Journal of Nonlinear and Convex Analysis
Volume 23, Number 11, 2022, 2593–2604

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ALTERNATED INERTIAL SUBGRADIENT EXTRAGRADIENT METHODS FOR SOLVING VARIATIONAL INEQUALITIES

ZHENG ZHOU, BING TAN, AND SUN YOUNG CHO

ABSTRACT. The goal of this paper is to study some iterative algorithms for solving a pseudomonotone variational inequality in Hilbert spaces. The iterative algorithms presented in this paper are based on the alternated inertial method and the subgradient extragradient method. Weak convergence of the algorithms is established by the adaptive stepsize criterion in Hilbert spaces.

1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$, and let C be a nonempty, closed, and convex subset of $\mathcal{H}$. The variational inequality (for short, VI) is to find a point $x^* \in C$ such that $\langle Ax^*, x-x^* \rangle \geq 0, \forall x \in C$, where $A : \mathcal{H} \to \mathcal{H}$ is a given mapping. For simplicity, let $VI(C, A)$ denote the solution set of VI from now on. It is known that many researches, such as constrained minimization problems, equilibrium problems, and complementarity problems, are regarded as a special form of the VI, and many solution methods have been introduced to study the solutions of the VI recently; see, e.g., [2, 7–9, 20, 31] and the references therein. The most well-known extragradient method was proposed by Korpelevich [13] and Antipin [1]. Unfortunately, this method is limited from the viewpoint of computation. Thus, Censor et al. [6] designed a subgradient extragradient method (for short, SEG). For any a starting point $x_1 \in \mathcal{H}$, the sequence $\{x_n\}$ generated by the SEG is constructed by

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{u \in \mathcal{H} \mid \langle x_n - \lambda A x_n - y_n, u - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \end{cases}$$

where mapping A is monotone and L-Lipschitz continuous and P_C is a metric projection from $\mathcal{H}$ onto C. If $\lambda \in (0, \frac{1}{L})$, then $\{x_n\}$ and $\{y_n\}$ converge weakly to a point in $VI(C, A)$ according to Censor et al. [6]. In addition to these improvements, the condition of the employed mapping A in the VI has been weakened from strong monotonicity to monotonicity, or even pseudo-monotonicity. From the above research, many interesting results have been established; see, e.g., [4, 5, 14, 24, 27–29] and the references therein. In the large-scale data calculations, the improvement of the convergence rate of the iterative algorithms is also a particularly important work. For this reason, based on the heavy ball with friction dynamical system, Alvarez and Attouch [3] proposed an inertial proximal point algorithm (for short, IPPA) to solve the inclusion problem with maximal monotone mappings. By

---

2020 *Mathematics Subject Classification.* 47H05, 47J20, 47J25.

Key words and phrases. Pseudomonotone variational inequality, alternated inertial, subgradient extragradient method, adaptive stepsize.

virtue of this feature, many inertial algorithms were introduced in recent years; see, e.g., [19, 25, 26, 32] for more details. Recently, Mu and Peng [17] pointed out that the sequence generated by IPPA loses the Fejér monotonicity of the sequence generated by the proximal point algorithm, and even fluctuates around the exact solution of the related problem. To overcome this shortcoming, they immediately proposed an alternated inertial proximal point algorithm and got the convergence of this algorithm in R^n. Some recent results on alternated inertial can be found in [10, 21, 22].

Inspiration by Censor et al. [6] and Mu and Peng [17], we construct several new alternated inertial algorithms to find solutions of a pseudomonotone variational inequality and prove their convergence under the condition of the adaptive stepsize criterion.

2. Preliminaries

Some basic definitions and related lemmas are listed below. Let $\mathcal{H}$ be a Hilbert space and C be a nonempty, closed and convex subset of $\mathcal{H}$. The symbol $x_n \rightharpoonup x$ represents weak convergence of $\{x_n\}$ to x. The metric projection of $\mathcal{H}$ onto C is expressed as P_C, i.e., $P_C(x) := \operatorname{argmin}_{y\in C} \|x-y\|$, $\forall x \in \mathcal{H}$. In addition, P_C satisfies the following property: $\langle P_C x-x, P_C x-y\rangle \leq 0, \forall y \in C \Leftrightarrow \|y-P_C x\|^2+\|x-P_C x\|^2 \leq \|x-y\|^2$. Meanwhile, for any $x, y \in \mathcal{H}$ and $\sigma \in R$, the following statement holds:

$$\|\sigma x + (1-\sigma)y\|^2 = \sigma\|x\|^2 + (1-\sigma)\|y\|^2 - \sigma(1-\sigma)\|x-y\|^2. \tag{2.1}$$

Definition 2.1. For any $x, y \in \mathcal{H}$, a mapping $A : \mathcal{H} \to \mathcal{H}$ is said to be (1) monotone if $\langle Ax - Ay, x - y\rangle \geq 0$; (2) pseudomonotone if $\langle Ay, x - y\rangle \geq 0 \Rightarrow \langle Ax, x - y\rangle \geq 0$; (3) L-Lipschitz continuous if there exists $L > 0$ such taht $\|Ax - Ay\| \leq L\|x - y\|$.

Lemma 2.2 ([16]). *Let C be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ and $A : C \to \mathcal{H}$ be a pseudomonotone and continuous mapping. Then, $x^* \in VI(C, A)$ if and only if $\langle Ax, x - x^*\rangle \geq 0,\ \forall x \in C$.*

Lemma 2.3 ([18]). *Let C be a nonempty subset of $\mathcal{H}$ and $\{x_n\}$ be a sequence in $\mathcal{H}$ such that the following conditions hold: (i) for every $x \in C$, $\lim_{n\to\infty} \|x_n-x\|$ exists; (ii) every sequentially weak cluster point of $\{x_n\}$ is in C. Then $\{x_n\}$ converges weakly to a point in C.*

Lemma 2.4 ([11]). *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Suppose that $A : \mathcal{H}_1 \to \mathcal{H}_2$ is uniformly continuous on bounded subsets of $\mathcal{H}_1$ and Q is a bounded subset of $\mathcal{H}_1$. Then, $A(Q)$ is bounded.*

3. Alternated inertial subgradient extragradient methods

In this section, we introduce two iterative algorithms to solve the pseudomonotone variational inequality based on the subgradient extragradient method by using the idea of alternated inertial terms and the adaptive stepsize. To begin with, suppose that the solution set $VI(C, A)$ is nonempty and some standard assumptions are as follows:

(A1) $\mathcal{H}$ is a Hilbert space and C is a nonempty, closed and convex subset of $\mathcal{H}$.

(A2) The mapping $A : \mathcal{H} \to \mathcal{H}$ is pseudomonotone and L-Lipschitz continuous but L is unknown.

(A2†) The mapping $A : \mathcal{H} \to \mathcal{H}$ is pseudomonotone on $\mathcal{H}$ and uniformly continuous on bounded subsets of $\mathcal{H}$.

(A3) The mapping $A : \mathcal{H} \to \mathcal{H}$ satisfies the following condition

$$\text{whenever } \{x_n\} \subset C \text{ and } x_n \rightharpoonup x, \text{one has } \|Ax\| \leq \liminf_{n\to\infty} \|Ax_n\|.$$

Based on the above setting conditions, the improved algorithms are of the form:

Algorithm 3.1. Give $\lambda_1 > 0$, $\mu \in (0,1)$ and two nonnegative real numbers sequences $\{\alpha_n\}$ and $\{\xi_n\}$ satisfying $\alpha_n \in [0,\alpha] \subset [0, \frac{1-\mu}{2})$ and $\sum_{n=1}^{\infty} \xi_n < +\infty$. Choose initial values x_0, $x_1 \in \mathcal{H}$ and set $n := 1$.
Step 1. Compute

$$w_n = \begin{cases} x_n, & n = \text{even}, \\ x_n + \alpha(x_n - x_{n-1}), & n = \text{odd}. \end{cases} \tag{3.1}$$

Step 2. Compute $y_n = P_C(w_n - \lambda_n A w_n)$. If $w_n = y_n$ or $Ay_n = 0$, then stop. Otherwise, go to **Step 3**.
Step 3. Compute $x_{n+1} = P_{T_n}(w_n - \lambda_n A y_n)$, where $T_n := \{x \in \mathcal{H} \mid \langle w_n - \lambda_n A w_n - y_n, x - y_n\rangle \leq 0\}$.
Step 4. Compute

$$\lambda_{n+1} = \begin{cases} \min\left\{\mu \frac{\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2}{2\langle Aw_n - Ay_n, x_{n+1} - y_n\rangle}, \lambda_n + \xi_n\right\}, & \text{if } \langle Aw_n - Ay_n, x_{n+1} - y_n\rangle > 0, \\ \lambda_n + \xi_n, & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and return **Step 1**.

Algorithm 3.2. Give $l, \mu \in (0,1)$, $\gamma > 0$ and a nonnegative real numbers sequence $\{\alpha_n\}$ satisfying $\alpha_n \in [0,\alpha] \subset [0, \frac{1-\mu}{2})$. Choose initial values x_0, $x_1 \in \mathcal{H}$ and set $n := 1$.
Step 1. Compute

$$w_n = \begin{cases} x_n, & n = \text{even}, \\ x_n + \alpha(x_n - x_{n-1}), & n = \text{odd}. \end{cases}$$

Step 2. Compute $y_n = P_C(w_n - \lambda_n A w_n)$, where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m satisfying $\gamma l^m \|Aw_n - Ay_n\| \leq \mu \|w_n - y_n\|$. If $w_n = y_n$ or $Ay_n = 0$, then stop. Otherwise, go to **Step 3**.
Step 3. Compute $x_{n+1} = P_{T_n}(w_n - \lambda_n A y_n)$, where $T_n := \{x \in \mathcal{H} \mid \langle w_n - \lambda_n A w_n - y_n, x - y_n\rangle \leq 0\}$, and set $n := n + 1$ and return **Step 1**.

Remark 3.3. In the past, when dealing with the pseudomonotone variational inequality (see, e.g., [27, 28]), the assigned mapping was always endowed with the sequentially weakly continuity, (for each sequence $\{x_n\}$ with $x_n \rightharpoonup x$ implies Ax_n converges weakly to Ax). It is worth noting that a weak condition setting, Condition (A3), is taken into consideration on the mapping A. In fact, if A is sequentially weakly continuous, Condition (A3) holds under the weak lower semicontinuity of the norm. On the contrary, it is not true. Some existing examples, which suffice to illustrate this point can be found in [29].

Remark 3.4. If $w_n = y_n$ or $Ay_n = 0$ for some n (that is, Algorithms 3.1 and 3.2 stop at the n-th iteration), then y_n is a solution of the VI. This stop criterion is also considered in many other subgradient extragradient algorithms; see, e.g., [27–29].

3.1. **Convergence of Algorithm 3.1.**

Lemma 3.5 ([15, 24])**.** *Assume that Conditions (A1) and (A2) hold. Then the adaptive stepsize* $\{\lambda_n\}$ *in Algorithm 3.1 is well-defined and has the following property:* $\lim_{n\to\infty} \lambda_n = \lambda$ *with* $\lambda \in [\min\{\frac{\mu}{L}, \lambda_1\}, \lambda_1 + \xi]$, *and* $\xi = \sum_{n=1}^{\infty} \xi_n$.

Lemma 3.6 ([29])**.** *Suppose that Conditions (A1) and (A2) hold. For any* $n \geq 1$ *and* $x^* \in VI(C, A)$, *the sequence* $\{x_n\}$ *generated by Algorithm 3.1 has the following property:* $\|x_{n+1} - x^*\|^2 \leq \|w_n - x^*\|^2 - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\left(\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2\right)$.

Lemma 3.7. *Suppose that Conditions (A1) and (A2) hold. The sequence* $\{x_n\}$ *generated by Algorithm 3.1 has the following properties:*

(I) *For any* $n \geq 1$ *and* $x^* \in VI(C, A)$,

$$\begin{aligned}&\|x_{2n+2} - x^*\|^2\\ \leq&\|x_{2n} - x^*\|^2 - \left(1 - \mu\frac{\lambda_{2n+1}}{\lambda_{2n+2}}\right)\left(\|w_{2n+1} - y_{2n+1}\|^2 + \|x_{2n+2} - y_{2n+1}\|^2\right)\\ &- (1 + \alpha_{2n+1})\left(1 - \mu\frac{\lambda_{2n}}{\lambda_{2n+1}} - 2\alpha_{2n+1}\right)\left(\|x_{2n} - y_{2n}\|^2 + \|x_{2n+1} - y_{2n}\|^2\right).\end{aligned}$$

(II) $\lim_{n\to\infty} \|x_{2n} - x^*\|$ *exists and the sequence* $\{x_{2n}\}$ *is bounded.*

Proof. (I) According to $VI(C, A)$ and Algorithm 3.1, it is obvious that $VI(C, A) \subset C \subset T_n$. Setting $n := 2n + 1$ and $x^* \in VI(C, A)$, we have from Lemma 3.6 that

$$\begin{aligned}&\|x_{2n+2} - x^*\|^2\\ \leq&\|w_{2n+1} - x^*\|^2 - \left(1 - \mu\frac{\lambda_{2n+1}}{\lambda_{2n+2}}\right)\left(\|x_{2n+2} - y_{2n+1}\|^2 + \|y_{2n+1} - w_{2n+1}\|^2\right).\end{aligned} \tag{3.2}$$

Applying (2.1) to (3.1), we obatin

$$\begin{aligned}\|w_{2n+1} - x^*\|^2 =& (1 + \alpha_{2n+1})\|x_{2n+1} - x^*\|^2 - \alpha_{2n+1}\|x_{2n} - x^*\|^2\\ &+ \alpha_{2n+1}(1 + \alpha_{2n+1})\|x_{2n+1} - x_{2n}\|^2.\end{aligned} \tag{3.3}$$

Besides, using Lemma 3.6 again, and setting $n := 2n$, we have

$$\begin{aligned}&\|x_{2n+1} - x^*\|^2\\ \leq&\|x_{2n} - x^*\|^2 - \left(1 - \mu\frac{\lambda_{2n}}{\lambda_{2n+1}}\right)\left(\|x_{2n+1} - y_{2n}\|^2 + \|y_{2n} - x_{2n}\|^2\right).\end{aligned} \tag{3.4}$$

Adding (3.3) and (3.4), we see that

$$\begin{aligned}&\|w_{2n+1} - x^*\|^2\\ \leq&\|x_{2n} - x^*\|^2 + \alpha_{2n+1}(1 + \alpha_{2n+1})\|x_{2n+1} - x_{2n}\|^2\\ &- (1 + \alpha_{2n+1})\left(1 - \mu\frac{\lambda_{2n}}{\lambda_{2n+1}}\right)\left(\|x_{2n+1} - y_{2n}\|^2 + \|y_{2n} - x_{2n}\|^2\right).\end{aligned} \tag{3.5}$$

Further, put (3.5) into (3.2), we obtain

$$\begin{aligned}&\|x_{2n+2}-x^*\|^2\\ \le&\|x_{2n}-x^*\|^2-\left(1-\mu\frac{\lambda_{2n+1}}{\lambda_{2n+2}}\right)\left(\|w_{2n+1}-y_{2n+1}\|^2+\|x_{2n+2}-y_{2n+1}\|^2\right)\\ &-(1+\alpha_{2n+1})\left(1-\mu\frac{\lambda_{2n}}{\lambda_{2n+1}}-2\alpha_{2n+1}\right)\left(\|x_{2n}-y_{2n}\|^2+\|x_{2n+1}-y_{2n}\|^2\right).\end{aligned}$$

(II) Using Lemma 3.5 and $\alpha_n\in[0,\alpha]\subset[0,\frac{1-\mu}{2})$, we observe that $1-\mu\frac{\lambda_{2n}}{\lambda_{2n+1}}-2\alpha_{2n+1}\ge 1-\mu\frac{\lambda_{2n}}{\lambda_{2n+1}}-2\alpha$ and

$$\lim_{n\to\infty}\left(1-\mu\frac{\lambda_{2n+1}}{\lambda_{2n+2}}\right)=1-\mu>0,\ \lim_{n\to\infty}\left(1-\mu\frac{\lambda_{2n}}{\lambda_{2n+1}}-2\alpha\right)=1-\mu-2\alpha>0.$$

Therefore, there exists a nonnegative integer N such that $1-\mu\frac{\lambda_{2n}}{\lambda_{2n+1}}-2\alpha>0$, $n\ge N$. In the light of Lemma 3.6, we have $\|x_{2n+2}-x^*\|\le\|x_{2n}-x^*\|$, $n\ge N$, which implies that $\lim_{n\to\infty}\|x_{2n}-x^*\|$ exists and $\{x_{2n}\}$ is bounded. □

Lemma 3.8. *Assume that Conditions (A1) and (A2) hold and $\{x_n\}$ is generated by Algorithm 3.1. Then all weak cluster points of $\{x_{2n}\}$ are contained in $VI(C,A)$.*

Proof. First, using Lemma 3.7 and setting $n\to\infty$, we obtain

$$\lim_{n\to\infty}\|x_{2n}-y_{2n}\|=\lim_{n\to\infty}\|x_{2n+1}-y_{2n}\|=0. \tag{3.6}$$

In view of the Lipschitz continuity of A, we have

$$\lim_{n\to\infty}\|Ax_{2n}-Ay_{2n}\|\le\lim_{n\to\infty}L\|x_{2n}-y_{2n}\|=0. \tag{3.7}$$

Further, without loss of generality, we assume that a subsequence $\{x_{2n_i}\}$ of $\{x_{2n}\}$ converges weakly $\tilde{x}\in\mathcal{H}$. It follows from $\lim_{n\to\infty}\|x_{2n}-y_{2n}\|=0$ that $\{y_{2n_i}\}$ also converges weakly to $\tilde{x}$. Since C is closed and convex subset in $\mathcal{H}$ and $\{y_{2n_i}\}$ is contained in C, then $\tilde{x}\in C$. Using the definition of y_n and $w_{2n}=x_{2n}$, we have $\langle x_{2n_i}-\lambda_{2n_i}Ax_{2n_i}-y_{2n_i},x-y_{2n_i}\rangle\le 0$, $\forall x\in C$. Equivalently, we can derive the following form

$$\frac{1}{\lambda_{2n_i}}\langle x_{2n_i}-y_{2n_i},x-y_{2n_i}\rangle+\langle Ax_{2n_i},y_{2n_i}-x_{2n_i}\rangle\le\langle Ax_{2n_i},x-x_{2n_i}\rangle,\ \forall x\in C. \tag{3.8}$$

Since A is Lipschitz continuous, then $\{Ax_{2n_i}\}$ is bounded. Furthermore, letting $i\to\infty$ in inequality (3.8), we obtain

$$\liminf_{i\to\infty}\langle Ax_{2n_i},x-x_{2n_i}\rangle\ge 0,\ \forall x\in C. \tag{3.9}$$

Moreover, by (3.6), (3.7) and (3.9), we obtain

$$\langle Ay_{2n_i},x-y_{2n_i}\rangle=\langle Ay_{2n_i}-Ax_{2n_i},x-x_{2n_i}\rangle+\langle Ax_{2n_i},x-x_{2n_i}\rangle+\langle Ay_{2n_i},x_{2n_i}-y_{2n_i}\rangle$$

and

$$\liminf_{i\to\infty}\langle Ay_{2n_i},x-y_{2n_i}\rangle\ge 0,\ \forall x\in C. \tag{3.10}$$

Next, choose a decreasing positive numbers sequence $\{\theta_i\}$ such that $\lim_{i\to\infty}\theta_i = 0$. For each $i \geq 1$, we can always find the smallest positive integer M_i such that

$$\langle Ay_{2n_j}, x - y_{2n_j}\rangle + \theta_i \geq 0,\ \forall j \geq M_i, \tag{3.11}$$

where the existence of M_i follows from (3.10). Meanwhile, since $\{\theta_i\}$ is decreasing, then $\{M_i\}$ is increasing. On the other hand, for each i, suppose $Ay_{2n_{M_i}} \neq 0$, otherwise, $y_{2n_{M_i}}$ is a solution of VI. Setting $z_{2n_{M_i}} = \frac{Ay_{2n_{M_i}}}{\|Ay_{2n_{M_i}}\|^2}$, one get $\langle Ay_{2n_{M_i}}, z_{2n_{M_i}}\rangle = 1$ for each i. Naturally, it follows from (3.11) that $\langle Ay_{2n_{M_i}}, x+\theta_i z_{2n_{M_i}} - y_{2n_{M_i}}\rangle \geq 0$. By the fact that the mapping A is pseudomonotone, we have

$$\langle A(x+\theta_i z_{2n_{M_i}}), x+\theta_i z_{2n_{M_i}} - y_{2n_{M_i}}\rangle \geq 0. \tag{3.12}$$

Suppose $A\tilde{x} \neq 0$ (otherwise, $\tilde{x}$ is a solution). By means of Condition (A3), we obtain $0 < \|Ax^*\| \leq \liminf_{i\to\infty}\|Ay_{2n_i}\|$. Since $\{y_{2n_{M_i}}\} \subset \{y_{2n_i}\}$ and $\theta_i \to 0$ as $i \to \infty$, we obtain

$$0 \leq \limsup_{i\to\infty}\left\|\theta_i z_{2n_{M_i}}\right\| = \limsup_{i\to\infty}\left(\frac{\theta_i}{\|Ay_{2n_i}\|}\right) \leq \frac{\limsup_{i\to\infty}\theta_i}{\liminf_{i\to\infty}\|Ay_{2n_i}\|} = 0,$$

which implies that $\lim_{i\to\infty}\theta_i z_{2n_{M_i}} = 0$. Further, for all $x \in C$, we have

$$\begin{aligned}\langle Ax, x-\tilde{x}\rangle &= \liminf_{i\to\infty}\langle Ax, x - y_{2n_{M_i}}\rangle \\ &\geq \liminf_{i\to\infty}\langle Ax - A(x+\theta_i z_{2n_{M_i}}), x+\theta_i z_{2n_{M_i}} - y_{2n_{M_i}}\rangle - \liminf_{i\to\infty}\theta_i\langle Ax, z_{2n_{M_i}}\rangle = 0.\end{aligned} \tag{3.13}$$

As a consequence, we have $\tilde{x} \in VI(C, A)$ by Lemma 2.2. □

Theorem 3.9. *Assume that Conditions (A1), (A2), and (A3) hold. The sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to a point in $VI(C, A)$.*

Proof. Based on the results in Lemmas 3.7 and 3.8, we can obtain that the whole sequence $\{x_{2n}\}$ converges weakly to a point in $VI(C, A)$ by Lemma 2.3. In addition, suppose that $\{x_{2n}\}$ converges weakly to $\tilde{x} \in VI(C, A)$ and $\{x_{2n}\}$ converges weakly to $\bar{x} \in VI(C, A)$. Then $\tilde{x} = \bar{x}$ by Remark 5.1.12 in [23], which implies that $\tilde{x}$ is unique. Last, for all $y \in \mathcal{H}$, we get

$$\begin{aligned}|\langle x_{2n+1} - \tilde{x}, y\rangle| &\leq |\langle x_{2n} - \tilde{x}, y\rangle| + |\langle x_{2n+1} - x_{2n}, y\rangle| \\ &\leq |\langle x_{2n} - \tilde{x}, y\rangle| + \|x_{2n+1} - x_{2n}\|\|y\| \to 0, \text{ as } n \to \infty\end{aligned}$$

Therefore, $\{x_{2n+1}\}$ converges weakly to $\tilde{x}$ in $VI(C, A)$. In summary, $\{x_n\}$ converges weakly to a point $\tilde{x} \in VI(C, A)$. This completes the proof. □

3.2. **Convergence of Algorithm 3.2.**

Lemma 3.10 ([12])**.** *Suppose that Conditions (A1) and (A2†) hold.*

(I) *The Armijo line-search stepsize sequence $\{\lambda_n\}$ in Algorithm 3.2 is well-defined and $\lambda_n \leq \gamma$ for all $n \geq 1$.*

(II) *For any $n \geq 1$ and $x^* \in VI(C, A)$, the sequence $\{x_n\}$ generated by Algorithm 3.2 has the following property:*

$$\|x_{n+1} - x^*\|^2 \leq \|w_n - x^*\|^2 - (1-\mu)\left(\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2\right).$$

Lemma 3.11. *Suppose that Conditions (A1) and (A2†) hold. The sequence $\{x_n\}$ generated by Algorithm 3.2 has the following properties:*

(I) *For any $n \geq 1$ and $x^* \in VI(C, A)$,*

$$\|x_{2n+2} - x^*\|^2 \leq \|x_{2n} - x^*\|^2 - (1-\mu)\left(\|w_{2n+1} - y_{2n+1}\|^2 + \|x_{2n+2} - y_{2n+1}\|^2\right) - (1 + \alpha_{2n+1})(1 - \mu - 2\alpha_{2n+1})\left(\|x_{2n} - y_{2n}\|^2 + \|x_{2n+1} - y_{2n}\|^2\right).$$

(II) $\lim_{n\to\infty} \|x_{2n} - x^*\|$ *exists and $\{x_{2n}\}$ is bounded.*

Proof. The proof is similar to that of Lemma 3.7. □

Lemma 3.12. *Assume that Conditions (A1) and (A2†) hold and the sequence $\{x_n\}$ is generated by Algorithm 3.2. Then all weak cluster points of $\{x_{2n}\}$ are contained in the solution set $VI(C, A)$.*

Proof. By means of Lemma 3.11, there exists a subsequence $\{x_{2n_i}\}$ of $\{x_{2n}\}$ that converges weakly to a point in $\mathcal{H}$, and $\lim_{n\to\infty} \|x_{2n} - y_{2n}\| = 0$. Following the exists proof as in [12, Lemma 3.2], we have $\liminf_{i\to\infty}\langle Ax_{2n_i}, x - x_{2n_i}\rangle \geq 0$, $\forall x \in C$. Combining the uniformly continuity of A and the proofs of Lemma 3.8, we also assert that all weak cluster points of $\{x_{2n}\}$ are contained in $VI(C, A)$. □

Theorem 3.13. *Assume that Conditions (A1), (A2†), and (A3) hold. The sequence $\{x_n\}$ generated by Algorithm 3.2 converges weakly to a point in $VI(C, A)$.*

Proof. Using the method of Theorem 3.9, it follows from Lemmas 3.11 and 3.12 that $\{x_n\}$ generated by Algorithm 3.2 converges weakly to a point in $VI(C, A)$. This completes the proof. □

4. Alternated inertial subgradient extragradient methods with relaxed terms

Under two different adaptive stepsize rules, two modified subgradient extragradient algorithms are given by combining the alternated inertial method, and the even sequence generated by Algorithm 3.1 (or Algorithm 3.2) has Fejér monotonicity. Theoretically, the weak convergence theorems are proved under the weak constraint assumption for A. But, the shortcoming is that the coefficient α_n in the alternated inertial is always between 0 and $\frac{1-\mu}{2}$, and even approaches 0 due to the value of μ. To solve this problem, we introduce the following relaxed algorithms to weaken the constraint.

Algorithm 4.1. Give $\lambda_1 > 0$, $\mu \in (0, 1)$, $\alpha \in [0, 1]$, $\theta \in (0, \frac{1}{3})$, and a nonnegative real numbers sequences $\{\xi_n\}$ satisfying $\sum_{n=1}^{\infty} \xi_n < +\infty$. Choose initial values x_0, $x_1 \in \mathcal{H}$ and set $n := 1$.
Step 1. Compute

$$w_n = \begin{cases} x_n, & n = \text{even}, \\ x_n + \alpha_n(x_n - x_{n-1}), & n = \text{odd}. \end{cases}$$

Step 2. Compute $y_n = P_C(w_n - \lambda_n Aw_n)$. If $w_n = y_n$ or $Ay_n = 0$, then stop. Otherwise, go to **Step 3**.
Step 3. Compute $u_n = P_{T_n}(w_n - \lambda_n Ay_n)$, where $T_n := \{x \in \mathcal{H} \mid \langle w_n - \lambda_n Aw_n -$

$y_n, x - y_n\rangle \leq 0\}$.
Step 4. Compute $x_{n+1} = (1-\theta)x_n + \theta u_n$. Update

$$\lambda_{n+1} = \begin{cases} \min\left\{\mu\frac{\|w_n-y_n\|^2+\|u_n-y_n\|^2}{2\langle Aw_n-Ay_n,u_n-y_n\rangle}, \lambda_n+\xi_n\right\}, & \text{if } \langle Aw_n - Ay_n, u_n - y_n\rangle > 0, \\ \lambda_n + \xi_n, & \text{otherwise,} \end{cases}$$

and set $n := n+1$ and return **Step 1**.

Algorithm 4.2. Given $l, \mu \in (0,1)$, $\gamma > 0$, $\alpha \in [0,1]$, and $\theta \in (0, \frac{1}{3})$. Choose initial values x_0, $x_1 \in \mathcal{H}$ and set $n := 1$.
Step 1. Compute

$$w_n = \begin{cases} x_n, & n = \text{even}, \\ x_n + \alpha_n(x_n - x_{n-1}), & n = \text{odd}. \end{cases}$$

Step 2. Compute $y_n = P_C(w_n - \lambda_n Aw_n)$, where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m satisfying $\gamma l^m \|Aw_n - Ay_n\| \leq \mu \|w_n - y_n\|$. If $w_n = y_n$ or $Ay_n = 0$, then stop. Otherwise, go to **Step 3**.
Step 3. Compute $u_n = P_{T_n}(w_n - \lambda_n Ay_n)$, where $T_n := \{x \in \mathcal{H} \mid \langle w_n - \lambda_n Aw_n - y_n, x - y_n\rangle \leq 0\}$.
Step 4. Compute $x_{n+1} = (1-\theta)x_n + \theta u_n$ and set $n := n+1$ and return **Step 1**.

Similarly, the corresponding convergence theorems are given below and verified with the lemmas and proofs in Section 3.

Theorem 4.3. *Assume that Conditions (A1), (A2), and (A3) hold. The sequence $\{x_n\}$ generated by Algorithm 4.1 converges weakly to a point in $VI(C,A)$.*

Proof. Let $x^* \in VI(C,A)$. From x_{n+1} and (2.1), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= (1-\theta)\|x_n - x^*\|^2 + \theta\|u_n - x^*\|^2 - \theta(1-\theta)\|x_n - u_n\|^2 \\ &= (1-\theta)\|x_n - x^*\|^2 + \theta\|u_n - x^*\|^2 - \frac{1-\theta}{\theta}\|x_{n+1} - x_n\|^2. \end{aligned} \tag{4.1}$$

Further, it follows from Lemma 3.6 that

$$\|u_n - x^*\|^2 \leq \|w_n - x^*\|^2 - \Big(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\Big)(\|w_n - y_n\|^2 + \|u_n - y_n\|^2). \tag{4.2}$$

Applying (4.2) to (4.1), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \leq &(1-\theta)\|x_n - x^*\|^2 + \theta\|w_n - x^*\|^2 \\ &- \theta\Big(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\Pi_n - \frac{1-\theta}{\theta}\|x_{n+1} - x_n\|^2, \end{aligned} \tag{4.3}$$

where $\Pi_n = \|w_n - y_n\|^2 + \|u_n - y_n\|^2$. Setting $n := 2n+1$ yields

$$\begin{aligned} \|x_{2n+2} - x^*\|^2 \leq &(1-\theta)\|x_{2n+1} - x^*\|^2 + \theta\|w_{2n+1} - x^*\|^2 \\ &- \theta\Big(1 - \mu\frac{\lambda_{2n+1}}{\lambda_{2n+2}}\Big)\Pi_{2n+1} - \frac{1-\theta}{\theta}\|x_{2n+2} - x_{2n+1}\|^2. \end{aligned} \tag{4.4}$$

In addition, using the definition of w_n and setting $n := 2n$ in (4.3), we have

$$\begin{aligned} &\|w_{2n+1} - x^*\|^2 \\ =&(1+\alpha)\|x_{2n+1} - x^*\|^2 - \alpha\|x_{2n} - x^*\|^2 + \alpha(1+\alpha)\|x_{2n+1} - x_{2n}\|^2 \end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\|x_{2n+1}-x^*\|^2 \leq&(1-\theta)\|x_{2n}-x^*\|^2+\theta\|w_{2n}-x^*\|^2\\&-\theta\Big(1-\mu\frac{\lambda_{2n}}{\lambda_{2n+1}}\Big)\Pi_{2n}-\frac{1-\theta}{\theta}\|x_{2n+1}-x_{2n}\|^2\\=&\|x_{2n}-x^*\|^2-\theta\Big(1-\mu\frac{\lambda_{2n}}{\lambda_{2n+1}}\Big)\Pi_{2n}-\frac{1-\theta}{\theta}\|x_{2n+1}-x_{2n}\|^2.\end{aligned}\tag{4.6}$$

So, (4.5) and (4.6) are applied to (4.4) to obtain

$$\begin{aligned}&\|x_{2n+2}-x^*\|^2\\\leq&(1+\theta\alpha)\|x_{2n+1}-x^*\|^2-\theta\alpha\|x_{2n}-x^*\|^2+\theta\alpha(1+\alpha)\|x_{2n+1}-x_{2n}\|^2\\&-\theta\Big(1-\mu\frac{\lambda_{2n+1}}{\lambda_{2n+2}}\Big)\Pi_{2n+1}-\frac{1-\theta}{\theta}\|x_{2n+2}-x_{2n+1}\|^2\\\leq&\|x_{2n}-x^*\|^2-\Big(\frac{(1-\theta)(1+\theta\alpha)}{\theta}-\theta\alpha(1+\alpha)\Big)\|x_{2n+1}-x_{2n}\|^2\\&-(1+\theta\alpha)\theta\Big(1-\mu\frac{\lambda_{2n}}{\lambda_{2n+1}}\Big)\Pi_{2n}-\theta\Big(1-\mu\frac{\lambda_{2n+1}}{\lambda_{2n+2}}\Big)\Pi_{2n+1}-\frac{1-\theta}{\theta}\|x_{2n+2}-x_{2n+1}\|^2.\end{aligned}$$

By Lemma 3.5, $\alpha\in[0,1]$, $\mu\in(0,1)$ and $\theta\in(0,\frac{1}{3})$, we obtain $\frac{(1-\theta)(1+\theta\alpha)}{\theta}-\theta\alpha(1+\alpha)>0$, $\lim_{n\to\infty}\big(1-\mu\frac{\lambda_{2n+1}}{\lambda_{2n+2}}\big)=1-\mu>0$, and $\lim_{n\to\infty}\big(1-\mu\frac{\lambda_{2n}}{\lambda_{2n+1}}\big)=1-\mu>0$. In other words, there exists a nonnegative integer N_0 such that $1-\mu\frac{\lambda_{2n}}{\lambda_{2n+1}}>0$ and $1-\mu\frac{\lambda_{2n+1}}{\lambda_{2n+2}}>0$ for all $n\geq N_0$. Then, $\|x_{2n+2}-x^*\|\leq\|x_{2n}-x^*\|$, $n\geq N_0$, which implies that $\lim_{n\to\infty}\|x_{2n}-x^*\|$ exists and $\{x_{2n}\}$ is bounded. In addition, similar to the proofs of Lemma 3.8 and Theorem 3.9, it is easy to see that $\{x_n\}$ generated by Algorithm 4.1 converges weakly to a point in $VI(C,A)$. □

Theorem 4.4. *Assume that Conditions (A1), (A2†), and (A3) hold. The sequence $\{x_n\}$ generated by Algorithm 4.2 converges weakly to a point in $VI(C,A)$.*

Proof. Fix $x^*\in VI(C,A)$. It follows from Lemma 3.10 that

$$\|u_n-x^*\|^2\leq\|w_n-x^*\|^2-(1-\mu)(\|w_n-y_n\|^2+\|u_n-y_n\|^2).\tag{4.7}$$

Further, (4.7) is merged into (4.1), we have

$$\begin{aligned}&\|x_{n+1}-x^*\|^2\\\leq&(1-\theta)\|x_n-x^*\|^2+\theta\|w_n-x^*\|^2-\theta(1-\mu)\Psi_n-\frac{1-\theta}{\theta}\|x_{n+1}-x_n\|^2,\end{aligned}\tag{4.8}$$

where $\Psi_n=\|w_n-y_n\|^2+\|u_n-y_n\|^2$. Let $n:=2n+1$ and $n:=2n$. We obtain

$$\begin{aligned}\|x_{2n+2}-x^*\|^2\leq&(1-\theta)\|x_{2n+1}-x^*\|^2+\theta\|w_{2n+1}-x^*\|^2\\&-\theta(1-\mu)\Psi_{2n+1}-\frac{1-\theta}{\theta}\|x_{2n+2}-x_{2n+1}\|^2\end{aligned}\tag{4.9}$$

and

$$\|x_{2n+1}-x^*\|^2\leq\|x_{2n}-x^*\|^2-\theta(1-\mu)\Psi_{2n}-\frac{1-\theta}{\theta}\|x_{2n+1}-x_{2n}\|^2.\tag{4.10}$$

So, (4.5) and (4.10) are applied to (4.9) to see that

$$\begin{aligned}
&\|x_{2n+2} - x^*\|^2 \\
\le&(1+\theta\alpha)\|x_{2n+1} - x^*\|^2 - \theta\alpha\|x_{2n} - x^*\|^2 + \theta\alpha(1+\alpha)\|x_{2n+1} - x_{2n}\|^2 \\
&- \theta(1-\mu)\Psi_{2n+1} - \frac{1-\theta}{\theta}\|x_{2n+2} - x_{2n+1}\|^2 \\
\le&\|x_{2n} - x^*\|^2 - \Big(\frac{(1-\theta)(1+\theta\alpha)}{\theta} - \theta\alpha(1+\alpha)\Big)\|x_{2n+1} - x_{2n}\|^2 \\
&- (1+\theta\alpha)\theta(1-\mu)\Psi_{2n} - \theta(1-\mu)\Psi_{2n+1} - \frac{1-\theta}{\theta}\|x_{2n+2} - x_{2n+1}\|^2.
\end{aligned}$$

By Lemma 3.5, $\alpha \in [0,1]$, $\mu \in (0,1)$ and $\theta \in (0, \frac{1}{3})$, we have $\frac{(1-\theta)(1+\theta\alpha)}{\theta} - \theta\alpha(1+\alpha) > 0$ and $1-\mu > 0$. Hence, $\|x_{2n+2} - x^*\| \le \|x_{2n} - x^*\|$, $n \ge 1$, which implies that $\lim_{n\to\infty}\|x_{2n} - x^*\|$ exists and $\{x_{2n}\}$ is bounded. Naturally, using the proofs as in Lemma 3.12 and Theorem 3.13, we obtain that $\{x_n\}$ generated by Algorithm 4.2 converges weakly to a point in $VI(C, A)$. □

5. Some significant remarks

Remark 5.1. In Algorithms 3.1 and 4.1, the adaptive stepsize sequence $\{\lambda_n\}$ has two remarkable characteristics. First, the selected stepsize in a large number of existing algorithms is usually a fixed constant or a non-increasing sequence. However, the stepsize $\{\lambda_n\}$ in our algorithms is generated adaptively and can be increased with the number of iterations. The other is to insert a summable nonnegative real numbers sequence $\{\xi_n\}$ into the calculation of $\{\lambda_n\}$. In fact, the limit of such a sequence $\{\xi_n\}$ is equal to 0 as $n \to \infty$, which means that this sequence is nonincreasing if n is large enough.

Derived from Remark 5.1, the following cases can be used as a minor modification or a special case of $\{\lambda_n\}$ in Algorithms 3.1 and 4.1.

Case 1: The adaptive stepsize sequence $\{\lambda_n\}$ is updated in the following form

$$\lambda_{n+1} = \begin{cases} \min\Big\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n + \xi_n\Big\}, & \text{if } Aw_n \ne Ay_n, \\ \lambda_n + \xi_n, & \text{otherwise.} \end{cases}$$

From Liu and Yang [15], the same conclusion as Lemma 3.5 is obtained, i.e., the sequence $\{\lambda_n\}$ has the property that $\lim_{n\to\infty}\lambda_n = \lambda$ with $\lambda \in [\min\{\frac{\mu}{L}, \lambda_1\}, \lambda_1 + \xi]$ and $\xi = \sum_{n=1}^{\infty}\xi_n$. Meanwhile, using the same method in [29], Lemma 3.6 is still valid for such a stepsize sequence. Further, the conclusions in Theorems 3.9 and 4.3 are still obtained.

Case 2: For $\{\lambda_n\}$ defined in Algorithm 3.1 and Case 1, if $\xi_n \equiv 0$, then $\{\lambda_n\}$ is converted to a non-increasing stepsize sequence with a lower bound $\min\{\frac{\mu}{L}, \lambda_1\}$, which directly indicates that the limit of $\{\lambda_n\}$ exists. The sequence $\{\lambda_n\}$ in this case is also widely used in iterative algorithms to solve VI. For more information, see the literatures [12,29,30]. Naturally, the same weak convergence is proved when such stepsize sequence $\{\lambda_n\}$ is added to Algorithms 3.1 and 4.1.

Case 3: In Algorithms 3.1 and 4.1, $\{\lambda_n\}$ is a constant sequence, i.e., $\lambda_n \equiv \lambda$ with $\lambda > 0$. Inevitably, the Lipschitz constant L of the involved mapping A must be known. In this case, our results can be still established under the following circumstances.

- $\lambda \in (0, \frac{1}{L})$ and $\{\alpha_n\} \subset [0, \alpha] \subset [0, \frac{1-\lambda L}{2})$ in Algorithm 3.1;
- $\lambda \in (0, \frac{1}{L})$ in Algorithm 4.1.

Remark 5.2. Two different types of stepsize selection in our algorithms have different characteristics. The first one is $\{\lambda_n\}$ in Algorithms 3.1 and 4.1 as described in Remark 5.1, and the other is the Armijo line-search rule in Algorithms 3.2 and 4.2, which is to find a suitable stepsize in each iteration (for this reason, the projection value y_n will be repeatedly calculated until λ_n is found). It is worth noting that under the Armijo line-search rule, mapping A only needs to satisfy uniformly continuity that is a weaker condition of the L-Lipschitz continuous mapping.

References

[1] A. S. Antipin, *On a method for convex programs using a symmetrical modification of the Lagrange function*, Ekon. Mat. Metody **12** (1976), 1164–1173.

[2] N. T. An, P. D. Dong and X. Qin, *Robust feature selection via nonconvex sparsity-based methods,* J. Nonlinear Var. Anal. **5** (2021), 59–77.

[3] F. Alvarez and H. Attouch, *An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping*, Set-Valued Anal. **9** (2001), 3–11.

[4] B. A. Bin Dehaish, *The Fibonacci–Mann iteration for monotone asymptotically pointwise nonexpansive mappings*, J. Fixed Point Theory Appl. **21** (2019): Article ID 17.

[5] L. C. Ceng, *A modified inertial subgradient extragradient method for solving pseudomonotone variational inequalities and common fixed point problems,* Fixed Point Theory **21** (2020), 93–108.

[6] Y. Censor, A. Gibali and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl. **148** (2011), 318–335.

[7] P. Cubiotti, *Existence theorem for the discontinuous generalized quasivariational inequality problem*, J. Optim. Theory Appl. **119** (2003), 623–633.

[8] T. H. Cuong, J. C. Yao and N. D. Yen, *Qualitative properties of the minimum sum-of-squares clustering problem,* Optimization, **69** (2020), 2131–2154.

[9] J. Fan, L. Liu and X. Qin, *A subgradient extragradient algorithm with inertial effects for solving strongly pseudomonotone variational inequalities* Optimization **69** (2020), 2199-–2215.

[10] F. Iutzeler and J. Malick, *On the proximal gradient algorithm with alternated inertia*, J. Optim. Theory Appl. **176** (2018), 688–710.

[11] A. N. Iusem and M. Nasri, *Korpelevichs method for variational inequality problems in Banach spaces*, J. Global Optim. **50** (2011), 59–76.

[12] P. Q. Khanh, D. V. Thong and N. T. Vinh, *Versions of the subgradient extragradient method for pseudomonotone variational inequalities*, Acta Appl. Math. **170** (2020), 319–345.

[13] G. M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Ekon. Mat. Metody **12** (1976), 747–756.

[14] L. Liu, and X. Qin, Strong convergence theorems for solving pseudo-monotone variational inequality problems and applications, Optimization, doi: 10.1080/02331934.2021.1905641.

[15] H. Liu and J. Yang, *Weak convergence of iterative methods for solving quasimonotone variational inequalities*, Comput. Optim. Appl. **77** (2020), 491–508.

[16] J. Mashreghi and M. Nasri, *Forcing strong convergence of Korpelevich ' s method in Banach spaces with its applications in game theory*, Nonlinear Anal. **72** (2010), 2086–2099.

[17] Z. Mu and Y. Peng, *A note on the inertial proximal point method*, Stat. Optim. Inf. Comput. **3** (2015), 241–248.

[18] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Am. Math. Soc. **73** (1967), 591–597.

[19] X. Qin, L. Wang and J. C. Yao, *Inertial splitting method for maximal monotone mappings*, J. Nonlinear Convex Anal. **21** (2020), 2325–2333.
[20] Y. Shehu, *Iterative methods for split feasibility problems in certain Banach spaces*, J. Nonlinear Convex Anal. **16** (2015), 2351–2364.
[21] Y. Shehu and J. C. Yao, *Rate of convergence for inertial iterative method for countable family of certain quasi-nonexpansive mappings,* J. Nonlinear Convex Anal. **21** (2020), 533–541.
[22] Y. Shehu and O. S. Iyiola, *Projection methods with alternating inertial steps for variational inequalities: Weak and linear convergence*, Appl. Numer. Math. **157** (2020), 315–337.
[23] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.
[24] B. Tan, J. Fan and S. Li, *Self-adaptive inertial extragradient algorithms for solving variational inequality problems,* Comput. Appl. Math. **40** (2021), 19.
[25] B. Tan and S. Li, *Strong convergence of inertial Mann algorithms for solving hierarchical fixed point problems*, J. Nonlinear Var. Anal. **4** (2020), 337–355.
[26] B. Tan and S. X. Li, *Modified inertial projection and contraction algorithms with non-monotonic step sizes for solving variational inequalities and their applications*, Optimization, doi: 10.1080/02331934.2022.2123705.
[27] D. V. Thong, Y. Shehu and O. S. Iyiola, *Weak and strong convergence theorems for solving pseudo-monotone variational inequalities with non-Lipschitz mappings*, Numer. Algo. **84** (2020), 795–823.
[28] D. V. Thong and P. T. Vuong, *Improved subgradient extragradient methods for solving pseudomonotone variational inequalities in Hilbert spaces*, Appl. Numer. Math. **163** (2021), 221–238.
[29] D. V. Thong, J. Yang, Y. J. Cho and T. M. Rassias, *Explicit extragradient-like method with adaptive stepsizes for pseudomonotone variational inequalities*, Optim. Lett. **15** (2021), 2181–2199.
[30] J. Yang and H. Liu, *Strong convergence result for solving monotone variational inequalities in Hilbert space*, Numer. Algo. **80** (2019), 741–752.
[31] S. Zeng, S. Migorski, Z. Liu and J.C. Yao, *Convergence of a generalized penalty method for variational-hemivariational inequalities,* Commun. Nonlinear Sci. Numer. Simul. **92** (2021), 105476.
[32] Z. Zhou, B. Tan and S. Li, *An accelerated hybrid projection method with a self-adaptive step-size sequence for solving split common fixed point problems*, Math. Methods Appl. Sci. **44** (2021), 7294–7303.

Manuscript received May 8, 2021
revised August 30, 2021

Z. Zhou
College of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong 643000, China
E-mail address: zhouzheng2272@163.com

B. Tan
Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China
E-mail address: bingtan72@gmail.com

S.Y. Cho
(Corresponding Author) Department of Human Health Care, Gyeongsang National University, Jinju-Si, Korea
E-mail address: chosy@gnu.ac.kr