n_k}\right)z_{n_k}+\beta_{n_k}Uz_{n_k}$, one sees that

$$\left\|t_{n_k}-z_{n_k}\right\|=\beta_{n_k}\left\|Uz_{n_k}-z_{n_k}\right\|\le(1-\lambda)\left\|Uz_{n_k}-z_{n_k}\right\|\,.$$

In view of (3.13), we get

$$\lim_{k\to\infty}\left\|t_{n_k}-z_{n_k}\right\|=0\,. \tag{3.16}$$

From (3.15) and (3.16), we deduce that

$$\begin{aligned}
\left\|x_{n_k+1}-x_{n_k}\right\|&=\left\|\alpha_{n_k}f\left(x_{n_k}\right)+\left(1-\alpha_{n_k}\right)t_{n_k}-x_{n_k}\right\|\\
&\le\alpha_{n_k}\left\|f\left(x_{n_k}\right)-x_{n_k}\right\|+\left(1-\alpha_{n_k}\right)\left\|t_{n_k}-x_{n_k}\right\|\\
&\le\alpha_{n_k}\left\|f\left(x_{n_k}\right)-x_{n_k}\right\|+\left\|t_{n_k}-z_{n_k}\right\|+\left\|z_{n_k}-x_{n_k}\right\|\to 0,\ \ \text{as } k\to\infty\,.
\end{aligned} \tag{3.17}$$

Since the sequence $\left\{x_{n_k}\right\}$ is bounded, one infers that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\left\{x_{n_k}\right\}$ such that $x_{n_{k_j}} \rightharpoonup z$, which further yields that

$$\limsup_{k\to\infty}\left\langle f(q)-q, x_{n_k}-q\right\rangle = \lim_{j\to\infty}\langle f(q)-q, x_{n_{k_j}}-q\rangle = \langle f(q)-q, z-q\rangle . \tag{3.18}$$

From (3.14), one gets $w_{n_k} \rightharpoonup z$. Combining (3.13), $\lim_{n\to\infty}\tau_n = \tau$ and Lemma 2.1, one concludes that $z \in \mathrm{VI}(C, A)$. It follows from (3.15) that $z_{n_k} \rightharpoonup z$. By the demiclosedness of $(I-U)$, we get that $z \in \mathrm{Fix}(U)$. Thus, $z \in \mathrm{Fix}(U) \cap \mathrm{VI}(C, A)$. Combining (3.18), the definition of q and $z \in \mathrm{Fix}(U) \cap \mathrm{VI}(C, A)$, we obtain

$$\limsup_{k\to\infty}\left\langle f(q)-q, x_{n_k}-q\right\rangle = \langle f(q)-q, z-q\rangle \leq 0 , \tag{3.19}$$

which, together with (3.17) and (3.19), yields that

$$\begin{aligned} &\limsup_{k\to\infty}\left\langle f(q)-q, x_{n_k+1}-q\right\rangle \\ &\quad\leq \limsup_{k\to\infty}\left\langle f(q)-q, x_{n_k+1}-x_{n_k}\right\rangle + \limsup_{k\to\infty}\langle f(q)-q, x_{n_k}-q\rangle \\ &\quad= \langle f(q)-q, z-q\rangle \leq 0 . \end{aligned} \tag{3.20}$$

Combining Claim 3, (3.20) and Remark 3.1, in the light of Lemma 2.2, we observe that $x_n \to q$ as $n \to \infty$. The proof of Theorem 3.1 is completed. □

In particular, considering when $U = I$ in Algorithm 3.1, where I is the identity operator, we can obtain a new algorithm to solve the variational inequality problem (VI). More precisely, we have the following Corollary 3.1.

Corollary 3.1 *Assume that mapping $A : \mathscr{H} \to \mathscr{H}$ is Lipschitz continuous and monotone, and mapping $f : \mathscr{H} \to \mathscr{H}$ is ρ-contraction with constant $\rho \in [0, 1)$. Let $\{\epsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty}\frac{\epsilon_n}{\alpha_n} = 0$, where $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Assume that the solution set of the variational inequality problem is non-empty. Let $x_0, x_1 \in \mathscr{H}$ and the sequence $\{x_n\}$ be generated by*

$$\left\{\begin{array}{l} w_n = x_n + \theta_n\left(x_n - x_{n-1}\right) , \\ y_n = P_C\left(w_n - \tau_n A w_n\right) , \\ z_n = P_{T_n}\left(w_n - \tau_n A y_n\right) , \\ T_n = \{x \in \mathscr{H} \mid \langle w_n - \tau_n A w_n - y_n, x - y_n\rangle \leq 0\} , \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) z_n , \end{array}\right. \tag{3.21}$$

where θ_n and τ_n are defined in (3.1) and (3.2), respectively. Then the iterative sequence $\{x_n\}$ generated by (3.21) converges to $q \in \mathrm{VI}(C, A)$ in norm, where $q = P_{\mathrm{VI}(C,A)}(f(q))$.

3.2 The viscosity-type inertial Tseng's extragradient algorithm

In this subsection, we introduce a viscosity-type inertial Tseng's extragradient algorithm for solving variational inequality problems and fixed point problems. Our Algorithm 3.2 is as follows.

Algorithm 3.2 The viscosity-type inertial Tseng's extragradient algorithm

Initialization: Take $\theta > 0$, $\tau_1 > 0$, $\mu \in (0, 1)$. Let $x_0, x_1 \in \mathscr{H}$ be arbitrary.
Iterative Steps: Calculate x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$. Set $w_n = x_n + \theta_n \left(x_n - x_{n-1}\right)$, where θ_n is defined in (3.1).
Step 2. Compute $y_n = P_C \left(w_n - \tau_n A w_n\right)$.
Step 3. Compute $z_n = y_n - \tau_n \left(A y_n - A w_n\right)$.
Step 4. Compute $x_{n+1} = \alpha_n f\left(x_n\right) + \left(1 - \alpha_n\right)\left[\left(1 - \beta_n\right) z_n + \beta_n U z_n\right]$, and update τ_{n+1} by (3.2).
Set $n := n + 1$ and go to **Step 1**.

The following lemma is very helpful for analyzing the convergence of the Algorithm 3.2.

Lemma 3.3 *([35]) Assume that Condition (C2) holds. Let $\{z_n\}$ be a sequence generated by Algorithm 3.2. Then*

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu^2 \frac{\tau_n^2}{\tau_{n+1}^2}\right) \|w_n - y_n\|^2, \quad \forall p \in \mathrm{VI}(C, A),$$

and

$$\|z_n - y_n\| \leq \mu \frac{\tau_n}{\tau_{n+1}} \|w_n - y_n\| .$$

Theorem 3.2 *Assume that Conditions (C1)–(C5) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges to $q \in \mathrm{Fix}(U) \cap \mathrm{VI}(C, A)$ in norm, where $q = P_{\mathrm{Fix}(U) \cap \mathrm{VI}(C,A)}(f(q))$.*

Proof We also divided the proof into four steps.
Claim 1. The sequence $\{x_n\}$ is bounded. Take $t_n = \left(1 - \beta_n\right) z_n + \beta_n U z_n$. Combining Lemma 2.3 (ii) and Lemma 3.3, we obtain

$$\begin{aligned}
\|t_n - q\|^2 &\leq \|z_n - q\|^2 - \beta_n \left(1 - \lambda - \beta_n\right) \|U z_n - z_n\|^2 \\
&\leq \|w_n - q\|^2 - \left(1 - \mu^2 \frac{\tau_n^2}{\tau_{n+1}^2}\right) \|w_n - y_n\|^2 - \beta_n \left(1 - \lambda - \beta_n\right) \|U z_n - z_n\|^2 .
\end{aligned} \tag{3.22}$$

By Lemma 3.1, there exists $n_1 \in \mathbb{N}$ such that $1 - \mu^2 \frac{\tau_n^2}{\tau_{n+1}^2} > 0, \forall n \geq n_1$. From (3.22), we have

$$\|t_n - q\| \leq \|w_n - q\|, \quad \forall n \geq n_1 . \tag{3.23}$$

Using the same arguments with the Claim 1 in the Theorem 3.1, we get that the sequence $\{x_n\}$ is bounded. Consequently, the sequences $\{w_n\}$, $\{f(x_n)\}$, $\{y_n\}$ and $\{z_n\}$ are also bounded.

Claim 2.

$$
\begin{aligned}
&(1-\alpha_n)\left(1-\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}\right)\|w_n-y_n\|^2+(1-\alpha_n)\beta_n(1-\lambda-\beta_n)\|Uz_n-z_n\|^2\\
&\quad\le\|x_n-q\|^2-\|x_{n+1}-q\|^2+\alpha_n\|f(x_n)-q\|^2+\alpha_n M_2\,.
\end{aligned}
$$

Indeed, using (3.9), (3.22) and (3.23), we have

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &\le \alpha_n\|f(x_n)-q\|^2+(1-\alpha_n)\|t_n-q\|^2\\
&\le \alpha_n\|f(x_n)-q\|^2+\|x_n-q\|^2+\alpha_n M_2\\
&\quad-(1-\alpha_n)\left(1-\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}\right)\|w_n-y_n\|^2\\
&\quad-(1-\alpha_n)\beta_n(1-\lambda-\beta_n)\|Uz_n-z_n\|^2\,,
\end{aligned}
$$

where M_2 is defined in Claim 2 of Theorem 3.1.

Claim 3.

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &\le (1-(1-\rho)\alpha_n)\|x_n-q\|^2\\
&\quad+(1-\rho)\alpha_n\Big[\frac{2}{1-\rho}\langle f(q)-q,x_{n+1}-q\rangle\\
&\quad+\frac{3M\theta_n}{(1-\rho)\alpha_n}\|x_n-x_{n-1}\|\Big],\quad \forall n\ge n_1\,.
\end{aligned}
$$

The desired result can be obtained by using the same arguments as in the Claim 3 of Theorem 3.1.

Claim 4. The sequence $\{\|x_n-q\|\}$ converges to zero. The proof is similar to the Claim 4 of Theorem 3.1. So we omit it here. □

As stated in Corollary 3.1, we have the following result by setting $U=I$ in Algorithm 3.2.

Corollary 3.2 *Let A, f, α_n and ϵ_n be the same as in Corollary 3.1. Let $x_0, x_1\in\mathscr{H}$ and the sequence $\{x_n\}$ be created by*

$$
\begin{cases}
w_n=x_n+\theta_n(x_n-x_{n-1})\,,\\
y_n=P_C(w_n-\tau_n Aw_n)\,,\\
z_n=y_n-\tau_n(Ay_n-Aw_n)\,,\\
x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)z_n\,,
\end{cases}
\tag{3.24}
$$

where θ_n and τ_n are defined in (3.1) and (3.2), respectively. Then the iterative sequence $\{x_n\}$ formed by (3.24) converges to $q\in\mathrm{VI}(C,A)$ in norm, where $q=P_{\mathrm{VI}(C,A)}(f(q))$.

4 Numerical examples

In this section, we provide some numerical examples to illustrate the numerical behavior of the proposed algorithms (Algorithm 3.1 (shortly, iVSEGM) and Algorithm 3.2 (shortly, iVTEGM)) and also to compare them with some existing strongly convergent algorithms, which including the Halpern subgradient extragradient method (HSEGM) [21], the modified subgradient extragradient algorithm (MSEGM) [22, Algorithm 1], the viscosity-type subgradient extragradient method (VSEGM) [23] and the viscosity-type Tseng's extragradient method (VTEGM) [23]. All the programs are performed in MATLAB 2018a on a Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz computer with RAM 8.00 GB.

The parameters of all algorithms are set as follows. In all algorithms, we set $\alpha_n = 1/(n+1)$ and $\beta_n = n/(2n+1)$. For the proposed algorithms and the algorithms (VSEGM) and (VTEGM), we choose $\tau_1 = 1$, $\mu = 0.5$ and $f(x) = 0.5x$. Take $\theta = 0.3$, $\epsilon_n = 100/(n+1)^2$ in our proposed algorithms. For the algorithms (HSEGM) and (MSEGM), we choose the step size as $\tau_n = 0.99/L$. In our experiment examples, the maximum number of iterations 200 as a common stopping criterion. The solution x^* of the problems are known, so we use $D_n = \|x_n - x^*\|$ to measure the n-th iteration error. The convergence of $\{D_n\}$ to 0 implies that $\{x_n\}$ converges to the solution of the problem.

4.1 Theoretical examples

Example 4.1 Consider a nonlinear operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$A(x, y) = (x + y + \sin x; -x + y + \sin y)$$

and the feasible set C is a box defined by $C = [-1, 1] \times [-1, 1]$. It is easy to check that A is monotone and Lipschitz continuous with the constant $L = 3$. Let E be a 2×2 matrix defined by

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

We consider the mapping $U : \mathbb{R}^2 \to \mathbb{R}^2$ by $Uz = \|E\|^{-1}Ez$, where $z = (x, y)^\mathsf{T}$. It is obvious to see that U is 0-demicontractive and thus $\lambda = 0$. The solution of the problem is $x^* = (0, 0)^\mathsf{T}$. The initial values $x_0 = x_1$ are randomly generated by *k*rand(2,1)* in MATLAB. The numerical results of all the algorithms with four different initial values are described in Fig. 1.

Example 4.2 Consider the linear operator $A : \mathbb{R}^m \to \mathbb{R}^m$ $(m = 50, 100, 150, 200)$ in the form $A(x) = Mx + q$, where $q \in \mathbb{R}^m$ and $M = NN^\mathsf{T} + Q + D$, N is a $m \times m$ matrix, Q is a $m \times m$ skew-symmetric matrix, and D is a $m \times m$ diagonal matrix with its diagonal entries being nonnegative (hence M is positive symmetric definite). The feasible set C is given by $C = \{x \in \mathbb{R}^m : -2 \le x_i \le 5,\ i = 1, \ldots, m\}$.

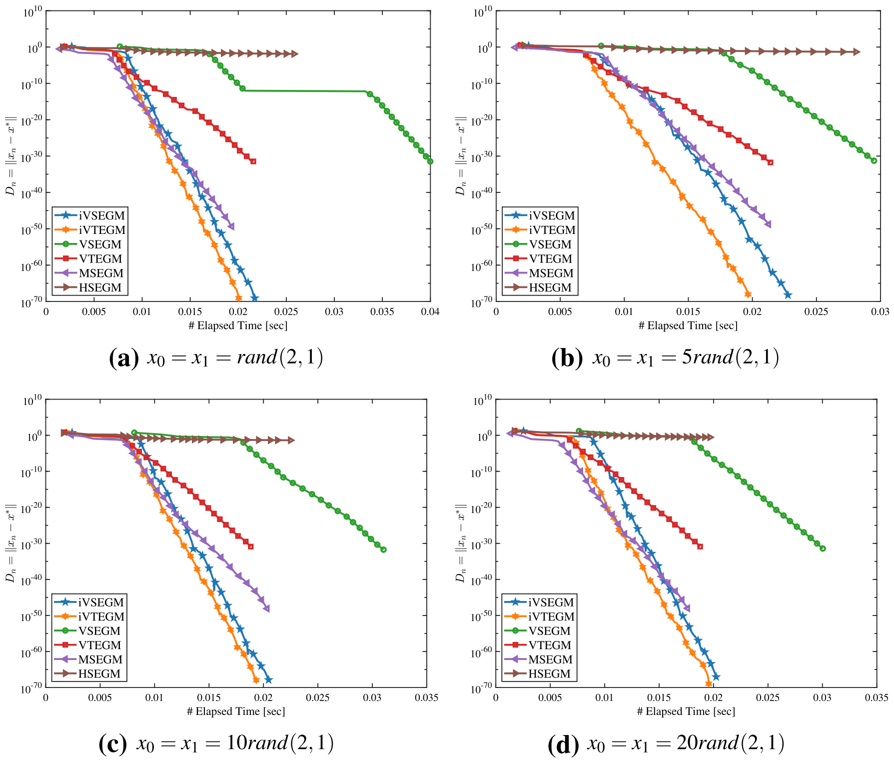


Fig. 1 Numerical results of all algorithms for Example 4.1

It is clear that A is monotone and Lipschitz continuous with constant $L = \|M\|$. In this experiment, all entries of N, D are generated randomly in $[0, 2]$, Q is generated randomly in $[-2, 2]$ and $q = \mathbf{0}$. Let $U : \mathcal{H} \to \mathcal{H}$ be given by $Ux := 0.5x$. It is easy to see that the solution of the problem in this case is $x^* = \{\mathbf{0}\}$. The initial values $x_0 = x_1$ are randomly generated by *10rand(2,1)* in MATLAB. Figure 2 shows the numerical behavior of all the algorithms in different dimensions.

Example 4.3 Finally, we consider our problem in the infinite-dimensional Hilbert space $\mathcal{H} = L^2([0,1])$ with inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)\mathrm{d}t$ and norm $\|x\| := (\int_0^1 |x(t)|^2 \mathrm{d}t)^{1/2}, \forall x, y \in \mathcal{H}$. Let the feasible set be the unit ball $C := \{x \in \mathcal{H} : \|x\| \leq 1\}$. Define an operator $A : C \to \mathcal{H}$ by

$$(Ax)(t) = \int_0^1 (x(t) - G(t,s)g(x(s)))\,\mathrm{d}s + h(t), \quad t \in [0,1],\ x \in C,$$

where

$$G(t,s) = \frac{2ts\mathrm{e}^{t+s}}{e\sqrt{\mathrm{e}^2 - 1}}, \quad g(x) = \cos x, \quad h(t) = \frac{2t\mathrm{e}^t}{e\sqrt{\mathrm{e}^2 - 1}}.$$

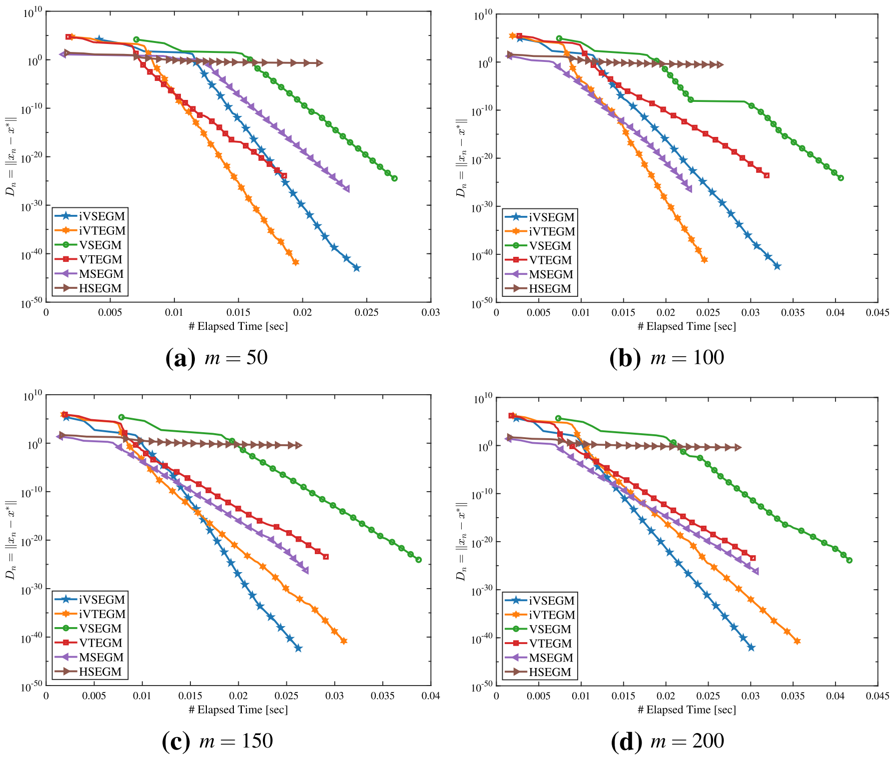


Fig. 2 Numerical results of all algorithms for Example 4.2

It is known that A is monotone and L-Lipschitz continuous with $L = 2$ (see [36]). The projection on C is inherently explicit, that is,

$$P_C(x) = \begin{cases} \frac{x}{\|x\|}, & \text{if } \|x\| > 1\,; \\ x, & \text{if } \|x\| \leq 1\,. \end{cases}$$

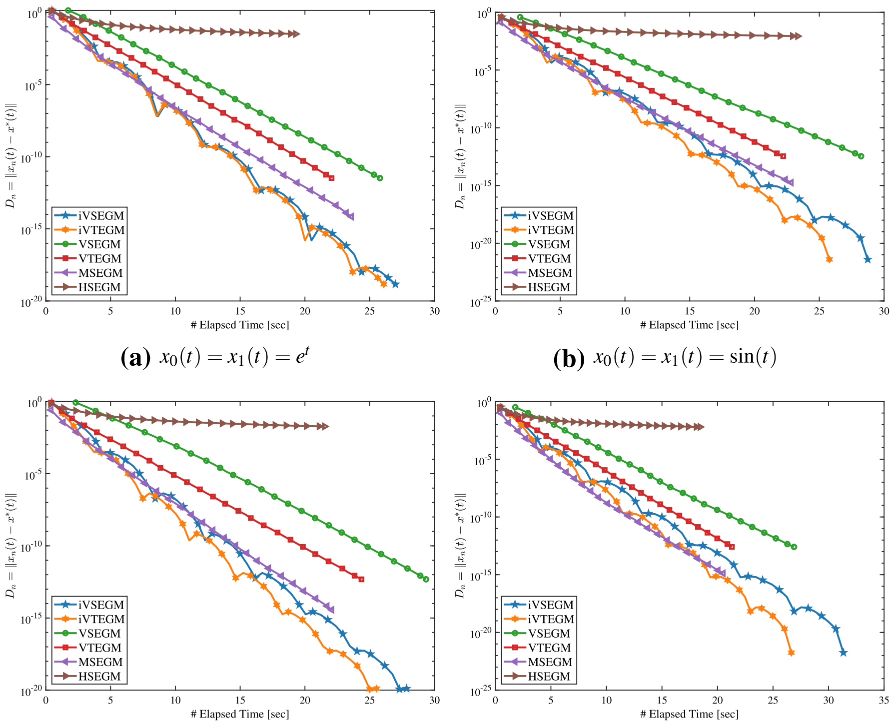
The mapping $U : L^2([0, 1]) \rightarrow L^2([0, 1])$ is of form

$$(Ux)(t) = \int_0^1 tx(s)\,\mathrm{d}s, \quad t \in [0, 1]\,.$$

A straightforward computation implies that U is 0-demicontractive. The solution of the problem is $x^*(t) = 0$. The maximum number of iterations 50 is used as a common stopping criterion for all algorithms. Figure 3 shows the behaviors of $D_n = \|x_n(t) - x^*(t)\|$ generated by all the algorithms with four starting points.

Remark 4.1 We have the following observations for Examples 4.1–4.3.

(1) From Example 4.1 and Example 4.2, we see that our proposed iterative schemes outperformance the existing algorithms [21–23] in terms of the elapsed time and



(a) $x_0(t) = x_1(t) = e^t$ **(b)** $x_0(t) = x_1(t) = \sin(t)$

(c) $x_0(t) = x_1(t) = \log(t)$ **(d)** $x_0(t) = x_1(t) = t^2$

Fig. 3 Numerical results of all algorithms for Example 4.3

accuracy. Since Example 4.3 occurs in an infinite-dimensional Hilbert space, it can be seen from Fig. 3 that the proposed algorithms have a higher accuracy and longer execution time. The increase in execution time is due to the fact that our algorithms need to calculate the value of the inertial parameter in each iteration.

(2) It is worth noting that our algorithms are robust and converge very quickly. However, there are still some oscillations because the inertial selection is too large.

(3) The maximum number of iterations we choose is only 200. It is noted that the iteration error of the Algorithm (HSEGM) is very big. In actual applications, it may require more iterations to meet the accuracy requirements.

4.2 Applications to optimal control problems

We use the proposed Algorithms (3.21) and (3.24) to solve variational inequalities that appears in optimal control problems. Assume that $L_2([0, T], \mathbb{R}^m)$ represents the square-integrable Hilbert space with inner product $\langle p, q\rangle = \int_0^T \langle p(t), q(t)\rangle \, \mathrm{d}t$ and norm $\|p\| = \sqrt{\langle p, p\rangle}$. The optimal control problem is described as follows:

$$p^*(t) \in \operatorname{Argmin}\{g(p) \mid p \in V\}, \quad t \in [0, T], \tag{3.1}$$

where V represents a set of feasible controls composed of m piecewise continuous functions. Its form is expressed as follows:

$$V = \left\{ p(t) \in L_2\left([0, T], \mathbb{R}^m\right) : p_i(t) \in \left[p_i^-, p_i^+\right], i = 1, 2, \ldots, m \right\}. \quad (3.2)$$

In particular, the control $p(t)$ may be a piecewise constant function (bang-bang type). The terminal objective function has the form

$$g(p) = \Phi(x(T)), \quad (3.3)$$

where Φ is a convex and differentiable defined on the attainability set. Assume that the trajectory $x(t) \in L_2([0, T])$ satisfies the constraints of the linear differential equation system:

$$\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t} x(t) = Q(t)x(t) + W(t)p(t), \quad 0 \le t \le T, \quad x(0) = x_0, \quad (3.4)$$

where $Q(t) \in \mathbb{R}^{n\times n}$, $W(t) \in \mathbb{R}^{n\times m}$ are given continuous matrices for every $t \in [0, T]$. By the solution of problem (3.1)–(3.4), we mean a control $p^*(t)$ and a corresponding (optimal) trajectory $x^*(t)$ such that its terminal value $x^*(T)$ minimizes objective function (3.3). It is known that the optimal control problem (3.1)–(3.4) can be transformed into a variational inequality problem (see [37]). We first use the classical Euler discretization method to decompose the optimal control problem (3.1)–(3.4) and then apply the proposed algorithms to solve the variational inequality problem corresponding to the discretized version of the problem (see [5,38,39] for more details).

Next, we present several mathematical examples to illustrate the computational performance of the proposed algorithms. Our parameters are set as follows. We set $N = 100$, $\alpha_n = 10^{-4}/(n+1)$, $\theta = 0.01$, $\epsilon_n = 10^{-4}/(n+1)^2$, $\tau_1 = 0.4$, $\mu = 0.1$ and $f(x) = 0.1x$ for the suggested Algorithms (3.21) and (3.24). The initial controls $p_0(t) = p_1(t)$ are randomly generated in $[-1, 1]$. The stopping criterion is either $D_n = \|p_{n+1} - p_n\| \le 10^{-4}$, or the maximum number of iterations is performed 1000.

Example 4.4 (Control of a harmonic oscillator, see [40])

$$\begin{aligned} \text{minimize} \quad & x_2(3\pi) \\ \text{subject to} \quad & \dot{x}_1(t) = x_2(t), \\ & \dot{x}_2(t) = -x_1(t) + p(t), \quad \forall t \in [0, 3\pi], \\ & x(0) = 0, \\ & p(t) \in [-1, 1]. \end{aligned}$$

The exact optimal control of Example 4.4 is known:

$$p^*(t) = \begin{cases} 1, & \text{if } t \in [0, \pi/2) \cup (3\pi/2, 5\pi/2); \\ -1, & \text{if } t \in (\pi/2, 3\pi/2) \cup (5\pi/2, 3\pi]. \end{cases}$$

Algorithm (3.21) and Algorithm (3.24) take 0.11806 seconds and 0.082014 seconds to reach the allowable error, respectively. Fig. 4 shows the approximate optimal control and the corresponding trajectories of the stated Algorithm (3.21).

We now consider an example in which the terminal function is not linear.

Example 4.5 (see [41])

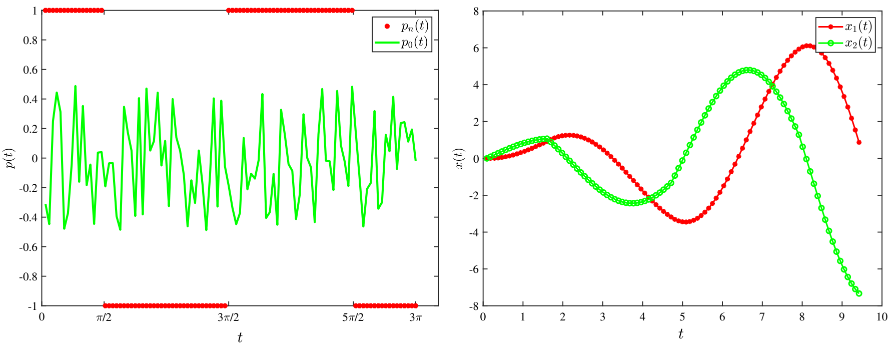
$$
\begin{aligned}
\text{minimize} \quad & -x_1(2) + (x_2(2))^2 \,, \\
\text{subject to} \quad & \dot{x}_1(t) = x_2(t)\,, \\
& \dot{x}_2(t) = p(t), \quad \forall t \in [0, 2]\,, \\
& x_1(0) = 0, \quad x_2(0) = 0\,, \\
& p(t) \in [-1, 1]\,.
\end{aligned}
$$

The exact optimal control of Example 4.5 is

$$
p^*(t) = \begin{cases} 1, & \text{if } t \in [0, 1.2)\,; \\ -1, & \text{if } t \in (1.2, 2]\,. \end{cases}
$$

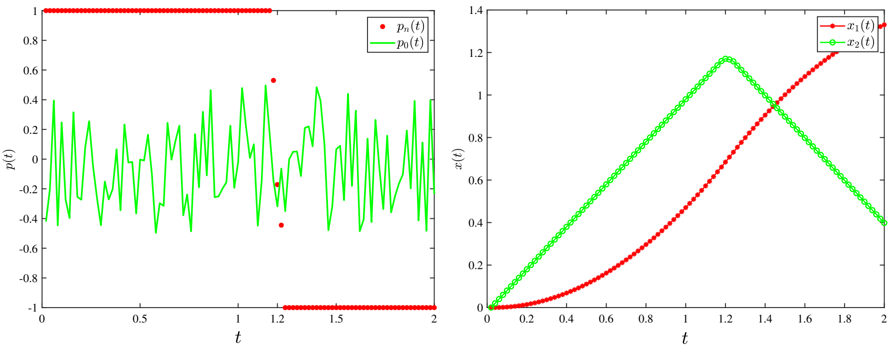
Both Algorithm (3.21) and Algorithm (3.24) need to perform a maximum of 1000 iterations and they take 0.41784 and 0.34624 seconds, respectively. The approximate optimal control and the corresponding trajectories of the suggested Algorithm (3.24) are plotted in Fig. 5.

Remark 4.2 As can be seen from Example 4.4 and Example 4.5, the suggested algorithms can be used to solve the optimal control problems described by the variational inequality model, and they perform well when the terminal function is linear or non-linear.



(a) Initial and optimal controls **(b)** Optimal trajectories

Fig. 4 Numerical results of the proposed Algorithm (3.21) for Example 4.4



(a) Initial and optimal controls **(b)** Optimal trajectories

Fig. 5 Numerical results of the proposed Algorithm (3.24) for Example 4.5

5 Conclusions

In this paper, we introduced two new extragradient algorithms to solve variational inequality problems and fixed point problems in real Hilbert spaces. The algorithms are constructed around the inertial method, the viscosity method, the subgradient extragradient method and the Tseng's extragradient method. Strong convergence theorems of the proposed algorithms are established without the prior knowledge of the Lipschitz constant of the operator. Some numerical experiments were conducted to illustrate the performance of the proposed algorithms over previously known ones. Our presented iterative schemes extended and improved some existing results in the literature.

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