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Research article

Two self-adaptive inertial projection algorithms for solving split variational inclusion problems

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Abstract: This paper is to analyze the approximation solution of a split variational inclusion problem in the framework of Hilbert spaces. For this purpose, inertial hybrid and shrinking projection algorithms are proposed under the effect of a self-adaptive stepsize which does not require information of the norms of the given operators. The strong convergence properties of the proposed algorithms are obtained under mild constraints. Finally, a numerical experiment is given to illustrate the performance of proposed methods and to compare our algorithms with an existing algorithm.

Keywords: self-adaptive stepsize; projection algorithm; inertial technique; split variational inclusion problem
Mathematics Subject Classification: 47H05, 49J40, 65K10, 65Y10

1. Introduction

Inspired by the split variational inequality problem proposed by Censor et al. [1], Moudafi [2] introduced a more general form of this problem, that is, the split monotone variational inclusion problem. It is worth noting that an important special case of the split monotone variation inclusion problem is the split variational inclusion problem (for short, SVIP), which is to find a zero of a maximal monotone mapping in one space, and the image of which under a given bounded linear transformation is a zero of another maximal monotone mapping in another space. As well as, the split variational inclusion problem is also a generalized form of many problems, such as the split variational inequality problem, the split minimization problem, the split equilibrium problem, the split saddle point problem and the split feasibility problem; see, for instance, [2–7] and the references therein. As applications, these problems are also widely applied to radiation therapy treatment planning, image recovery and signal recovery; for detail, we refer to [8–10]. In the SVIP, when the two spaces are the same and the given bounded linear operator is an identity mapping, it is equivalent to the well-known common solution problem, i.e., the common solution of two variational inclusion problems. Naturally, common solution problems of other aspects can be obtained, such as the variational inequality problem, the minimization problem and the equilibrium problem. In general, the above common solution problems can be regarded as the distinguished convex feasibility problem.

In particular, finding the zero of a maximal monotone mapping is known as the variational inclusion problem (for short, VIP), which is a special case of the SVIP. Since the resolvent mapping of the maximal monotone mapping is an important tool for solving the VIP, the variational inclusion problem and the split variational inclusion problem has obtained quite a few remarkable results; for example, see, [11–16]. On the other hand, based on the idea of the time implicit discretization of a second-order differential equation, Alvarez and Attouch [17] introduced an inertial proximal point algorithm to approximate a solution of the VIP. Under the effect of the inertial technique, the iterative sequence of the SVIP and other problems rapidly converges to the approximation solution of the corresponding problems, such as the split variational inclusion problem [3, 6, 7, 16, 18], the split common fixed point problem [10, 19], the monotone inclusion problem [20, 21], the fixed point problem [22–24] and the variational inequality problem [25–28].

From the existing results of the split variational inclusion problem, we find that it is easy to get the weak convergence property, and sometimes its strong convergence is proved in the case of other methods, such as the viscosity method, the Halpern method, the Mann-type method, the hybrid steepest descent method, and so on; for detail, see [3, 4, 6, 15]. Unfortunately, the stepsize sequences in these existing results often depend on the norm of bounded linear operators. Hence, the work of this paper can be summarized in two aspects. The first one is to construct new inertial iterative algorithms that converge strongly to a solution of the SVIP. For this purpose, we consider two projection methods in our algorithms, namely hybrid projection [29] and shrinking projection [30]. The second one is to design a new stepsize sequence which does not need prior knowledge of the bounded linear operator in our algorithms.

The remainder of this paper is organized as follows. Section 2 introduces the split variational inclusion problem and some preliminaries. Two new iterative algorithms and their convergence theorems for the SVIP are proposed in Section 3. Theoretical applications on other mathematical problems are given in Section 4. Finally, in Section 5, the validity and authenticity of the convergence behavior of the proposed algorithms are demonstrated by some applicable numerical examples.

2. State of problem and preliminaries

2.1. Split variational inclusion problem

Let H_1 and H_2 be Hilbert spaces, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mappings. Let $A : H_1 \to H_2$ be a bounded linear operator. The split variational inclusion problem is to find a point $x^* \in H_1$ such that

$$0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*). \tag{SVIP}$$

The solution set of the SVIP is denoted by Ω, i.e.,

$$\Omega := \{x^* \in H_1 : 0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*)\}.$$

2.2. Preliminaries

To standardize, the notations $\to$ and $\rightharpoonup$ stand for strong convergence and weak convergence, respectively. The symbol $Fix(S)$ denotes the fixed point set of a mapping S, and $\omega_w(x_n)$ represents the set of weak cluster point of a sequence $\{x_n\}$. Let H be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$ induced by the inner product. Let $B : H \to 2^H$ be a set-valued mapping with domain $\mathcal{D}(B) = \{x \in H : B(x) \neq \emptyset\}$ and graph $\mathcal{G}(B) = \{(x, w) \in H \times H : x \in \mathcal{D}(B), w \in B(x)\}$. Recall that a mapping $B : H \to 2^H$ is monotone if and only if $\langle x - y, w - v\rangle \geq 0$ for any $w \in B(x)$ and $v \in B(y)$. A monotone mapping $B : H \to 2^H$ is maximal, that is, the graph $\mathcal{G}(B)$ is not properly contained in the graph of any other monotone mapping. In this case, B is a maximal monotone mapping if and only if for any $(x, w) \in \mathcal{G}(B)$ and $(y, v) \in H \times H$, $\langle x - y, w - v\rangle \geq 0$ implies $v \in B(y)$. In addition, the metric projection from H onto C, denoted P_C, is defined as $P_C x = \operatorname{argmin}_{y\in C} \|x - y\|,\ \forall x \in H$. Naturally, the following properties of P_C hold:

$$\langle P_C x - x, P_C x - y\rangle \leq 0, \forall y \in C \Leftrightarrow \|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2.$$

Lemma 2.1 ([31, 32]). *The resolvent mapping J^B_β of a maximal monotone mapping B with $\beta > 0$ is defined as $J^B_\beta(x) = (I + \beta B)^{-1}(x), \forall x \in H$. The following properties associated with J^B_β hold.*

(1) The mapping J^B_β is single-valued and firmly nonexpansive;

(2) The fixed point set of J^B_β is equivalent to

$$B^{-1}(0) = \{x \in \mathcal{D}(B) : 0 \in B(x)\}.$$

Lemma 2.2 ([33]). *Let $B : \mathcal{D}(B) \subset H \to 2^H$ be a maximal monotone mapping. For any $0 < \beta \leq r$, we have*

$$\|x - J^B_\beta(x)\| \leq 2\|x - J^B_r(x)\|,\ \forall x \in H.$$

Definition 2.3. The mapping $S : H \to H$ is said to be

(1) nonexpansive if $\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in H$;

(2) firmly nonexpansive if $\|Sx - Sy\|^2 \leq \langle Sx - Sy, x - y\rangle, \forall x, y \in H$.

Remark 2.4. If S is a firmly nonexpansive mapping, then it is also nonexpansive and $I - S$ is a firmly nonexpansive mapping.

Lemma 2.5 ([32]). *Let C be a nonempty closed convex subset of H and $S : C \to C$ be a nonexpansive mapping with $Fix(S) \neq \emptyset$. $I - S$ is demiclosed at zero, that is, for any sequence $\{x_n\}$ in C, satisfying $x_n \rightharpoonup x$ and $(I - S)x_n \to 0$, then $x \in Fix(S)$.*

Lemma 2.6 ([34]). *Let C be a nonempty closed convex subset of H. Let a sequence $\{x_n\}$ in H and $u = P_C v,\ v \in H$. If $\omega_w(x_n) \subset C$ and $\|x_n - v\| \leq \|u - v\|$, then $\{x_n\}$ converges strongly to u.*

3. Self-adaptive inertial hybrid and shrinking projection algorithms

Combining the inertial technique with the projection methods, two types of projection algorithms are given for approximating a solution of the split variational inclusion problem. Before this, we always assume that the following conditions are satisfied:

(C1) H_1, H_2 are two Hilbert spaces and $A : H_1 \to H_2$ is a bounded linear operator with the adjoint operator A^*;

(C2) $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are two set-valued maximal monotone mappings.

An inertial hybrid projection algorithm and an inertial shrinking projection algorithm are introduced below and the strong convergence of these algorithms are guaranteed by the following appropriate parameter conditions:

(P1) $\{\alpha_n\} \subset [a, b] \subset (-\infty, \infty)$ and $\{\beta_n\} \subset (0, \infty)$ with $\inf_n\{\beta_n\} \geq \beta > 0$;

(P2) If $Az_n \notin B_2^{-1}0$, the stepsize $\gamma_n = \frac{\sigma_n \|(I-J_{\beta_n}^{B_2})Az_n\|^2}{\|A^*(I-J_{\beta_n}^{B_2})Az_n\|^2}$ with $0 < c \leq \sigma_n \leq d < 2$. Otherwise, $\gamma_n = 0$.

3.1. The strong convergence of inertial hybrid projection algorithm

Algorithm 3.1 Given appropriate parameter sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$, for any $x_0,\ x_1 \in H_1$, the sequence $\{x_n\}$ is constructed by the following iterative form.

$$\begin{cases} z_n = x_n + \alpha_n(x_n - x_{n-1}), \\ u_n = J_{\beta_n}^{B_1}\left(z_n - \gamma_n A^*(I - J_{\beta_n}^{B_2})Az_n\right), \\ C_n = \{x \in H_1 : \|u_n - x\|^2 \leq \|z_n - x\|^2 - \theta_n\}, \\ Q_n = \{x \in H_1 : \langle x_n - x_1, x_n - x\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_1,\ n \geq 1, \end{cases}$$

where

$$\theta_n = \gamma_n \left(2\|(I - J_{\beta_n}^{B_2})Az_n\|^2 - \gamma_n \|A^*(I - J_{\beta_n}^{B_2})Az_n\|^2\right).$$

Lemma 3.1. *Assumed that (C1)-(C2) hold. For any $\gamma_n > 0$, $\beta_n > 0$ and set $u_n = J_{\beta_n}^{B_1}(z_n - \gamma_n A^*(I - J_{\beta_n}^{B_2})Az_n)$, $n \geq 1$, we have*

$$\|u_n - x\|^2 \leq \|z_n - x\|^2 - \gamma_n \left(2\|(I - J_{\beta_n}^{B_2})Az_n\|^2 - \gamma_n\|A^*(I - J_{\beta_n}^{B_2})Az_n\|^2\right),\ \forall x \in \Omega.$$

Proof. Choose any $x \in \Omega$, we have $x \in B_1^{-1}(0)$ and $Ax \in B_2^{-1}(0)$. Since $J_{\beta_n}^{B_1}$, $J_{\beta_n}^{B_2}$ and $I - J_{\beta_n}^{B_2}$ are firmly nonexpansive mappings, we have

$$\begin{aligned} &\|u_n - x\|^2 \\ \leq&\|z_n - \gamma_n A^*(I - J_{\beta_n}^{B_2})Az_n - x\|^2 \\ =&\|z_n - x\|^2 + \gamma_n^2\|A^*(I - J_{\beta_n}^{B_2})Az_n\|^2 - 2\gamma_n\langle z_n - x, A^*(I - J_{\beta_n}^{B_2})Az_n\rangle \\ \leq&\|z_n - x\|^2 + \gamma_n^2\|A^*(I - J_{\beta_n}^{B_2})Az_n\|^2 - 2\gamma_n\|(I - J_{\beta_n}^{B_2})Az_n\|^2 \\ =&\|z_n - x\|^2 - \gamma_n\left(2\|(I - J_{\beta_n}^{B_2})Az_n\|^2 - \gamma_n\|A^*(I - J_{\beta_n}^{B_2})Az_n\|^2\right). \end{aligned}$$

The proof is complete. □

Theorem 3.2. *Assumed that (C1)-(C2) and (P1)-(P2) hold. If the solution set Ω is nonempty, then $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $x^* = P_\Omega x_1 \in \Omega$.*

Proof. Step 1: Firstly, we show that $P_{C_n \bigcap Q_n}$ is well defined and $\Omega \subset C_n \bigcap Q_n$.

From the definition of C_n and Q_n, it is obvious that the sets C_n and Q_n are convex and closed, which implies that $P_{C_n \bigcap Q_n}$ is well defined. For any $p \in \Omega$, it follows from Lemma 3.1 that $\Omega \subset C_n$. In addition, $Q_1 = \{x \in H_1 : \langle x_1 - x_1, x_1 - x\rangle \leq 0\} = H_1$, then $\Omega \subset Q_1$. Further, suppose $\Omega \subset C_{n-1} \bigcap Q_{n-1}$, using the property of metric projection and $x_n = P_{C_{n-1} \bigcap Q_{n-1}} x_1$, we get

$$\langle x_n - x_1, x_n - x\rangle \leq 0\,,\ \forall x \in C_{n-1} \cap Q_{n-1};$$

$$\langle x_n - x_1, x_n - p\rangle \leq 0\,,\ \forall p \in \Omega.$$

This implies that $\Omega \subset Q_n$. Hence, $\Omega \subset C_n \bigcap Q_n$, $n \geq 1$.

Step 2: Afterwards, we show that iterative sequence $\{x_n\}$ is bounded and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Since Ω is a nonempty closed convex set, there exists a point $x^* = P_\Omega x_1 \in \Omega$. Combining $x_{n+1} = P_{C_n \cap Q_n} x_1$ with $\Omega \subset C_n \cap Q_n$, we have $\|x_1 - x_{n+1}\| \leq \|x_1 - x^*\|$. Accordingly, the sequence $\{\|x_1 - x_n\|\}$ is bounded, i.e., the sequence $\{x_n\}$ is bounded. From the definition of Q_n and $x_{n+1} = P_{C_n \cap Q_n} x_1 \in Q_n$, we get $x_n = P_{Q_n} x_1$ and $\|x_1 - x_n\| \leq \|x_1 - x_{n+1}\|$. These indicate that $\lim_{n\to\infty} \|x_1 - x_n\|$ exists. Further, it follows from the property of metric projection P_{Q_n} that

$$\|x_n - x_{n+1}\|^2 \leq \|x_1 - x_{n+1}\|^2 - \|x_1 - x_n\|^2.$$

This implies $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

Step 3: Lastly, we prove that the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega x_1$.

From the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ that converges weakly to q, for any $q \in \omega_w(x_n)$. Furthermore, $\|z_n - x_n\| = \alpha_n \|x_n - x_{n-1}\| \to 0$, as $n \to \infty$. This implies that $\{z_n\}$ is bounded and $z_{n_l} \rightharpoonup q$. From (P2) and Algorithm 3.1, we have $\|u_n - x_{n+1}\|^2 \leq \|z_n - x_{n+1}\|^2 - \theta_n \leq \|z_n - x_{n+1}\|^2$. In addition,

$$\begin{aligned}
\|u_n - z_n\| &\leq \|u_n - x_n\| + \|x_n - z_n\| \\
&\leq \|u_n - x_{n+1}\| + \|x_n - x_{n+1}\| + \|x_n - z_n\| \\
&\leq 2\|z_n - x_n\| + 2\|x_n - x_{n+1}\| \to 0,\ n \to \infty.
\end{aligned}$$

Hence, the sequence $\{u_n\}$ is bounded. Using Lemma 3.1, for any $p \in \Omega$,

$$\begin{aligned}
\theta_n &\leq \|z_n - p\|^2 - \|u_n - p\|^2 \\
&\leq (\|z_n - p\| - \|u_n - p\|)(\|z_n - p\| + \|u_n - p\|) \\
&\leq \|z_n - u_n\|(\|z_n - p\| + \|u_n - p\|) \to 0,\ n \to \infty.
\end{aligned}$$

If $Az_n \notin B_2^{-1}0$, from the definition of θ_n, $\lim_{n\to\infty} \|(I - J_{\beta_n}^{B_2})Az_n\| = 0$. On the other hand, from the definition of u_n and the firmly nonexpansive property of $J_{\beta_n}^{B_1}$, we obtain

$$\|u_n - J_{\beta_n}^{B_1} z_n\| \leq \|\gamma_n A^*(I - J_{\beta_n}^{B_2})Az_n\| \leq \gamma_n \|A\| \|(I - J_{\beta_n}^{B_2})Az_n\| \to 0,\ \text{as } n \to \infty.$$

Therefore, we also have $\lim_{n\to\infty} \|z_n - J_{\beta_n}^{B_1} z_n\| = 0$. Further, using Lemma 2.2 and $\inf_n\{\beta_n\} \geq \beta > 0$, we have

$$\|z_n - J_\beta^{B_1} z_n\| \leq 2\|z_n - J_{\beta_n}^{B_1} z_n\| \to 0,\ \|(I - J_\beta^{B_2})Az_n\| \leq 2\|(I - J_{\beta_n}^{B_2})Az_n\| \to 0.$$

Since A is a bounded linear operator, we get $Az_{n_l} \rightharpoonup Aq$. By Remark 2.4 and Lemma 2.5, it follows that $q \in Fix(J_\beta^{B_1})$ and $Aq \in Fix(J_\beta^{B_2})$, that is, $q \in \Omega$. Meanwhile, if $Az_n \in B_2^{-1}0$, we can also get the same result. In summary, we have $\omega_w(x_n) \subset \Omega$ and $\|x_n - x_1\| \leq \|x^* - x_1\|$. By virtue of Lemma 2.6, we obtain that $\{x_n\}$ converges strongly to $x^* = P_\Omega x_1$. □

3.2. The strong convergence of inertial shrinking projection algorithm

Algorithm 3.2 Given appropriate parameter sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$. Choose any $x_0,\ x_1 \in H_1$ and $C_1 := H_1$, the sequence $\{x_n\}$ is constructed by the following iterative process.

$$\begin{cases} z_n = x_n + \alpha_n(x_n - x_{n-1}), \\ u_n = J_{\beta_n}^{B_1}\left(z_n - \gamma_n A^*(I - J_{\beta_n}^{B_2})Az_n\right), \\ x_{n+1} = P_{C_{n+1}}x_1, n \geq 1, \end{cases}$$

where

$$C_{n+1} = \{x \in C_n : \|u_n - x\|^2 \leq \|z_n - x\|^2 - \theta_n\}$$

and θ_n is defined as in Algorithm 3.1.

Theorem 3.3. *Assumed that (C1)-(C2) and (P1)-(P2) hold. If the solution set Ω is nonempty, then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to $x^* = P_\Omega x_1 \in \Omega$.*

Proof. Firstly, it is obvious that the half space C_n ($n \geq 1$) is convex and closed and P_{C_n} is well defined. By Lemma 3.1, we can easily get that the solution set $\Omega \subset C_n$. Using $x_n = P_{C_n}x_1$, $x_{n+1} = P_{C_{n+1}}x_1$ and $C_{n+1} \subset C_n$, we have $\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$, which implies that $\{\|x_n - x_1\|\}$ is nondecreasing. Furthermore, $\|x_n - x_1\| \leq \|p - x_1\|$, for any $p \in \Omega$, that is, $\{x_n\}$ is bounded. These imply that $\lim_{n\to\infty} \|x_n - x_1\|$ exists. Similarly to the proof of Theorem 3.2, we can also prove that the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega x_1$. □

4. Theoretical applications

In this section, we give several interesting special cases of the split variation inclusion problem. At the same time, Algorithms 3.1 and 3.2 are applied to these problems.

4.1. Split variational inequality problems

Let C and Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $F : H_1 \to H_1$ and $G : H_2 \to H_2$ be given operators, $A : H_1 \to H_2$ be a bounded linear operator. The split variational inequality problem is to find a point $x^* \in C$ such that

$$\langle F(x^*), x - x^*\rangle \geq 0,\ \forall x \in C \text{ and } \langle G(Ax^*), y - Ax^*\rangle \geq 0,\ \forall y \in Q.$$

Especially, when $H_1 = H_2$, $F = G$ and $A = I$, the split variational inequality problem is transformed into the classical variational inequality problem which is to find a point $x^* \in C$ such that $\langle F(x^*), x - x^*\rangle \geq 0,\ \forall x \in C$, and the solution set of the variational inequality problem is represented by $VI(F, C)$. Then, the split variational inequality problem is formulated as

$$\text{find } x^* \in C \text{ such that } x^* \in VI(F, C) \text{ and } Ax^* \in VI(G, Q). \tag{4.1}$$

Meanwhile, the solution set of problem (4.1) is denoted by Θ. Before this, the normal cone $N_C(x)$ of C at a point $x \in C$ is defined as follows:

$$N_C(x) = \{z \in H : \langle z, v - x\rangle \leq 0,\ \forall v \in C\}.$$

Further, the set-valued mapping S_F related to the normal cone $N_C(x)$ is defined by

$$S_F(x) := \begin{cases} F(x) + N_C(x), & x \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$$

In the sense, if F is a α-inverse strongly monotone operator (i.e., for any $x, z \in C$, $\langle F(x) - F(z), x - z\rangle \geq \alpha\|F(x) - F(z)\|^2$), then S_F is a maximal monotone mapping. More importantly, $x \in VI(F, C)$ if and only if $0 \in S_F(x)$. Let F and G be α-inverse strongly monotone operators. The set-valued mappings S_F and S_G are associated with F and G, respectively. The split variational inequality problem is equivalent to the following form:

$$\text{find } x^* \in H_1 \text{ such that } 0 \in S_F(x^*) \text{ and } 0 \in S_G(Ax^*).$$

Therefore, the following theorem can naturally arise to solve the split variational inequality problem.

Theorem 4.1. *Choose real numbers sequences* $\{\alpha_n\} \subset [a, b] \subset (-\infty, \infty)$, $\{\sigma_n\} \subset [c, d] \subset (0, 2)$ *and* $\{\beta_n\} \subset (0, \infty)$ *with* $\inf_n\{\beta_n\} \geq \beta > 0$. *For any* $x_0,\ x_1 \in H_1$, *let the sequence* $\{x_n\}$ *be constructed by the following iterative form.*

$$\begin{cases} z_n = x_n + \alpha_n(x_n - x_{n-1}), \\ u_n = J_{\beta_n}^{S_F}\left(z_n - \gamma_n A^*(I - J_{\beta_n}^{S_G})Az_n\right), \\ C_n = \{x \in H_1 : \|u_n - x\|^2 \leq \|z_n - x\|^2 - \hat{\theta}_n\}, \\ Q_n = \{x \in H_1 : \langle x_n - x_1, x_n - x\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_1,\ n \geq 1, \end{cases} \tag{4.2}$$

where $\hat{\theta}_n := \gamma_n\left(2\|(I - J_{\beta_n}^{S_G})Az_n\|^2 - \gamma_n\|A^*(I - J_{\beta_n}^{S_G})Az_n\|^2\right)$ *and*

$$\gamma_n := \begin{cases} \dfrac{\sigma_n\|(I - J_{\beta_n}^{S_G})Az_n\|^2}{\|A^*(I - J_{\beta_n}^{S_G})Az_n\|^2}, & Az_n \notin VI(G, Q), \\ 0, & \textit{otherwise.} \end{cases}$$

If the solution set Θ *is nonempty, then the iterative sequence* $\{x_n\}$ *generated by algorithm* (4.2) *converges strongly to* $x^* = P_\Theta x_1$.

Theorem 4.2. *Choose real numbers sequences* $\{\alpha_n\} \subset [a, b] \subset (-\infty, \infty)$, $\{\sigma_n\} \subset [c, d] \subset (0, 2)$ *and* $\{\beta_n\} \subset (0, \infty)$ *with* $\inf_n\{\beta_n\} \geq \beta > 0$. *For any* $x_0,\ x_1 \in H_1$ *and* $C_1 := H_1$, *let the sequence* $\{x_n\}$ *be generated by the following algorithm.*

$$\begin{cases} z_n = x_n + \alpha_n(x_n - x_{n-1}), \\ u_n = J_{\beta_n}^{S_F}\left(z_n - \gamma_n A^*(I - J_{\beta_n}^{S_G})Az_n\right), \\ C_{n+1} = \{x \in C_n : \|u_n - x\|^2 \leq \|z_n - x\|^2 - \hat{\theta}_n\}, \\ x_{n+1} = P_{C_{n+1}} x_1,\ n \geq 1, \end{cases} \tag{4.3}$$

where $\hat{\theta}_n$ *and* γ_n *are defined as in algorithm* (4.2). *If the solution set* Θ *is nonempty, then the sequence* $\{x_n\}$ *generated by algorithm* (4.3) *converges strongly to* $x^* = P_\Theta x_1$.

4.2. Split saddle point problems

Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. A bifunction $L : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{-\infty, \infty\}$ is convex-concave if and only if $L(x, \cdot)$ is convex for any $x \in \mathcal{X}$ and $L(\cdot, y)$ is concave for any $y \in \mathcal{Y}$. The operator T_L is defined as follows:

$$T_L(x, y) = (\partial_1 L(x, y), \partial_2(-L)(x, y)),$$

where ∂_1 is the subdifferential of L with respect to x and ∂_2 is the subdifferential of $-L$ with respect to y. It is worth noting that T_L is maximal monotone if and only if L is closed and proper, for detail, see, [35]. Naturally, the zeros of T_L coincide with the saddle points of L. Therefore, let $\mathcal{X}_i (i = 1, 2)$, $\mathcal{Y}_i$ $(i = 1, 2)$ be Hilbert spaces. Let $A : \mathcal{X}_1 \times \mathcal{Y}_1 \to \mathcal{X}_2 \times \mathcal{Y}_2$ be a bounded linear operator with the adjoint operator A^*. Let $L_1 : \mathcal{X}_1 \times \mathcal{Y}_1 \to \mathbb{R} \cup \{-\infty, \infty\}$ and $L_2 : \mathcal{X}_2 \times \mathcal{Y}_2 \to \mathbb{R} \cup \{-\infty, \infty\}$ be closed proper convex-concave bifunctions. Then, the split saddle point problem is to find a point $(x^*, y^*) \in \mathcal{X}_1 \times \mathcal{Y}_1$ such that

$$(x^*, y^*) \in \operatorname{argminmax}_{(x,y)\in\mathcal{X}_1\times\mathcal{Y}_1} L_1(x, y)$$

and

$$A(x^*, y^*) \in \operatorname{argminmax}_{(z,w)\in\mathcal{X}_2\times\mathcal{Y}_2} L_2(z, w).$$

For convenience, the solution set of the split saddle point problem is expressed as Φ. Let $H_i = \mathcal{X}_i \times \mathcal{Y}_i$ $(i = 1, 2)$ and $T_{L_i} = B_i$ $(i = 1, 2)$, the split saddle point problem is regarded as a special case of the split variational inclusion problem, and the following theorems can be derived naturally.

Theorem 4.3. *Let real numbers sequences $\{\alpha_n\} \subset [a, b] \subset (-\infty, \infty)$, $\{\sigma_n\} \subset [c, d] \subset (0, 2)$ and $\{\beta_n\} \subset (0, \infty)$ with $\inf_n\{\beta_n\} \geq \beta > 0$. For any initial points $x_0,\ x_1 \in H_1$, the sequence $\{x_n\}$ is obtained by the following process.*

$$\begin{cases} z_n = x_n + \alpha_n(x_n - x_{n-1}), \\ u_n = J_{\beta_n}^{T_{L_1}}\left(z_n - \gamma_n A^*(I - J_{\beta_n}^{T_{L_2}})Az_n\right), \\ C_n = \{x \in H_1 : \|u_n - x\|^2 \leq \|z_n - x\|^2 - \varrho_n\}, \\ Q_n = \{x \in H_1 : \langle x_n - x_1, x_n - x\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1,\ n \geq 1, \end{cases} \tag{4.4}$$

where $\varrho_n := \gamma_n\left(2\|(I - J_{\beta_n}^{T_{L_2}})Az_n\|^2 - \gamma_n\|A^*(I - J_{\beta_n}^{T_{L_2}})Az_n\|^2\right)$ and

$$\gamma_n := \begin{cases} \dfrac{\sigma_n\|(I - J_{\beta_n}^{T_{L_2}})Az_n\|^2}{\|A^*(I - J_{\beta_n}^{T_{L_2}})Az_n\|^2}, & Az_n \notin \operatorname{argminmax}_{y\in H_2} L_2(y), \\ 0, & \textit{otherwise}. \end{cases}$$

If the solution set Φ is nonempty, then the iterative sequence $\{x_n\}$ generated by algorithm (4.4) converges strongly to $x^* = P_\Phi x_1$.

Theorem 4.4. *Let real numbers sequences $\{\alpha_n\} \subset [a, b] \subset (-\infty, \infty)$, $\{\sigma_n\} \subset [c, d] \subset (0, 2)$ and $\{\beta_n\} \subset (0, \infty)$ with $\inf_n\{\beta_n\} \geq \beta > 0$. For any initial points $x_0,\ x_1 \in H_1$ and $C_1 := H_1$, the sequence $\{x_n\}$ is* *constructed by the following iterative form.*

$$\begin{cases} z_n = x_n + \alpha_n(x_n - x_{n-1}), \\ u_n = J_{\beta_n}^{T_{L_1}}\left(z_n - \gamma_n A^*(I - J_{\beta_n}^{T_{L_2}})Az_n\right), \\ C_{n+1} = \{x \in C_n : \|u_n - x\|^2 \leq \|z_n - x\|^2 - \varrho_n\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_1, \ n \geq 1, \end{cases} \tag{4.5}$$

where ϱ_n and γ_n are defined as in algorithm (4.4)*. If the solution set Φ is nonempty, then $\{x_n\}$ generated by algorithm* (4.5) *converges strongly to $x^* = P_\Phi x_1$.*

4.3. Split minimization problems

Let H_1 and H_2 be Hilbert spaces. Let $\phi : H_1 \to \mathbb{R}$ and $\psi : H_2 \to \mathbb{R}$ be proper lower semi-continuous convex functions, $A : H_1 \to H_2$ be a bounded linear operator. The split minimization problem is to find $x^* \in H_1$ such that

$$x^* \in \operatorname{argmin}_{x \in H_1} \phi(x) \text{ and } Ax^* \in \operatorname{argmin}_{y \in H_2} \psi(y).$$

It is well know that $x^* \in \operatorname{argmin}_{x \in H_1} \phi(x)$ if and only if $0 \in \partial\phi(x^*)$, where $\partial\phi$ is the subdifferential of ϕ defined by

$$\partial\phi(x^*) := \{\hat{x} \in H_1 : \phi(x^*) + \langle z - x^*, \hat{x}\rangle \leq \phi(z), \ \forall z \in H_1\}.$$

Recall that the proximal operator prox_ϕ of ϕ is defined as follows:

$$\operatorname{prox}_{\beta,\phi}(x) = \operatorname{argmin}_{z \in H_1} \left\{\phi(z) + \frac{1}{2\beta}\|z - x\|^2\right\}, \ \forall \beta > 0.$$

It is very important that $\operatorname{prox}_{\beta,\phi}(x) = (I + \beta\partial\phi)^{-1}(x) = J_\beta^{\partial\phi}(x)$. In addition, $\partial\phi$ is a maximal monotone mapping and prox_ϕ is a firmly nonexpansive mapping. In view of this, when $B_1 = \partial\phi$ and $B_2 = \partial\psi$ in (SVIP), the split variational inclusion problem is transformed into the split minimization problem. Based on our Theorems 3.2 and 3.3, we also have the following results.

Theorem 4.5. *Given real numbers sequences $\{\alpha_n\} \subset [a, b] \subset (-\infty, \infty)$, $\{\sigma_n\} \subset [c, d] \subset (0, 2)$ and $\beta > 0$. For any $x_0, \ x_1 \in H_1$, the sequence $\{x_n\}$ is constructed by the following iterative form.*

$$\begin{cases} z_n = x_n + \alpha_n(x_n - x_{n-1}), \\ u_n = \operatorname{prox}_{\beta,\phi}\left(z_n - \gamma_n A^*(I - \operatorname{prox}_{\beta,\psi})Az_n\right), \\ C_n = \{x \in H_1 : \|u_n - x\|^2 \leq \|z_n - x\|^2 - \chi_n\}, \\ Q_n = \{x \in H_1 : \langle x_n - x_1, x_n - x\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_1, \ n \geq 1, \end{cases} \tag{4.6}$$

where $\chi_n := \gamma_n\left(2\|(I - \operatorname{prox}_{\beta,\psi})Az_n\|^2 - \gamma_n\|A^*(I - \operatorname{prox}_{\beta,\psi})Az_n\|^2\right)$ and

$$\gamma_n := \begin{cases} \dfrac{\sigma_n\|(I - \operatorname{prox}_{\beta,\psi})Az_n\|^2}{\|A^*(I - \operatorname{prox}_{\beta,\psi})Az_n\|^2}, & Az_n \notin \operatorname{argmin}_{y \in H_2} \psi(y), \\ 0, & \textit{otherwise}. \end{cases}$$

If the solution set Υ of the split minimization problem is nonempty, then $\{x_n\}$ generated by algorithm (4.6) *converges strongly to $x^* = P_\Upsilon x_1$.*

Theorem 4.6. *Given real numbers sequences* $\{\alpha_n\} \subset [a, b] \subset (-\infty, \infty)$, $\{\sigma_n\} \subset [c, d] \subset (0, 2)$ *and* $\beta > 0$. *For any* $x_0,\ x_1 \in H_1$ *and* $C_1 := H_1$, *the sequence* $\{x_n\}$ *is constructed by the following iterative form.*

$$\begin{cases} z_n = x_n + \alpha_n(x_n - x_{n-1}), \\ u_n = \operatorname{prox}_{\beta,\phi}\left(z_n - \gamma_n A^*(I - \operatorname{prox}_{\beta,\psi})Az_n\right), \\ C_n = \{x \in H_1 : \|u_n - x\|^2 \le \|z_n - x\|^2 - \chi_n\}, \\ Q_n = \{x \in H_1 : \langle x_n - x_1, x_n - x\rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1,\ n \ge 1, \end{cases} \tag{4.7}$$

where χ_n *and* γ_n *are defined as in algorithm* (4.6). *If the solution set* Υ *of the split minimization problem is nonempty, then the iterative sequence* $\{x_n\}$ *generated by algorithm* (4.7) *converges strongly to* $x^* = P_\Upsilon x_1$.

Remark 4.7. Through the above results, the split variational inclusion problem, which includes the split variational inequality problem, the split saddle point problem and the split minimization problem as special cases, is quite general. Using the same methods as in Theorems 3.2 and 3.3, the strong convergence of Theorems 4.1–4.6 are obtained under the above corresponding conditions in Subsections 4.1, 4.2 and 4.3.

5. Numerical example

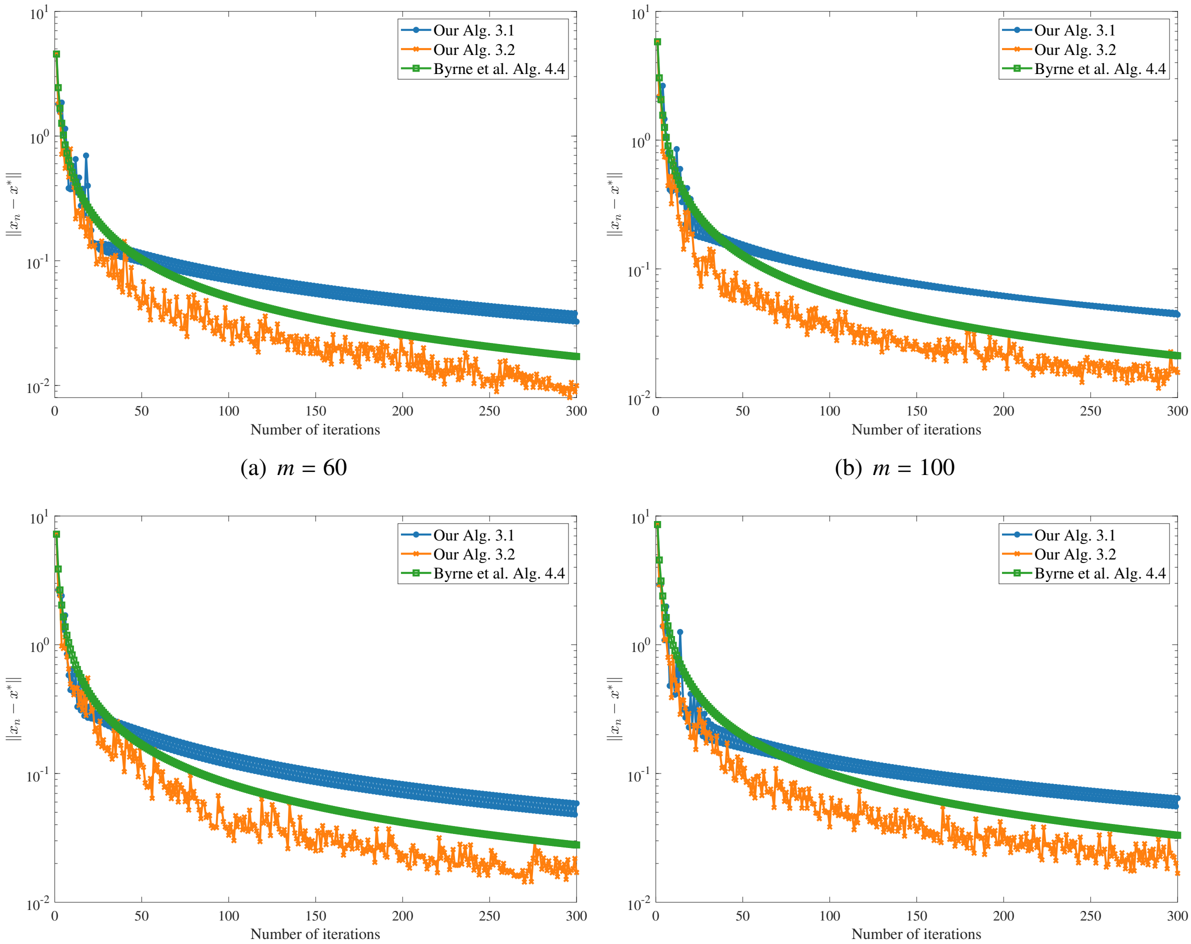
In this section, a numerical example is provided to illustrate the effectiveness and realization of convergence behavior of Algorithms 3.1 and 3.2. All codes were written in Matlab 2018a on a Intel(R) Core(TM) i5-8250U CPU @1.60 GHz computer with RAM 8.00 GB. Our results compare the existing conclusion below.

Theorem 5.1. (Byrne et al. [4, Algorithm 4.4]) *Let* H_1 *and* H_2 *be Hilbert spaces,* $A : H_1 \to H_2$ *be a bounded linear operator with the adjoint operator* A^*. *Let* $B_1 : H_1 \to 2^{H_1}$ *and* $B_2 : H_2 \to 2^{H_2}$ *be two set-valued maximal monotone mappings. Take any initial point* $x_1 \in H_1$, $\delta_n \in (0, 1)$ *and* $\beta > 0$, *the iterative sequence* $\{x_n\}$ *is generated by the following iterative scheme.*

$$x_{n+1} = \delta_n x_1 + (1 - \delta_n)J_\beta^{B_1}\left(x_n - \gamma A^*(I - J_\beta^{B_2})Ax_n\right),\ n \ge 1.$$

If $\{\delta_n\}$ *satisfies* $\lim_{n\to\infty}\delta_n = 0$ *and* $\sum_{n=1}^{\infty}\delta_n = \infty$, $0 < \gamma < 2/\|A^*A\|$, *then the iterative sequence* $\{x_n\}$ *converges strongly to a point* $x^* \in \Omega$.

Example 5.2. Assume that $A, A_1, A_2 : \mathbb{R}^m \to \mathbb{R}^m$ are created from a normal distribution with mean zero and unit variance. Let $B_1 : \mathbb{R}^m \to \mathbb{R}^m$ and $B_2 : \mathbb{R}^m \to \mathbb{R}^m$ be defined by $B_1(x) = A_1^*A_1x$ and $B_2(y) = A_2^*A_2y$, respectively. Consider the problem of finding a point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)^T \in \mathbb{R}^m$ such that $B_1(\bar{x}) = (0, \ldots, 0)^T$ and $B_2(A\bar{x}) = (0, \ldots, 0)^T$. It is easy to see that the minimum norm solution of the mentioned above problem is $x^* = (0, \ldots, 0)^T$. Our parameter settings are as follows. In our algorithms 3.1 and 3.2, set $\alpha_n = 0.5, \beta_n = 1$ and $\sigma_n = 1.5$. Take $\beta = 1$, $\delta_n = \frac{1}{n+1}$ and $\gamma_n = \frac{1.5}{\|A^*A\|}$ in the Algorithm 4.4 proposed by Byrne et al. [4]. We use $E_n = \|x_n - x^*\|$ to measure the iteration error of all algorithms. The stopping condition is that the maximum number of iterations is 300 times. Figure 1 describes the numerical behavior of all algorithms in different dimensions.



(a) $m = 60$

(b) $m = 100$

(c) $m = 150$

(d) $m = 200$

Figure 1. Numerical behavior of all algorithms in different dimensions.

It can be seen from the above results that our Algorithms 3.1 and 3.2 are efficient and robust. These results are independent of the selection of initial values and dimensions. Moreover, the convergence performance and the iteration error of the suggested Algorithm 3.2 are better than the existing Algorithm 4.4 in [4].

6. Conclusions

In this paper, our innovations are twofold. One is to provide a self-adaptive stepsize selection which does not require the norm of the bounded linear operator. The other is to propose two types of projection algorithms (i.e., a hybrid projection algorithm and a shrinking projection algorithm), which combine inertial technique with the proposed self-adaptive stepsize. Under mild constraints, the corresponding strong convergence theorems of SVIP are obtained in the framework of Hilbert spaces. At the same time, our results are also extended to the split variational inequality problem, the split saddle point problem and the split minimization problem. In terms of numerical experiments, the effectiveness of our proposed algorithms is showed by comparing with some existing results.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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