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SELF-ADAPTIVE-TYPE CQ ALGORITHMS FOR SPLIT EQUALITY PROBLEMS

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Abstract. The purpose of this paper is concerned with the approximate solution of split equality problems. Two types of CQ algorithms with a new self-adaptive stepsize criterion are presented without prior knowledge of operator norms. The corresponding strong convergence theorems are obtained under mild conditions. Moreover, some numerical examples including an application in signal recovery problems, are provided to show the effectiveness of the proposed algorithms.

Key Words and Phrases: Halpern-type algorithm, viscosity-type algorithm, self-adaptive stepsize, split equality problem, strong convergence theorem, fixed point.
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1. Introduction

As an extension of the split feasibility problem, in 2013, Moudafi [12, 13] introduced the following split equality problem (for short, SEP) that is applied to decomposition for PDEs [1] and intensity-modulated radiation therapy [4]. It is worth noting that the split equality problem contains many important optimization problems, such as the split feasibility problem, the fixed point problem, the variational inequality problem, the split common fixed point problem and the monotone inclusion problem; see, e.g., [5, 15, 22, 21, 27]. Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ be Hilbert spaces and $C \subset \mathcal{H}_1$, $Q \subset \mathcal{H}_2$ be nonempty closed convex subsets. Let $A : \mathcal{H}_1 \to \mathcal{H}_3$ and $B : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators. The split equality problem is to find

$$x^* \in C,\ y^* \in Q \ \text{ such that } \ Ax^* = By^*. \tag{1.1}$$

In particular, when $B = I$ and $\mathcal{H}_2 = \mathcal{H}_3$, SEP can be considered as well-known the split feasibility problem (Censor and Elfving introduced in [5]), which is to find $x^* \in C$ such that $Ax^* \in Q$. Naturally, x^* is a solution of this problem if and only if x^* is a solution of the equation $x^* = P_C(I - \gamma A^*(I - P_Q)A)x^*$, where $P_C : \mathcal{H}_1 \to C$ and $P_Q : \mathcal{H}_2 \to Q$ are metric projection operators, A^* is the adjoint operator of

A. By virtue of the fixed point algorithm, Byrne [2] came up with CQ algorithm to approximate a solution of the split feasibility problem by the recursive procedure $x_{n+1} = P_C(I - \gamma A^{\mathrm{T}}(I - P_Q)A)x_n$, where A^{T} is the matrix transposition of A, L is the largest eigenvalue of matrix $A^{\mathrm{T}}A$ and $\gamma \in (0, 2/L)$. Subsequently, Wang [24], Yao, Liou and Postolache [26] studied the following new iterative algorithm

$$x_{n+1} = x_n - \gamma_n[(I - P_C) + A^*(I - P_Q)A)]x_n, \ \forall n \geq 0, \tag{1.2}$$

where $\{\gamma_n\}$ is a self-adaptive stepsize sequence without prior knowledge of operator norms. It is worth noting that only the weak convergence of the split feasibility problem was obtained by Byrne's algorithm and Algorithm (1.2). To fill this gap, the Halpern algorithms and the viscosity algorithms were often employed to ensure strong convergence properties of the algorithms.

Based on the idea of Halpern algorithm (Halpern introduced in [9]), Xu [25] proposed the following modified algorithm and obtained strong convergence of the split feasibility problem

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) P_C(I - \gamma A^*(I - P_Q)A)x_n, \tag{1.3}$$

where γ is a constant in $(0, 2/\|A\|^2)$ and u is a fixed point. Further, Takahashi [18, 19] proposed a modified Halpern-type algorithm that uses a sequence $\{u_n\}$ converges strongly to u, and guaranteed the corresponding strong convergence. Moreover, by using the property of a contraction mapping, Moudafi [14] introduced the viscosity algorithm to approximate a solution of the fixed point problem. With the active help of these methods in [9, 14, 24, 26], many excellent results have been produced in other mathematical problems, such as the fixed point problem [6, 7, 20], the variational inequality problem [17, 23] and the split common fixed point problem [16, 27, 28].

On the other hand, to approximate the solutions of SEP, Moudafi [13] presented the following alternating CQ algorithm (for short, ACQA)

$$\begin{cases} x_{n+1} = P_C(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = P_Q(y_n + \gamma_n B^*(Ax_{n+1} - By_n)), \end{cases} \tag{1.4}$$

where $\{\gamma_n\}$ is a sequence in $(\varepsilon, \min\{\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}\} - \varepsilon)$ (ε is a small enough nonnegative real number). Further, Byrne and Moudafi [3] came up with the following simultaneous CQ algorithm (for short, SCQA) to solve SEP, for $\varepsilon < \gamma_n < \frac{2}{\|A\|^2 + \|B\|^2} - \varepsilon$,

$$\begin{cases} x_{n+1} = P_C(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = P_Q(y_n + \gamma_n B^*(Ax_n - By_n)). \end{cases} \tag{1.5}$$

Consistently, the iterative sequences generated by Algorithms (1.4) and (1.5) converge weakly to a solution of SEP. In addition, both of these algorithms involve metric projection operators, which are complicated and time-consuming in practical application.

Due to expensive calculation of the projection operators P_C and P_Q, Moudafi considered level sets to solve SEP, in which the level set of convex function is easy to implement, that is, C and Q are replaced with level sets of convex and subdifferentiable functions $\mathfrak{f} : \mathcal{H}_1 \to \mathbb{R}$ and $\mathfrak{g} : \mathcal{H}_2 \to \mathbb{R}$, respectively, i.e., $C = \{x \in \mathcal{H}_1 \mid \mathfrak{f}(x) \leq 0\}$

and $Q = \{y \in \mathcal{H}_2 \mid \mathfrak{g}(y) \leq 0\}$. In this situation, Moudafi [12] suggested the relaxed alternating CQ algorithm (for short, RACQA)

$$\begin{cases} x_{n+1} = P_{C_n}(x_n - \gamma A^*(Ax_n - By_n)), \\ y_{n+1} = P_{Q_n}(y_n + \gamma B^*(Ax_{n+1} - By_n)), \end{cases}$$

where γ is a constant in $(0, \min\{\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}\})$, $C_n = \{x \in \mathcal{H}_1 \mid \mathfrak{f}(x_n) + \langle \xi_n, x - x_n\rangle \leq 0\}$, $\xi_n \in \partial\mathfrak{f}(x_n)$, and $Q_n = \{y \in \mathcal{H}_2 \mid \mathfrak{g}(y_n) + \langle \eta_n, y - y_n\rangle \leq 0\}$, $\eta_n \in \partial\mathfrak{g}(y_n)$. Especially, RACQA still achieved weak convergence property. Inspired and motivated by Moudafi [12, 13, 14] and Takahashi [18, 19], we introduce two modified self-adaptive-type iterative algorithms to solve the split equality problem (1.1) in infinite-dimensional Hilbert spaces by the Halpern algorithm and the viscosity algorithm. The corresponding strong convergence theorems are obtained without prior knowledge of operator norms. Furthermore, some numerical experiments including signal recovery are used to demonstrate and show the efficiency of our main results.

The rest of this article is built up as follows. Some basic properties and relevant lemmas will be introduced in Section 2, which will be used in the proof for the convergence of the proposed algorithms. The main results and some corollaries of this paper are contained in Sections 3 and 4. Moreover, an application to signal recovery problem is given in Section 5. The last section, in Section 6, some numerical experiments demonstrate the efficiency of our results and compare them with some known algorithms, i.e., alternating CQ algorithm (ACQA) in Moudafi [13], simultaneous CQ algorithm (SCQA) in Byrne and Moudafi [3], and self-adaptive viscosity algorithm in Dong et al. [8].

2. Preliminaries

For the convenience and standard, we use the notations $\to$ and $\rightharpoonup$ to represent strong convergence and weak convergence, respectively. The fixed point set of the mapping T is represented by $F(T)$. Some well-known basic properties are as follows.

- (P1) P_C is denoted metric projection from $\mathcal{H}$ onto C, that is, $P_C x = \operatorname{argmin}_{y\in C} \|x - y\|$, $\forall x \in \mathcal{H}$. It has such an equivalent form $\langle P_C x - x, P_C x - y\rangle \leq 0, \forall y \in C$, and can also be converted to $\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2$;
- (P2) The mapping $T : \mathcal{H} \to \mathcal{H}$ with $F(T) \neq \emptyset$ and $I - T$ is demiclosed at 0, i.e., for any sequence $\{x_n\} \subset H$, if $\{x_n\}$ weakly converges to x and $(I - T)x_n$ strongly converges to 0, then $x \in F(T)$;
- (P3) The mapping $f : \mathcal{H} \to \mathcal{H}$ is a contraction with coefficient λ, that is,
$$\|f(x) - f(y)\| \leq \lambda\|x - y\|,\ \forall x, y \in \mathcal{H},\ \lambda \in [0, 1);$$
- (P4) $\partial\mathfrak{f}$ is denoted the subdifferential of convex function $\mathfrak{f} : \mathcal{H} \to \mathbb{R}$ at x, that is,
$$\partial\mathfrak{f}(x) = \{\varpi \in \mathcal{H} \mid \mathfrak{f}(y) \geq \mathfrak{f}(x) + \langle \varpi, y - x\rangle,\ \forall y \in \mathcal{H}\}.$$
- (P5) For any $x, y \in \mathcal{H}$, the following properties hold
$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle;$$
$$\|\kappa x + (1 - \kappa)y\|^2 = \kappa\|x\|^2 + (1 - \kappa)\|y\|^2 - \kappa(1 - \kappa)\|x - y\|^2,\ \forall \kappa \in \mathbb{R}.$$

Lemma 2.1. *Let the solution set of SEP be nonempty. For any $\gamma > 0$, a solution of SEP is equivalent to a solution of the following equations*

$$\begin{cases} x = x - \gamma\left((I - P_C)x + A^*(Ax - By)\right), \\ y = y - \gamma\left((I - P_Q)y - B^*(Ax - By)\right). \end{cases} \tag{2.1}$$

Proof. Obviously, any solution of SEP is the solution of equations (2.1). On the other hand, put any element (x, y) in the solution set of equations (2.1), we have

$$\begin{cases} 0 = (I - P_C)x + A^*(Ax - By), \\ 0 = (I - P_Q)y - B^*(Ax - By). \end{cases}$$

For any (x^*, y^*) in the solution set of SEP, that is, $x^* \in C$, $y^* \in Q$ and $Ax^* = By^*$, we have

$$\begin{aligned} 0 &= \langle (I - P_C)x + A^*(Ax - By), x - x^* \rangle \\ &= \langle x - P_C x, x - P_C x \rangle + \langle x - P_C x, P_C x - x^* \rangle + \langle Ax - By, Ax - Ax^* \rangle \\ &\geq \|x - P_C x\|^2 + \langle Ax - By, Ax - Ax^* \rangle, \end{aligned}$$

and

$$0 = \langle (I - P_Q)y - B^*(Ax - By), y - y^* \rangle \geq \|y - P_Q y\|^2 - \langle Ax - By, By - By^* \rangle.$$

Combining the above two formulas, we get

$$0 \geq \|x - P_C x\|^2 + \|y - P_Q y\|^2 + \|Ax - By\|^2.$$

This implies that $x \in C$, $y \in Q$ and $Ax = By$. Hence, (x, y) is also a solution of SEP. $\square$

Lemma 2.2. [10] *Let $\{\theta_n\}$ and $\{\eta_n\}$ be two nonnegative real numbers sequences such that*

$$\theta_{n+1} \leq (1 - \delta_n)\theta_n + \delta_n \tau_n, \ \text{and } \theta_{n+1} \leq \theta_n - \eta_n + \zeta_n, n \geq 0,$$

where $\{\tau_n\}$, $\{\zeta_n\}$ and $\{\delta_n\}$ are real sequences with $0 < \delta_n < 1$. If $\sum_{n=0}^{\infty} \delta_n = \infty$ and $\lim_{n\to\infty} \zeta_n = 0$ and $\lim_{k\to\infty} \eta_{n_k} = 0$ implies $\limsup_{k\to\infty} \ \tau_{n_k} \leq 0$, where $\{n_k\}$ is an arbitrary subsequence of $\{n\}$. The sequence $\{\theta_n\}$ is convergent to 0 as $n \to \infty$.

3. Halpern-type CQ algorithms

In what follows, $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are Hilbert spaces, $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ are nonempty closed convex subsets, $A : \mathcal{H}_1 \to \mathcal{H}_3$, $B : \mathcal{H}_2 \to \mathcal{H}_3$ are bounded linear operators, A^* and B^* are the adjoint operators of A and B, respectively. Meanwhile, we propose two Halpern-type algorithms to approximate a solution of SEP, and assume that the solution set of SEP is nonempty, i.e.,

$$\Omega = \{(x^*, y^*) \mid x^* \in C, y^* \in Q \text{ and } Ax^* = By^*\} \neq \emptyset.$$

In addition, the following assumption is presupposed.

(A1) $\{u_n\} \subset \mathcal{H}_1$, $\{v_n\} \subset \mathcal{H}_2$ are two convergence sequences such that

$$u_n \to u \in \mathcal{H}_1 \text{ and } v_n \to v \in \mathcal{H}_2.$$

3.1. **Self-adaptive Halpern-type CQ algorithm (SHCQA).** Through the aforementioned Halpern-type algorithm in [18, 19], the iterative sequence $\{(x_n, y_n)\}$ is generated by the following recursive procedure

$$\begin{cases} \widehat{x}_n = x_n - \gamma_n \left[(I - P_C)x_n + A^*(Ax_n - By_n)\right], \\ x_{n+1} = \delta_n u_n + (1-\delta_n)\widehat{x}_n, \\ \widehat{y}_n = y_n - \gamma_n \left[(I - P_Q)y_n - B^*(Ax_n - By_n)\right], \\ y_{n+1} = \delta_n v_n + (1-\delta_n)\widehat{y}_n, n \geq 0. \end{cases} \tag{3.1}$$

The corresponding parameters satisfy the following restrictions.
(R1) If $Ax_n \neq By_n$, the self-adaptive stepsize

$$\gamma_n = \alpha_n \min\left\{1, \frac{\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2}\right\}$$

with $\{\alpha_n\} \subset (0,1)$ and $\inf_n \alpha_n(1-\alpha_n) > 0$. Otherwise, $\gamma_n = \alpha_n$;
(R2) $\{\delta_n\}$ is a real numbers sequence in $(0,1)$ such that $\lim_{n\to\infty} \delta_n = 0$ and $\sum_{n=0}^{\infty} \delta_n = \infty$.

Theorem 3.1. *Given Assumption* (A1) *and Conditions* (R1)-(R2)*. For any $x_0 \in \mathcal{H}_1$ and $y_0 \in \mathcal{H}_2$, the sequence $\{(x_n, y_n)\}$ generated by Algorithm* (3.1) *converges strongly to $P_\Omega(u, v) \in \Omega$.*

Proof. Take $(x^*, y^*) = P_\Omega(u, v) \in \Omega$, that is, $x^* \in C$, $y^* \in Q$ and $Ax^* = By^*$. Using projection operator P_C, we have

$$\begin{aligned} &\|\widehat{x}_n - x^*\|^2 \\ =&\|x_n - x^*\|^2 - 2\gamma_n\langle (I-P_C)x_n + A^*(Ax_n - By_n), x_n - x^*\rangle \\ &+ \gamma_n^2\|(I-P_C)x_n + A^*(Ax_n - By_n)\|^2 \\ \leq&\|x_n - x^*\|^2 - 2\gamma_n\|(I-P_C)x_n\|^2 - 2\gamma_n\langle Ax_n - By_n, Ax_n - Ax^*\rangle \\ &+ 2\gamma_n^2\left(\|(I-P_C)x_n\|^2 + \|A^*(Ax_n - By_n)\|^2\right). \end{aligned} \tag{3.2}$$

Similarly, the following inequality is available

$$\begin{aligned} \|\widehat{y}_n - y^*\|^2 \leq \|y_n - y^*\|^2 - 2\gamma_n\|(I-P_Q)y_n\|^2 + 2\gamma_n\langle Ax_n - By_n, By_n - By^*\rangle \\ + 2\gamma_n^2\left(\|(I-P_Q)y_n\| + \|B^*(Ax_n - By_n)\|^2\right). \end{aligned} \tag{3.3}$$

On the other hand, we have

$$2\langle Ax_n - By_n, By_n - By^*\rangle - 2\langle Ax_n - By_n, Ax_n - Ax^*\rangle = -2\|Ax_n - By_n\|^2.$$

Combining (3.2), (3.3) and (R1), we get

$$\begin{aligned} &\|\widehat{x}_n - x^*\|^2 + \|\widehat{y}_n - y^*\|^2 \\ \leq& \|x_n - x^*\|^2 + \|y_n - y^*\|^2 - 2\gamma_n(1-\gamma_n)\left(\|(I-P_C)x_n\|^2 + \|(I-P_Q)y_n\|^2\right) \\ &- 2\gamma_n\left(\|Ax_n - By_n\|^2 - \gamma_n\|A^*(Ax_n - By_n)\|^2 - \gamma_n\|B^*(Ax_n - By_n)\|^2\right) \\ =& \|x_n - x^*\|^2 + \|y_n - y^*\|^2 - \Phi_n. \end{aligned} \tag{3.4}$$

From the definition of γ_n, we have $\gamma_n(1-\gamma_n) \geq \gamma_n(1-\alpha_n) > 0$ and

$$\begin{aligned}&\gamma_n\|Ax_n - By_n\|^2 - \gamma_n^2(\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2)\\ \geq&\gamma_n(1-\alpha_n)\|Ax_n - By_n\|^2 \geq 0.\end{aligned}$$

Thus,

$$\begin{aligned}\Phi_n &= 2\gamma_n(1-\gamma_n)\left(\|(I-P_C)x_n\|^2 + \|(I-P_Q)y_n\|^2\right) + 2\gamma_n\|Ax_n - By_n\|^2\\ &\quad - 2\gamma_n^2\left(\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2\right)\\ &\geq 2\gamma_n(1-\alpha_n)\left[\|(I-P_C)x_n\|^2 + \|(I-P_Q)y_n\|^2 + \|Ax_n - By_n\|^2\right] \geq 0.\end{aligned} \tag{3.5}$$

In addition, using the convexity of the squared norm and (3.4), we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2\\ \leq&\delta_n\left(\|u_n - x^*\|^2 + \|v_n - y^*\|^2\right) + (1-\delta_n)\left(\|x_n - x^*\|^2 + \|y_n - y^*\|^2\right).\end{aligned}$$

Since $\{u_n\}$ and $\{v_n\}$ are convergence sequences, there exists a non-negative constant G such that $\sup_{n\geq 0}\{\|u_n - x^*\|^2, \|v_n - y^*\|^2\} \leq G/2$. Let $\theta_n = \|x_n - x^*\|^2 + \|y_n - y^*\|^2$. The above formula can be converted to

$$\begin{aligned}\theta_{n+1} &\leq \delta_n\left(\|u_n - x^*\|^2 + \|v_n - y^*\|^2\right) + (1-\delta_n)\theta_n\\ &\leq \max\{G, \theta_n\} \leq \cdots \leq \max\{G, \theta_0\}.\end{aligned} \tag{3.6}$$

This implies that $\{\theta_n\}$ is bounded, that is, the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. From (P5), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\delta_n u_n + (1-\delta_n)\widehat{x}_n - x^*\|^2\\ &\leq \|(1-\delta_n)(\widehat{x}_n - x^*)\|^2 + 2\delta_n\langle u_n - x^*, x_{n+1} - x^*\rangle\\ &\leq (1-\delta_n)\|\widehat{x}_n - x^*\|^2 + 2\delta_n\langle u_n - x^*, x_{n+1} - x^*\rangle,\end{aligned}$$

$$\|y_{n+1} - y^*\|^2 \leq (1-\delta_n)\|\widehat{y}_n - y^*\|^2 + 2\delta_n\langle v_n - y^*, y_{n+1} - y^*\rangle.$$

Combining the above two inequalities and (3.4), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 &\leq (1-\delta_n)\left(\|x_n - x^*\|^2 + \|y_n - y^*\|^2\right) - (1-\delta_n)\Phi_n\\ &\quad + 2\delta_n\left(\langle u_n - x^*, x_{n+1} - x^*\rangle + \langle v_n - y^*, y_{n+1} - y^*\rangle\right).\end{aligned}$$

For each $n \geq 0$, set

$$\begin{aligned}&\eta_n = (1-\delta_n)\Phi_n,\ \tau_n = 2\left(\langle u_n - x^*, x_{n+1} - x^*\rangle + \langle v_n - y^*, y_{n+1} - y^*\rangle\right),\\ &\zeta_n = 2\delta_n\left(\langle u_n - x^*, x_{n+1} - x^*\rangle + \langle v_n - y^*, y_{n+1} - y^*\rangle\right).\end{aligned}$$

Then, the above formula is reduced to the following inequalities:

$$\theta_{n+1} \leq (1-\delta_n)\theta_n + \delta_n\tau_n \ \text{ and } \ \theta_{n+1} \leq \theta_n - \eta_n + \zeta_n,\ n \geq 0.$$

By (R2) and the boundedness of $\{u_n\}$, $\{v_n\}$, $\{x_n\}$, $\{y_n\}$, we see that $\lim_{n\to\infty}\zeta_n = 0$ and $\sum_{n=0}^{\infty}\delta_n = \infty$. By virtue of Lemma 2.2, this proof remains to show that $\lim_{k\to\infty}\eta_{n_k} = 0$ implies $\limsup_{k\to\infty}\tau_{n_k} \leq 0$ for any subsequence $\{n_k\}$ of $\{n\}$. Let $\{\eta_{n_k}\}$ be a any subsequence of $\{\eta_n\}$ such that $\lim_{k\to\infty}\eta_{n_k} = 0$. This implies that

$\lim_{k\to\infty}\Phi_{n_k} = 0$. Hence, suppose that $Ax_n \neq By_n$, it follows from (R1) that $\lim_{n\to\infty}\gamma_n(1-\alpha_n) \neq 0$. Further, using (3.5), we obtain

$$\lim_{k\to\infty}\|(I-P_C)x_{n_k}\| = \lim_{k\to\infty}\|(I-P_Q)y_{n_k}\| = \lim_{k\to\infty}\|Ax_{n_k} - By_{n_k}\| = 0. \quad (3.7)$$

From the boundedness of $\{x_n\}$ and $\{y_n\}$, there exists two subsequences $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ and $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup \bar{x}$, $y_{n_{k_j}} \rightharpoonup \bar{y}$ and

$$\limsup_{k\to\infty}\langle u_{n_k} - x^*, x_{n_k} - x^*\rangle = \lim_{j\to\infty}\langle u_{n_{k_j}} - x^*, x_{n_{k_j}} - x^*\rangle,$$

$$\limsup_{k\to\infty}\langle v_{n_k} - y^*, y_{n_k} - y^*\rangle = \lim_{j\to\infty}\langle v_{n_{k_j}} - y^*, y_{n_{k_j}} - y^*\rangle.$$

Since $I - P_C$ and $I - P_Q$ are demiclosed at 0, we have $\bar{x} \in C$ and $\bar{y} \in Q$. In addition, it follows from the bounded linearity of A and B that $Ax_{n_{k_j}} - By_{n_{k_j}} \rightharpoonup A\bar{x} - B\bar{y}$. Using the weak lower semicontinuity of the squared norm, we have $\|A\bar{x} - B\bar{y}\|^2 \leq \liminf_{j\to\infty}\|Ax_{n_{k_j}} - By_{n_{k_j}}\|^2 = 0$, which implies that $(\bar{x}, \bar{y}) \in \Omega$. On the other hand, if $Ax_n = By_n$, we can also get the same result as above. Besides, from the property of projection and the strong convergence property of $\{u_n\}$ and $\{v_n\}$, it follows that

$$\begin{aligned}\limsup_{k\to\infty}\langle u_{n_k} - x^*, x_{n_k} - x^*\rangle &= \lim_{j\to\infty}\langle u_{n_{k_j}} - x^*, x_{n_{k_j}} - x^*\rangle \\ &= \langle u - x^*, \bar{x} - x^*\rangle \leq 0,\end{aligned} \quad (3.8)$$

$$\begin{aligned}\limsup_{k\to\infty}\langle v_{n_k} - y^*, y_{n_k} - y^*\rangle &= \lim_{j\to\infty}\langle v_{n_{k_j}} - y^*, y_{n_{k_j}} - y^*\rangle \\ &= \langle v - y^*, \bar{y} - y^*\rangle \leq 0.\end{aligned} \quad (3.9)$$

According to (R1) and (3.7), we have

$$\|\widehat{x}_{n_k} - x_{n_k}\| \leq \gamma_n(\|(I-P_C)x_{n_k}\| + \|A\|\|Ax_{n_k} - By_{n_k}\|) \to 0,$$

$$\|\widehat{y}_{n_k} - y_{n_k}\| \leq \gamma_n(\|(I-P_Q)y_{n_k}\| + \|B\|\|Ax_{n_k} - By_{n_k}\|) \to 0.$$

Further, we have

$$\|x_{n_k+1} - x_{n_k}\| \leq \delta_{n_k}\|u_{n_k} - x_{n_k}\| + (1-\delta_{n_k})\|\widehat{x}_{n_k} - x_{n_k}\| \to 0, \quad (3.10)$$

$$\|y_{n_k+1} - y_{n_k}\| \leq \delta_{n_k}\|v_{n_k} - y_{n_k}\| + (1-\delta_{n_k})\|\widehat{y}_{n_k} - y_{n_k}\| \to 0. \quad (3.11)$$

From (3.8), (3.9), (3.10) and (3.11), we have $\limsup_{k\to\infty}\langle u_{n_k} - x^*, x_{n_k+1} - x^*\rangle \leq 0$ and $\limsup_{k\to\infty}\langle v_{n_k} - y^*, y_{n_k+1} - y^*\rangle \leq 0$. This implies that $\limsup_{k\to\infty}\tau_{n_k} \leq 0$. By virtue of Lemma 2.2, we obtain $\lim_{n\to\infty}\theta_n = 0$, which implies that $(x_n, y_n) \to (x^*, y^*)$. □

Remark 3.2. In order to get the limits in (3.7), we must require $\inf_n \alpha_n(1-\alpha_n) > 0$ in (R1), which is to ensure that $\lim_{n\to\infty}\gamma_n(1-\alpha_n) \neq 0$. In addition, the selection method of α_n can also refer to the following forms.

$$\{\alpha_n\} \subset (\varepsilon, 1-\varepsilon) \subset (0,1) \text{ for some } \varepsilon > 0, \text{ or } \{\alpha_n\} \text{ is a fixed constant in } (0,1).$$

Remark 3.3. (I) The sequences $\{u_n\}$ and $\{v_n\}$ in Theorem 3.1 are easily chosen. For example, (1) the monotonically decreasing sequence $u_n = \frac{n^2}{(n+1)^2}u$; (2) the monotonically increasing sequence $u_n = \frac{(n+1)^2}{n^2}u$; (3) the non-monotonically convergent sequence $u_n = \frac{2n+(-1)^n}{2n}u$.

(II) In particular, when $\{u_n\}$ and $\{v_n\}$ are constant sequences, that is, $u_n \equiv u$ and $v_n \equiv v$, Algorithm (3.1) is equal to the following Halpern algorithm.

$$\begin{cases} \widehat{x}_n = x_n - \gamma_n \left[(I - P_C)x_n + A^*(Ax_n - By_n)\right], \\ x_{n+1} = \delta_n u + (1 - \delta_n)\widehat{x}_n, \\ \widehat{y}_n = y_n - \gamma_n \left[(I - P_Q)y_n - B^*(Ax_n - By_n)\right], \\ y_{n+1} = \delta_n v + (1 - \delta_n)\widehat{y}_n, n \geq 0. \end{cases} \tag{3.12}$$

From Remark 3.3 (II), we have the following important corollary of Theorem 3.1.

Corollary 3.4. *Assume that Conditions* (R1)-(R2) *are satisfied. For any* $x_0, u \in \mathcal{H}_1$ *and* $y_0, v \in \mathcal{H}_2$*, the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm* (3.12) *converges strongly to* $P_\Omega(u, v) \in \Omega$.

3.2. **Self-adaptive relaxed Halpern-type CQ algorithm (SRHCQA).** Here, we consider using the level sets of two convex functions $\mathfrak{f} : \mathcal{H}_1 \to \mathbb{R}$ and $\mathfrak{g} : \mathcal{H}_2 \to \mathbb{R}$ instead of closed convex sets C and Q in Theorem 3.1, i.e.,

$$C = \{x \in \mathcal{H}_1 \mid \mathfrak{f}(x) \leq 0\}, \ Q = \{y \in \mathcal{H}_2 \mid \mathfrak{g}(y) \leq 0\}.$$

For solving SEP, we construct the corresponding closed convex sets as follows.

$$C_n = \{x \in \mathcal{H}_1 \mid \mathfrak{f}(x_n) + \langle \xi_n, x - x_n \rangle \leq 0, \ \xi_n \in \partial\mathfrak{f}(x_n)\},$$

$$Q_n = \{y \in \mathcal{H}_2 \mid \mathfrak{g}(y_n) + \langle \eta_n, y - y_n \rangle \leq 0, \ \eta_n \in \partial\mathfrak{g}(y_n)\}.$$

Assume that both $\mathfrak{f}$ and $\mathfrak{g}$ are subdifferentiable on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and that $\partial\mathfrak{f}$ and $\partial\mathfrak{g}$ are bounded on bounded sets. It is obvious that $C \subset C_n$ and $Q \subset Q_n$. Under the above conditions, using the method of Halpern-type algorithm to promote the relaxed alternating CQ algorithm, the iterative sequence $\{(x_n, y_n)\}$ is generated by the following recursive procedure

$$\begin{cases} \widehat{x}_n = x_n - \gamma_n \left[(I - P_{C_n})x_n + A^*(Ax_n - By_n)\right], \\ \widehat{y}_n = y_n - \gamma_n \left[(I - P_{Q_n})y_n - B^*(Ax_n - By_n)\right], \\ x_{n+1} = \delta_n u_n + (1 - \delta_n)\widehat{x}_n, \\ y_{n+1} = \delta_n v_n + (1 - \delta_n)\widehat{y}_n, n \geq 0. \end{cases} \tag{3.13}$$

Theorem 3.5. *Given Assumption* (A1) *and Conditions* (R1)-(R2)*. For any* $x_0 \in \mathcal{H}_1$ *and* $y_0 \in \mathcal{H}_2$*, the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm* (3.13) *converges strongly to* $P_\Omega(u, v) \in \Omega$.

Proof. Take $(x^*, y^*) = P_\Omega(u, v) \in \Omega$, that is, $x^* \in C$, $y^* \in Q$ and $Ax^* = By^*$. As similar proof in Theorem 3.1, we obtain that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. On the other hand,

$$\begin{aligned} \Psi_n &= 2\gamma_n(1 - \gamma_n)(\|(I - P_{C_n})x_n\|^2 + \|(I - P_{Q_n})y_n\|^2) + 2\gamma_n\|Ax_n - By_n\|^2 \\ &\quad - 2\gamma_n^2 \left(\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2\right) \\ &\geq 2\gamma_n(1 - \alpha_n) \left[\|(I - P_{C_n})x_n\|^2 + \|(I - P_{Q_n})y_n\|^2 + \|Ax_n - By_n\|^2\right] \\ &\geq 0. \end{aligned} \tag{3.14}$$

Similarly, we have

$$\begin{aligned}&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\\ \le&(1-\delta_n)\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2\right)-(1-\delta_n)\Psi_n\\ &+2\delta_n\left(\langle u_n-x^*,x_{n+1}-x^*\rangle+\langle v_n-y^*,y_{n+1}-y^*\rangle\right).\end{aligned}$$

For each $n\ge 0$, set

$$\begin{aligned}&\theta_n=\|x_n-x^*\|^2+\|y_n-y^*\|,\ \eta_n=(1-\delta_n)\Psi_n,\\ &\tau_n=2\left(\langle u_n-x^*,x_{n+1}-x^*\rangle+\langle v_n-y^*,y_{n+1}-y^*\rangle\right),\\ &\zeta_n=2\delta_n\left(\langle u_n-x^*,x_{n+1}-x^*\rangle+\langle v_n-y^*,y_{n+1}-y^*\rangle\right).\end{aligned}$$

Naturally, we have the following inequalities

$$\theta_{n+1}\le(1-\delta_n)\theta_n+\delta_n\tau_n \text{ and } \theta_{n+1}\le\theta_n-\eta_n+\zeta_n, n\ge 0.$$

By the boundedness and (R2), we see that $\lim_{n\to\infty}\zeta_n=0$ and $\sum_{n=0}^{\infty}\delta_n=\infty$. By Lemma 2.2, this proof remains to show that $\lim_{k\to\infty}\eta_{n_k}=0$ implies $\limsup_{k\to\infty}\tau_{n_k}\le 0$ for any subsequence $\{n_k\}$ of $\{n\}$. Let $\{\eta_{n_k}\}$ be a any subsequence of $\{\eta_n\}$ such that $\lim_{k\to\infty}\eta_{n_k}=0$. This implies that $\lim_{k\to\infty}\Psi_{n_k}=0$. Hence, suppose that $Ax_n\ne By_n$, it follows from (R1) that $\lim_{n\to\infty}\gamma_n(1-\alpha_n)\ne 0$. Using (3.14), we have

$$\lim_{k\to\infty}\|(I-P_{C_n})x_{n_k}\|=\lim_{k\to\infty}\|(I-P_{Q_n})y_{n_k}\|=\lim_{k\to\infty}\|Ax_{n_k}-By_{n_k}\|=0.$$

Using the boundedness of $\{x_n\}$ and $\{y_n\}$, there exists two sequences $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ and $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ such that $x_{n_{k_j}}\rightharpoonup\bar{x}$, $y_{n_{k_j}}\rightharpoonup\bar{y}$ and

$$\begin{aligned}\limsup_{k\to\infty}\ \langle u_{n_k}-x^*,x_{n_k}-x^*\rangle=\lim_{j\to\infty}\langle u_{n_{k_j}}-x^*,x_{n_{k_j}}-x^*\rangle,\\ \limsup_{k\to\infty}\ \langle v_{n_k}-y^*,y_{n_k}-y^*\rangle=\lim_{j\to\infty}\langle v_{n_{k_j}}-y^*,y_{n_{k_j}}-y^*\rangle.\end{aligned}$$

Since $\partial\mathfrak{f}$ is bounded on bounded sets, there exists a constant $\vartheta>0$ such that $\|\xi_{n_k}\|\le\vartheta$, $\forall k\ge 0$. Using the definition of C_n, we get

$$\mathfrak{f}(x_{n_k})\le\langle\xi_{n_k},x_{n_k}-P_{C_{n_k}}x_{n_k}\rangle\le\vartheta\|x_{n_k}-P_{C_{n_k}}x_{n_k}\|\to 0,\ \text{as } k\to\infty.$$

By the weak lower semi-continuity of $\mathfrak{f}$, we have $\mathfrak{f}(\bar{x})\le\liminf_{j\to\infty}\mathfrak{f}(x_{n_{k_j}})\le 0$. So we have $\bar{x}\in C$. Similarly, we obtain $\bar{y}\in Q$. In addition, it follows from the bounded linearity of A and B that $Ax_{n_{k_j}}-By_{n_{k_j}}\rightharpoonup A\bar{x}-B\bar{y}$. By virtue of the weak lower semicontinuity of the squared norm, we have $\|A\bar{x}-B\bar{y}\|^2\le\liminf_{j\to\infty}\|Ax_{n_{k_j}}-By_{n_{k_j}}\|^2=0$, which implies that $(\bar{x},\bar{y})\in\Omega$. On the other hand, if $Ax_n=By_n$, we can also get the same result as above. Last, using the proof process in Theorem 3.1 and Lemma 2.2, we obtain $\lim_{n\to\infty}\theta_n=0$, which implies that $(x_n,y_n)\to(x^*,y^*)$. □

According to Remark 3.3 (II), Algorithm (3.13) can also be degraded to the following Halpern algorithm.

$$\begin{cases}\widehat{x}_n=x_n-\gamma_n\left[(I-P_{C_n})x_n+A^*(Ax_n-By_n)\right],\\ \widehat{y}_n=y_n-\gamma_n\left[(I-P_{Q_n})y_n-B^*(Ax_n-By_n)\right],\\ x_{n+1}=\delta_n u+(1-\delta_n)\widehat{x}_n,\\ y_{n+1}=\delta_n v+(1-\delta_n)\widehat{y}_n, n\ge 0.\end{cases}\tag{3.15}$$

Corollary 3.6. *Assume that Conditions* (R1)-(R2) *are established. For any* $x_0, u \in \mathcal{H}_1$ *and* $y_0, v \in \mathcal{H}_2$*, the iterative sequence* $\{(x_n, y_n)\}$ *generated by Algorithm* (3.15) *converges strongly to* $P_\Omega(u, v) \in \Omega$.

4. Viscosity-type CQ algorithms

In this section, we propose two viscosity-type algorithms to approximate a solution of SEP, and assume that the solution set of SEP is nonempty. Meanwhile, the following assumptions are presupposed.
(V1) $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ are contraction mappings with coefficient $\lambda_1 \in [0, 1/\sqrt{2})$, $\lambda_2 \in [0, 1/\sqrt{2})$, respectively.

4.1. **Self-adaptive viscosity-type CQ algorithm (SVCQA).** According to the mentioned viscosity-type algorithm in Moudafi [14], the iterative sequence $\{(x_n, y_n)\}$ is generated by the following recursive procedure

$$\begin{cases} \widehat{x}_n = x_n - \gamma_n \left[(I - P_C)x_n + A^*(Ax_n - By_n)\right], \\ x_{n+1} = \delta_n f(\widehat{x}_n) + (1 - \delta_n)\widehat{x}_n, \\ \widehat{y}_n = y_n - \gamma_n \left[(I - P_Q)y_n - B^*(Ax_n - By_n)\right], \\ y_{n+1} = \delta_n g(\widehat{y}_n) + (1 - \delta_n)\widehat{y}_n, n \geq 0, \end{cases} \tag{4.1}$$

where the corresponding parameters $\{\gamma_n\}$ and $\{\delta_n\}$ are defined as (R1) and (R2).

Theorem 4.1. *Given Assumption* (V1) *and Conditions* (R1)-(R2)*. For any* $x_0 \in \mathcal{H}_1$ *and* $y_0 \in \mathcal{H}_2$*, the iterative sequence* $\{(x_n, y_n)\}$ *generated by Algorithm* (4.1) *converges strongly to* $(x^*, y^*) = P_\Omega(f(x^*), g(y^*))$.

Proof. Take $(x^*, y^*) = P_\Omega(f(x^*), g(y^*)) \in \Omega$, that is, $x^* \in C$, $y^* \in Q$ and $Ax^* = By^*$. From the proof of Theorem 3.1, we can get

$$\begin{aligned} \|\widehat{x}_n - x^*\|^2 + \|\widehat{y}_n - y^*\|^2 &= \|x_n - x^*\|^2 + \|y_n - y^*\|^2 - \Phi_n \\ &\leq \|x_n - x^*\|^2 + \|y_n - y^*\|^2, \end{aligned} \tag{4.2}$$

where Φ_n is defined as in (3.5).
Set $\lambda = \max\{\lambda_1, \lambda_2\}$. From (4.2), we have

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 \\ &\leq \delta_n \left(\|f(\widehat{x}_n) - x^*\|^2 + \|g(\widehat{y}_n) - y^*\|^2\right) + (1 - \delta_n)\left(\|\widehat{x}_n - x^*\|^2 + \|\widehat{y}_n - y^*\|^2\right) \\ &\leq 2\delta_n \left(\|f(\widehat{x}_n) - f(x^*)\|^2 + \|f(x^*) - x^*\|^2 + \|g(\widehat{y}_n) - g(y^*)\|^2 + \|g(y^*) - y^*\|^2\right) \\ &\quad + (1 - \delta_n)\left(\|\widehat{x}_n - x^*\|^2 + \|\widehat{y}_n - y^*\|^2\right) \\ &\leq 2\delta_n(\lambda_1^2\|\widehat{x}_n - x^*\|^2 + \lambda_2^2\|\widehat{y}_n - y^*\|^2) + 2\delta_n(\|f(x^*) - x^*\|^2 + \|g(y^*) - y^*\|^2) \\ &\quad + (1 - \delta_n)\left(\|\widehat{x}_n - x^*\|^2 + \|\widehat{y}_n - y^*\|^2\right) \\ &\leq (1 - \delta_n(1 - 2\lambda^2))\left(\|x_n - x^*\|^2 + \|y_n - y^*\|^2\right) - (1 - \delta_n(1 - 2\lambda^2))\Phi_n \\ &\quad + 2\delta_n(\|f(x^*) - x^*\|^2 + \|g(y^*) - y^*\|^2). \end{aligned}$$

Let $\theta_n = \|x_n - x^*\|^2 + \|y_n - y^*\|^2$. Since $\lambda_1, \lambda_2 \in [0, 1/\sqrt{2})$ and (R1), we have

$$\begin{aligned}\theta_{n+1} \leq &(1-\delta_n(1-2\lambda^2))\theta_n - (1-\delta_n(1-2\lambda^2))\Phi_n \\ &+ 2\delta_n \left(\|f(x^*) - x^*\|^2 + \|g(y^*) - y^*\|^2\right) \\ \leq &(1-\delta_n(1-2\lambda^2))\theta_n + 2\delta_n(1-2\lambda^2)\frac{\|f(x^*) - x^*\|^2 + \|g(y^*) - y^*\|^2}{1-2\lambda^2} \\ \leq &\max\left\{\theta_n, \frac{2(\|f(x^*) - x^*\|^2 + \|g(y^*) - y^*\|^2)}{1-2\lambda^2}\right\} \\ \leq &\cdots \leq \max\left\{\theta_0, \frac{2(\|f(x^*) - x^*\|^2 + \|g(y^*) - y^*\|^2)}{1-2\lambda^2}\right\}.\end{aligned}$$

This implies that $\{\theta_n\}$ is bounded, that is, $\{x_n\}$ and $\{y_n\}$ are bounded. On the other hand, using (P5), we can obtain

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ =&\|\delta_n(f(\widehat{x}_n) - f(x^*)) + \delta_n(f(x^*) - x^*) + (1-\delta_n)(\widehat{x}_n - x^*)\|^2 \\ \leq&\|\delta_n(f(\widehat{x}_n) - f(x^*)) + (1-\delta_n)(\widehat{x}_n - x^*)\|^2 + 2\delta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ \leq&\delta_n\|f(\widehat{x}_n) - f(x^*)\|^2 + (1-\delta_n)\|\widehat{x}_n - x^*\|^2 + 2\delta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ \leq&(1-\delta_n(1-\lambda_1^2))\|\widehat{x}_n - x^*\|^2 + 2\delta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle.\end{aligned}$$

Similarly,

$$\|y_{n+1} - y^*\|^2 \leq (1-\delta_n(1-\lambda_2^2))\|\widehat{y}_n - y^*\|^2 + 2\delta_n\langle g(y^*) - y^*, y_{n+1} - y^*\rangle.$$

According to the above formulas, we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 \\ \leq&(1-\delta_n(1-\lambda^2))\left(\|\widehat{x}_n - x^*\|^2 + \|\widehat{y}_n - y^*\|^2\right) \\ &+ 2\delta_n\left(\langle f(x^*) - x^*, x_{n+1} - x^*\rangle + \langle g(y^*) - y^*, y_{n+1} - y^*\rangle\right) \\ \leq&(1-\delta_n(1-\lambda^2))\left(\|x_n - x^*\|^2 + \|y_n - y^*\|^2\right) - (1-\delta_n(1-\lambda^2))\Phi_n \\ &+ 2\delta_n\left(\langle f(x^*) - x^*, x_{n+1} - x^*\rangle + \langle g(y^*) - y^*, y_{n+1} - y^*\rangle\right).\end{aligned}$$

For each $n \geq 0$, we also set

$$\begin{aligned}\eta_n &= (1-\delta_n(1-\lambda^2))\Phi_n, \\ \tau_n &= 2\left(\langle f(x^*) - x^*, x_{n+1} - x^*\rangle + \langle g(y^*) - y^*, y_{n+1} - y^*\rangle\right)/(1-\lambda^2), \\ \zeta_n &= 2\delta_n\left(\langle f(x^*) - x^*, x_{n+1} - x^*\rangle + \langle g(y^*) - y^*, y_{n+1} - y^*\rangle\right).\end{aligned}$$

Then, the above formula is reduced to the following inequalities

$$\theta_{n+1} \leq (1-\delta_n)\theta_n + \delta_n\tau_n \text{ and } \ \theta_{n+1} \leq \theta_n - \eta_n + \zeta_n, n \geq 0.$$

By the boundedness of $\{x_n\}$ and $\{y_n\}$, and Condition (R2), we see that $\lim_{n\to\infty}\zeta_n = 0$ and $\sum_{n=0}^{\infty}\delta_n = \infty$. By virtue of Lemma 2.2, this proof remains to show that $\lim_{k\to\infty}\eta_{n_k} = 0$ implies $\limsup_{k\to\infty}\tau_{n_k} \leq 0$ for any subsequence $\{n_k\}$ of $\{n\}$. Let

$\{\eta_{n_k}\}$ be a any subsequence of $\{\eta_n\}$ such that $\lim_{k\to\infty}\eta_{n_k} = 0$. According to the proof in Theorem 3.1, if $Ax_n \neq By_n$, we have

$$\lim_{k\to\infty}\|(I-P_C)x_{n_k}\| = \lim_{k\to\infty}\|(I-P_Q)y_{n_k}\| = \lim_{k\to\infty}\|Ax_{n_k} - By_{n_k}\| = 0. \tag{4.3}$$

By the boundedness of $\{x_n\}$ and $\{y_n\}$, there exists two sequences $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ and $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup \bar{x}$, $y_{n_{k_j}} \rightharpoonup \bar{y}$ and

$$\limsup_{k\to\infty}\langle f(x^*) - x^*, x_{n_k} - x^*\rangle = \lim_{j\to\infty}\langle f(x^*) - x^*, x_{n_{k_j}} - x^*\rangle,$$

$$\limsup_{k\to\infty}\langle g(y^*) - y^*, y_{n_k} - y^*\rangle = \lim_{j\to\infty}\langle g(y^*) - y^*, y_{n_{k_j}} - y^*\rangle.$$

Since $I - P_C$ and $I - P_Q$ are demiclosed at 0, we have $\bar{x} \in C$ and $\bar{y} \in Q$ by (4.3). In addition, it follows from bounded linearity of A and B that $Ax_{n_{k_j}} - By_{n_{k_j}} \rightharpoonup A\bar{x} - B\bar{y}$. Using the weak lower semicontinuity of the squared norm implies $\|A\bar{x} - B\bar{y}\|^2 \leq \liminf_{j\to\infty}\|Ax_{n_{k_j}} - By_{n_{k_j}}\|^2 = 0$, which implies that $(\bar{x}, \bar{y}) \in \Omega$. On the other hand, if $Ax_n = By_n$, we can also get the same result as above. In addition, from the property of projection, it follows that

$$\begin{aligned}\limsup_{k\to\infty}\langle f(x^*) - x^*, x_{n_k} - x^*\rangle &= \lim_{j\to\infty}\langle f(x^*) - x^*, x_{n_{k_j}} - x^*\rangle\\ &= \langle f(x^*) - x^*, \bar{x} - x^*\rangle \leq 0,\\ \limsup_{k\to\infty}\langle g(y^*) - y^*, y_{n_k} - y^*\rangle &= \lim_{j\to\infty}\langle g(y^*) - y^*, y_{n_{k_j}} - y^*\rangle\\ &= \langle g(y^*) - y^*, \bar{y} - y^*\rangle \leq 0.\end{aligned}$$

According to (R1) and (4.3), we have

$$\|\widehat{x}_{n_k} - x_{n_k}\| \leq \gamma_n(\|(I - P_C)x_{n_k}\| + \|A\|\|Ax_{n_k} - By_{n_k}\|) \to 0, k \to \infty,$$

$$\|\widehat{y}_{n_k} - y_{n_k}\| \leq \gamma_n(\|(I - P_Q)y_{n_k}\| + \|B\|\|Ax_{n_k} - By_{n_k}\|) \to 0, k \to \infty.$$

Further, we obtain

$$\|x_{n_k+1} - x_{n_k}\| = \delta_{n_k}\|f(x_{n_k}) - x_{n_k}\| + (1 - \delta_{n_k})\|\widehat{x}_{n_k} - x_{n_k}\| \to 0, k \to \infty,$$

$$\|y_{n_k+1} - y_{n_k}\| = \delta_{n_k}\|g(y_{n_k}) - y_{n_k}\| + (1 - \delta_{n_k})\|\widehat{y}_{n_k} - y_{n_k}\| \to 0, k \to \infty.$$

Hence, we have $\limsup_{k\to\infty}\langle f(x^*) - x^*, x_{n_k+1} - x^*\rangle \leq 0$ and $\limsup_{k\to\infty}\langle g(y^*) - y^*, y_{n_k+1} - y^*\rangle \leq 0$, which implies $\limsup_{k\to\infty}\tau_{n_k} \leq 0$. By Lemma 2.2, we obtain $\lim_{n\to\infty}\theta_n = 0$. This implies that $(x_n, y_n) \to (x^*, y^*)$. □

4.2. **Self-adaptive relaxed viscosity-type CQ algorithm (SRVCQA).** In this subsection, we first set the same C, Q, C_n and Q_n as in Theorem 3.5. In addition, the combination of the relaxed CQ algorithm and the viscosity-type algorithm has the following algorithm

$$\begin{cases}\widehat{x}_n = x_n - \gamma_n\left[(I - P_{C_n})x_n + A^*(Ax_n - By_n)\right],\\ x_{n+1} = \delta_n f(\widehat{x}_n) + (1 - \delta_n)\widehat{x}_n,\\ \widehat{y}_n = y_n - \gamma_n\left[(I - P_{Q_n})y_n - B^*(Ax_n - By_n)\right],\\ y_{n+1} = \delta_n g(\widehat{y}_n) + (1 - \delta_n)\widehat{y}_n, n \geq 0,\end{cases} \tag{4.4}$$

where the corresponding parameters $\{\gamma_n\}$ and $\{\delta_n\}$ are defined as (R1) and (R2).

Theorem 4.2. *Given Assumption* (V1) *and Conditions* (R1)-(R2)*. For any* $x_0 \in \mathcal{H}_1$ *and* $y_0 \in \mathcal{H}_2$*, the iterative sequence* $\{(x_n, y_n)\}$ *generated by Algorithm* (4.4) *converges strongly to* $(x^*, y^*) = P_\Omega(f(x^*), g(y^*))$*.*

Proof. According to the proof of Theorems 3.5 and 4.1, it follows from Lemma 2.2 that the sequence $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) = P_\Omega(f(x^*), g(y^*))$. □

Remark 4.3. Obviously, when the contraction mappings are constant mappings, that is, $f(x) \equiv u$, $\forall x \in \mathcal{H}_1$ and $g(y) \equiv v$, $\forall y \in \mathcal{H}_2$. This shows that the viscosity algorithm is equivalent to the Halpern algorithm. It follows that the self-adaptive viscosity-type CQ algorithm (4.1) and the self-adaptive relaxed viscosity-type CQ algorithm (4.4) are actually Algorithm (3.12) in Corollary 3.4 and Algorithm (3.15) in Corollary 3.6, respectively.

Remark 4.4. When $B = I$ and $\mathcal{H}_2 = \mathcal{H}_3$, the split equality problem is equivalent to the split feasibility problem. So, based on the previous conclusion, SHCQA, SHRCQA, SVCQA and SRVCQA are applied to the split feasibility problems and their strong convergence are also guaranteed under certain conditions.

5. An application to signal recovery

Compressed sensing is a popular and effective method for recovering a clean signal from a polluted signal. To solve this problem, the following question needs to be considered:

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \varepsilon,$$

where $\mathbf{y} \in \mathbb{R}^M$ is the observed noise data, $\mathbf{A} : \mathbb{R}^{M\times N}$ is a bounded linear observation operator, $\mathbf{x} \in \mathbb{R}^N$ with k ($k \ll N$) non-zero elements is the original and clean data that needs to be restored, and ε is the noise observation encountered during data transmission. An important characteristic of such a problem is that the signal $\mathbf{x}$ is sparse which means that the number of non-zero elements in the signal $\mathbf{x}$ is much smaller than the dimension of the signal $\mathbf{x}$. To overcome this difficulty, a familiar and practical model of the problem, namely the following model of the convex constraint minimization problem, is considered to characterize the above problem:

$$\min_{\mathbf{x}\in\mathbb{R}^N} \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2 \quad \text{subject to} \quad \|\mathbf{x}\|_1 \le t\,, \tag{5.1}$$

where t is a positive constant and $\|\cdot\|_1$ is ℓ_1 norm. It should be pointed out that this problem is related to the least absolute shrinkage and selection operator problem. Note that the problem (5.1) described above can be regarded as a special case of the split equality problem when $C = \left\{x \in \mathbb{R}^N \mid \|\mathbf{x}\|_1 \le t\right\}$, $Q = \{\mathbf{y}\}$, $B = I$ and $\mathcal{H}_2 = \mathcal{H}_3$. Hence, the proposed algorithms (SHCQA and SVCQA) can be applied to the approximation solution of (5.1), for more detail, see [11].

6. Numerical experiments

In this section, all codes were written in Matlab R2018b, and ran on a Lenovo ideapad 720S with 1.6 GHz Intel Core i5 processor and 8GB of RAM. We consider some numerical experiments to demonstrate the efficiency of our results and compare them with the existing alternating CQ algorithm (ACQA) in Moudafi [13], simultaneous CQ algorithm (SCQA) in Byrne and Moudafi [3] and the following Dong et al. algorithm in [8].

Theorem 6.1. [8] *Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ be Hilbert spaces, C and Q be two nonempty closed subsets of $\mathcal{H}_1$, $\mathcal{H}_2$, respectively. Let $A : \mathcal{H}_1 \to \mathcal{H}_3$ and $B : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators, A^* and B^* be the adjoint operators of A, B, respectively. Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ be two contraction mappings with coefficients $\lambda_1 \in (0, \frac{\sqrt{2}}{2})$, $\lambda_2 \in (0, \frac{\sqrt{2}}{2})$, respectively. For any $x_0 \in \mathcal{H}_1$ and $y_0 \in \mathcal{H}_2$, the iterative sequence $\{(x_n, y_n)\}$ is generated by the following iterative scheme*

$$\begin{cases} x_{n+1} = \delta_n f(x_n) + (1-\delta_n)P_C(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = \delta_n g(y_n) + (1-\delta_n)P_Q(y_n + \gamma_n B^*(Ax_n - By_n)), \ \forall n \geq 0, \end{cases} \tag{6.1}$$

where $\delta_n \in (0,1)$ such that $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=0}^{\infty} \delta_n = \infty$ and the stepsize

$$\gamma_n = \alpha_n \min\left\{ \frac{\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2}, \frac{\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2} \right\}, \ \alpha_n \in (\varepsilon, 1-\varepsilon) \subset (0,1).$$

Then the iterative sequence $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) \in \Omega$.

Remark 6.2. From the numerical results of Dong et al. algorithm in [8], they considered a suitable stepsize selection based on Algorithm (6.1), that is,

$$\gamma_n = 0.65 \times \min\left\{ \frac{\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2}, \frac{\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2} \right\}.$$

Based on the above results, we will carry out the following work and obtain the corresponding numerical results to characterize the effectiveness and superiority of our algorithms.

(Test environment) According to the setting conditions of SEP, we choose the following conditions: $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \mathbb{R}^3$, $C = \{(x_1, x_2, x_3)^T \in \mathcal{H}_1 \mid x_2^2 + x_3^2 - 1 \leq 0\}$, and $Q = \{(y_1, y_2, y_3)^T \in \mathcal{H}_2 \mid y_1^2 - y_2 + 5 \leq 0\}$, in addition, $A = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

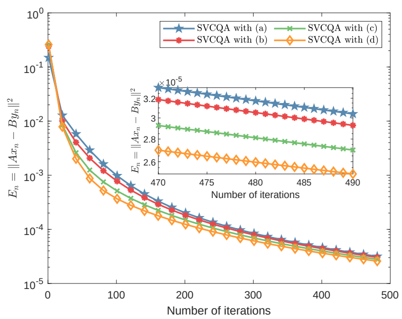
By the above matrixes A and B, we can easily get the corresponding adjoint operators of A and B, that is, $A^* = A^T$ and $B^* = B^T$. Under the above assumption, it is easy to prove that (x^*, y^*) is a unique solution of problem (1.1), where $x^* = (0,1,0)^T$, $y^* = (0,5,0)^T$. The norm $\|Ax_n - By_n\|^2$ as an error estimate and denoted by E_n for all of the following examples. Next, we study and analyze our numerical experiments in such an environment.

Example 6.3. In the above test environment, we will analyze the convergence behavior of Algorithm (4.1) (SVCQA) in Theorem 4.1. Firstly, initial points x_0, y_0 generated randomly in $\mathbb{R}^3$,

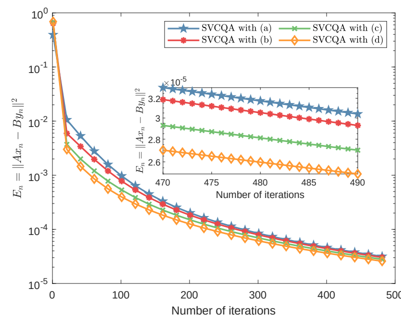
$$\gamma_n = \alpha_n \min\left\{ 1, \frac{\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} \right\} \ \text{ with } \ \alpha_n = \frac{99n}{100n+1},$$

and take contraction mappings $f(x) = 0.5x$, $g(y) = 0.5y$. We consider the following four cases of the parameter δ_n: (a) $\delta_n = \frac{1}{n+1}$, (b) $\delta_n = \frac{1}{n+10}$, (c) $\delta_n = \frac{1}{n+30}$, (d)

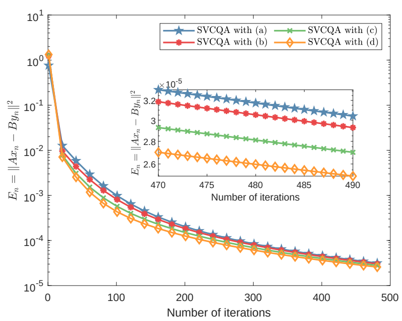
$\delta_n = \frac{1}{n+50}$. The numerical results of Algorithm (4.1) (SVCQA) for any initial points x_0, y_0 are shown in Figure 1.



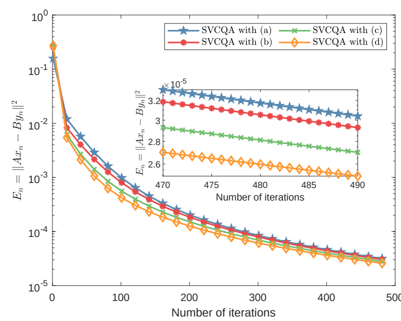
(a) Case I



(b) Case II



(c) Case III



(d) Case IV

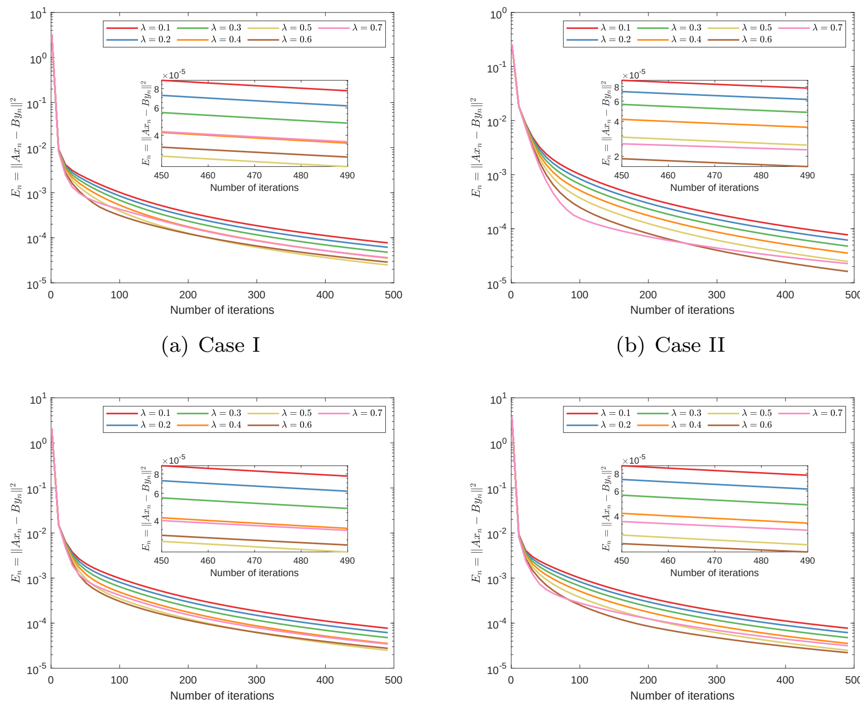
FIGURE 1. The numerical results of four parameter choices of SVCQA

Remark 6.4. In Figure 1, we can easily see that all the results are valid and convergent. Under the same number of iterations, the error accuracy of the fourth setting (d) is better than all other cases for the different parameters δ_n in Algorithm (4.1). In view of this, we choose $\delta_n = \frac{1}{n+50}$ in Algorithm (4.1).

Example 6.5. For Algorithm (4.1) (SVCQA) in Theorem 4.1, we further consider the choice of contraction mappings f and g. Firstly, initial points x_0, y_0 generated randomly in $\mathbb{R}^3$,

$$\gamma_n = \alpha_n \min\left\{1, \frac{\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2}\right\} \quad \text{with} \quad \alpha_n = \frac{99n}{100n+1},$$

we choose directly the parameter $\delta_n = \frac{1}{n+50}$ and consider different contraction mappings $f(x) = \lambda_1 x$ and $g(y) = \lambda_2 y$, $\lambda = \lambda_1 = \lambda_2 \in [0, 1/\sqrt{2})$ for any $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$. The numerical results of Algorithm (4.1) (SVCQA) for any initial points x_0, y_0 are shown in Figure 2.



(a) Case I (b) Case II

(c) Case III (d) Case IV

FIGURE 2. The numerical results of the different contraction mappings of SVCQA

Remark 6.6.
- In Figure 2, we can see that all the results are valid and convergent. Figure 2 shows that the coefficients λ_1 and λ_2 of contraction mappings f and g have better convergence results in the range $(0.5, 0.7)$ for any initial points under the parameter $\delta_n = \frac{1}{n+50}$.
- By virtue of the numerical results of Examples 5.1 and 5.2, we have analyzed the different choices of parameters δ_n and contractions mapping in Algorithm (4.1) (SVCQA) in Theorem 4.1. Further, we have got the best results, when

$$\delta_n = \frac{1}{n+50}, \ f(x) = 0.6x, \ g(y) = 0.6y.$$

Example 6.7. For the four algorithms mentioned in this paper: alternating CQ algorithm (ACQA) in Moudafi [13] (i.e., Algorithm (1.4)), simultaneous CQ algorithm (SCQA) in Byrne and Moudafi [3] (i.e., Algorithm (1.5)), Dong et al. algorithm in [8] (i.e., Algorithm (6.1)) and our algorithm (4.1) (SVCQA). We compare the number of iterations of four algorithms with different initial points at the same iteration error accuracy. Firstly, we set the corresponding parameters as follows:

(I) Take γ_n in ACQA and SCQA as $0.9\min(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2})$ and $0.9\frac{2}{\|A\|^2+\|B\|^2}$, respectively;
(II) Take the parameters in Algorithm (6.1) as $\delta_n = \frac{1}{n+50}$, $f(x) = 0.6x$, $g(y) = 0.6y$ and

$$\gamma_n = 0.65\min\left\{\frac{\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2}, \frac{\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2}\right\}.$$

(III) Take the parameters in Algorithm (4.1) (SVCQA) as $\delta_n = \frac{1}{n+50}$, $f(x) = 0.6x$, $g(y) = 0.6y$ and

$$\gamma_n = \alpha_n \min\left\{1, \frac{\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2}\right\} \text{ with } \alpha_n = \frac{99n}{100n+1}.$$

For four different initial values, Figure 3 shows the iteration error E_n of the four algorithms under the same number of iterations, and Table 1 shows the number of iterations of the four algorithms at the same iteration error accuracy.

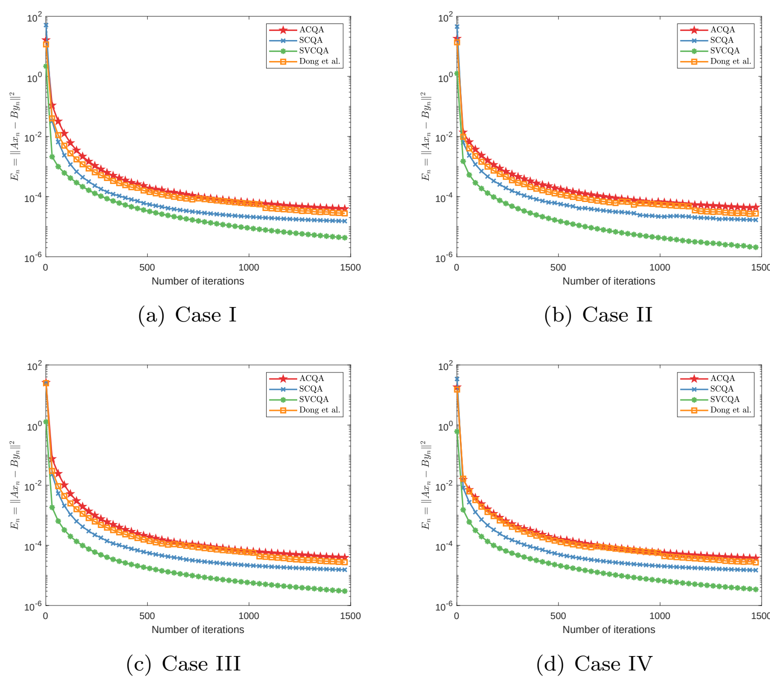


FIGURE 3. The numerical results of four algorithms

TABLE 1. Number of iterations for different error estimates

Initial point (x_0, y_0)	$\|Ax_n - By_n\|^2$	Number of iterations			
		SVCQA	ACQA	SCQA	Dong et al.
$x_0 = (0.7922, 0.9595, 0.6557)^{\mathrm{T}}$ $y_0 = (0.0357, 0.8491, 0.9340)^{\mathrm{T}}$	10^{-2}	10	100	52	65
	10^{-3}	61	249	129	197
	10^{-4}	277	750	373	627
$x_0 = (0.6787, 0.7577, 0.7431)^{\mathrm{T}}$ $y_0 = (0.3922, 0.6555, 0.1712)^{\mathrm{T}}$	10^{-2}	9	43	21	31
	10^{-3}	41	196	100	151
	10^{-4}	179	721	349	564
$x_0 = (0.7060, 0.0318, 0.2769)^{\mathrm{T}}$ $y_0 = (0.0462, 0.0971, 0.8235)^{\mathrm{T}}$	10^{-2}	8	92	47	59
	10^{-3}	47	241	124	192
	10^{-4}	180	757	366	624
$x_0 = (0.1190, 0.4984, 0.9597)^{\mathrm{T}}$ $y_0 = (0.3404, 0.5853, 0.2238)^{\mathrm{T}}$	10^{-2}	6	47	27	45
	10^{-3}	43	193	104	173
	10^{-4}	181	686	343	596

Remark 6.8. From Figure 3 and Table 1, we can see that our proposed algorithm (4.1) (SVCQA) outperforms alternating CQ algorithm (ACQA), simultaneous CQ algorithm (SCQA), Dong et al. algorithm (6.1) in both error accuracy and number of iterations.

Example 6.9. Consider $\mathcal{H}_1 = \mathcal{H}_2 = L_2([0, 2\pi])$ with the inner product $\langle x, y\rangle := \int_0^{2\pi} x(t)y(t)dt$ and with the associated norm which given by $\|x\|_2 := \left(\int_0^{2\pi} |x(t)|^2 dt\right)^{\frac{1}{2}}$, $\forall x, y \in L_2([0, 2\pi])$. The closed convex subsets are defined by

$$C = \left\{x \in L_2([0, 2\pi]) \mid \int_0^{2\pi} x(t)dt \le 1\right\}$$

and

$$Q = \left\{y \in L_2([0, 2\pi]) \mid \int_0^{2\pi} |y(t) - \sin(t)|^2 \, dt \le 16\right\}.$$

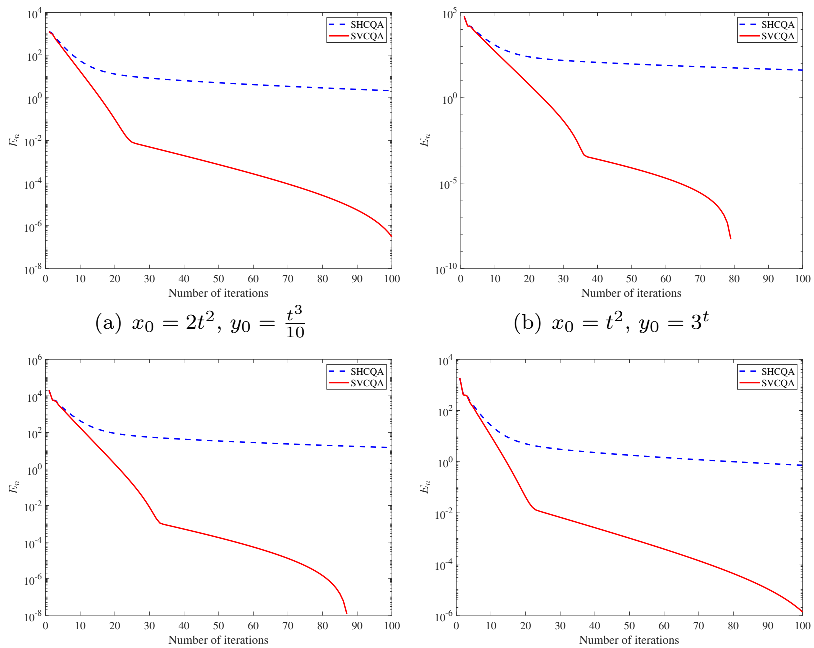
Let us define a linear continuous operator $A : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ by $(Ax)(t) := x(t)$. So $(A^*x)(t) = x(t)$ and $\|A\| = 1$. Now, we solve the split equality problem: find $x^* \in C$, $y^* \in Q$ and $Ax^* = By^*$, where $B = I$ and $\mathcal{H}_2 = \mathcal{H}_3$. The projection operators P_C and P_Q on the sets C and Q, respectively, are written as follows:

$$P_C(x) = \begin{cases} \frac{1-\int_0^{2\pi} x(t)dt}{4\pi^2} + x, & \int_0^{2\pi} x(t)dt > 1, \\ x, & \int_0^{2\pi} x(t)dt \le 1, \end{cases}$$

and

$$P_Q(y) = \begin{cases} \sin(t) + \frac{4}{\sqrt{\int_0^{2\pi} |y(t)-\sin(t)|^2 dt}}(y - \sin(t)), & \int_0^{2\pi} |y(t) - \sin(t)|^2 dt > 16, \\ y, & \int_0^{2\pi} |y(t) - \sin(t)|^2 dt \le 16. \end{cases}$$

$E_n = \|(I - P_C)x_n\|^2 + \|(I - P_Q)Ax_n\|^2$ is used to measure the iteration error. The stopping criterion is maximum number of iterations which is set to 100. Choose $u_n = x_0$ and $v_n = y_0$ in SHCQA, all the rest of the parameters in SHCQA and SVCQA are selected as set in Example 6.7. Figure 4 shows the numerical behavior of SHCQA and SVCQA with four different initial values.



(a) $x_0 = 2t^2$, $y_0 = \frac{t^3}{10}$ (b) $x_0 = t^2$, $y_0 = 3^t$

(c) $x_0 = t^2$, $y_0 = e^t$ (d) $x_0 = \sin t$, $y_0 = 3t^2$

FIGURE 4. The numerical results of SHCQA and SVCQA for four different initial choices

Example 6.10. According to the description of signal recovery problem in Section 5, SHCQA and SVCQA are used to solve the problem (5.1). Therefore, we consider the following computing environment: The observation $\mathbf{y}$ is formed by $\mathbf{y} = \mathbf{Ax} + \varepsilon$ with white Gaussian noise ϵ of variance 10^{-4}. The matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$ is created from a standard normal distribution with zero mean and unit variance and then orthonormalizing the rows. The clean signal $\mathbf{x} \in \mathbb{R}^N$ contains k ($k \ll N$) randomly generated ± 1 spikes. The recovery process starts with the initial signals $\mathbf{x_0} = \mathbf{0}$, $\mathbf{y_0} = \mathbf{0}$ and ends after 2000 iterations. The mean squared error $MSE = (1/N)\|\mathbf{x}^* - \mathbf{x}\|^2$ ($\mathbf{x}^*$ is an estimated signal of $\mathbf{x}$) to measure the restoration accuracy of our algorithms. In addition, set $M = 256$, $N = 512$ and $k = 50$. The parameters of SHCQA and SVCQA are set as in Example 6.9. Figure 5 displays the original signal and the contaminated signal. The recovery results of the suggested algorithms are shown in Figure 6.

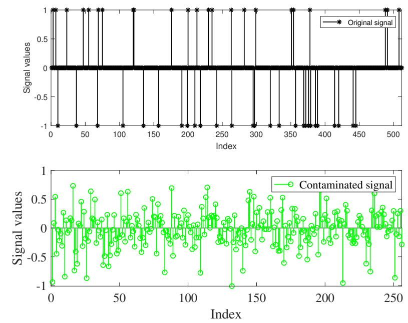
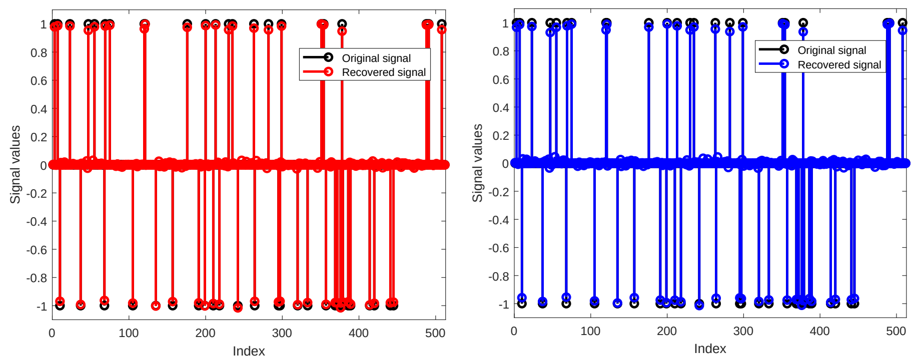


FIGURE 5. Original signal and contaminated signal



(a) Recovered by SVCOA (b) Recovered by SHCQA

FIGURE 6. The original signal and the signal recovered by SVCQA and SHCQA

7. CONCLUSION

The first conclusion from Sections 3 and 4 is that we propose four self-adaptive CQ algorithms by using the Halpern algorithm and the viscosity algorithm for solving SEP under the condition of a self-adaptive stepsize sequence. A point should be stressed is that such a self-adaptive stepsize sequence does not depend on the prior knowledge of operator norms. The second conclusion from the numerical results in Section 6 is that the convergence of our algorithm is validity and authenticity. Meanwhile, our proposed self-adaptive viscosity-type CQ algorithm (4.1) (SVCQA) improves and extends the existing results.

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