AN INERTIAL SHRINKING PROJECTION ALGORITHM FOR SPLIT COMMON FIXED POINT PROBLEMS

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Abstract In this paper, the purpose is to introduce and study a new modified shrinking projection algorithm with inertial effects, which solves split common fixed point problems in Banach spaces. The corresponding strong convergence theorems are obtained without the assumption of semi-compactness on mappings. Finally, some numerical examples are presented to illustrate the results in this paper.

Keywords Split common fixed point problem, shrinking projection method, inertial method, strong convergence.

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1. Introduction

To model inverse problems in phase retrievals and medical image reconstruction [2], Censor and Elfving [6] introduced the concept of the split feasibility problem (for short, SFP) in framework of finite-dimensional Hilbert spaces in 1994. It has been founded that the SFP can be used in many areas of applications, such as image restoration, computer tomograph, radiation therapy treatment planning and other areas of mathematical research [3,5,7–9,16,21].

As a generalization of the SFP in 2009, Censor and Segal [10] introduced the following split common fixed point problem (for short, SCFPP): Let $H_1$ and $H_2$ be two Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator, $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be two mappings ($F(T)$ and $F(S)$ denote the fixed point sets of $T$ and $S$, respectively). The split common fixed point problem for mappings $T$ and $S$ which is to find a point $x^*$ satisfying

$$x^* \in F(T) \text{ and } Ax^* \in F(S).$$

The solution set of (1.1) is denoted by $\Gamma$, i.e., $\Gamma = \{x^* | x^* \in F(T), Ax^* \in F(S)\}$.

Since then, the SCFPP has been widely studied by many authors in Hilbert spaces (see [11,12,22,23,31]). Usually in order to achieve strong convergence properties of the SCFPP, we often do this directly by considering the assumption of semi-compactness on the mappings. In addition, there are papers using some algorithms to replace the assumption of the mappings, such as Halpern iterative algorithm, viscosity iterative algorithm, shrinking projection algorithm. Ulteriorly,
there are a few studies on the split common fixed point problem in the framework of Banach spaces. For example in recent years, Takahashi and Yao [28] and Takahashi [26, 27] studied the split common fixed point problem in the setting of one Hilbert space and one Banach space, and obtained some weak convergence theorems and strong convergence theorems. Furthermore, in the framework of two Banach spaces, there are many studies established as follows:

In 2015, Tang et al. [29] studied the SCFPP for asymptotically nonexpansive mappings and quasi-strictly pseudo-contractive mappings. Then, the strong convergence theorem was also proved with the condition of semi-compactness on the mappings. In 2016, Shehu et al. [25] studied split feasibility problems and fixed point problems for left Bregman strongly nonexpansive mappings, and showed strong convergence theorems by Halpern iterative method.

Recently, Ma et al. [18] also studied split feasibility problems and fixed point problems in Banach spaces, and obtained the strong convergence theorem by the following shrinking projection iterative algorithm

$$\begin{align*}
z_n &= J_1^{-1}(J_1 x_n + \gamma A^* J_2 (P_Q - I) A x_n), \\
y_n &= J_1^{-1}((1 - \alpha_n) J_1 z_n + \alpha_n J_1 S z_n), \\
C_{n+1} &= \{ v \in C_n : \phi(v, y_n) \leq \phi(v, x_n), \phi(v, z_n) \leq \phi(v, x_n) \}, \\
x_{n+1} &= \Pi_{C_{n+1}} x_1, \quad n \geq 1,
\end{align*}$$

where $E_1$ is a 2-uniformly convex and 2-uniformly smooth real Banach space, $E_2$ is a smooth, strictly convex and reflective Banach space and $Q$ is a nonempty closed convex subset of $E_2$, $A : E_1 \to E_2$ is a bounded linear operator with the adjoint operator $A^*$, $S : E_1 \to E_1$ is a closed quasi-$\phi$-nonexpansive mapping, $P_Q : E_2 \to Q$ is the metric projection and $\Pi_{C_{n+1}} : E_1 \to C_{n+1}$ is the generalized projection.

In view of the above studies and methods and in order to accelerate better the convergence rate of the iterative algorithms. The inertial effects have been studied recently by many authors in terms of variational inequality problems, inclusion problems, equilibrium problems, etc., see [1, 4, 15, 17, 19, 24] and the references therein.

The main characteristic of the inertial method is that the new iterate process is produced by making use of two values of the previous iterative point. In 2001, Alvarez and Attouch [1] studied the problem of approximating the null point of a maximal monotone operator and proposed the following inertial proximal algorithm:

$$\begin{align*}
y_n &= x_n + \alpha_n (x_n - x_{n-1}), \\
x_{n+1} &= (I + \lambda T)^{-1} y_n, \forall n \geq 1.
\end{align*}$$

Then, they obtained the weak convergence of the algorithm.

For these research, the ideas of this paper are as follows: this article introduce a new shrinking projection iterative algorithm with inertial effects to solve problem (1.1) for firmly nonexpansive-like mappings in the framework of $p$-uniformly convex and uniformly smooth real Banach spaces. Meanwhile, the strong convergence theorems of this algorithm are obtained without assumption of semi-compactness on mappings. As applications, the results are utilized to split fixed point problems and variational inclusion problems, split fixed point problem and equilibrium problems. Furthermore, some numerical examples are used to demonstrate and show the efficiency of our main results. To this end, some basic properties and relevant lemmas will be introduced in next section which will be used in the proof for the convergence analysis of the proposed algorithm.
2. Preliminaries

Throughout this paper, we use notations: → to denote the strong convergence and
→ to denote the weak convergence. The set of all fixed points of T is denoted by
F(T).

Let E be a Banach space. A function \( \delta_E : [0, 2] \to [0, 1] \) is called the modulus
of convexity of E as follows:

\[
\delta_E(\varepsilon) = \inf \{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \}.
\]

A function \( \rho_E : [0, +\infty] \to [0, +\infty] \), which is called the modulus of smoothness of
E as follows:

\[
\rho_E(t) = \sup \{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t \}.
\]

**Definition 2.1.** A Banach space E is said to be

(I) uniformly convex if for any \( x, y \in E \) with \( \|x\| = \|y\| = 1 \) and \( \|x - y\| \geq \varepsilon \),
there exists \( \eta = \eta(\varepsilon) > 0 \) for all \( \varepsilon \in (0, 2] \) such that \( \frac{\|x + y\|}{2} \leq 1 - \eta \). This is equivalent to \( \delta_E(\varepsilon) > 0 \), for all \( \varepsilon \in (0, 2] \).

(II) uniformly smooth if and only if \( \lim_{t \to 0} \frac{\rho_E(t)}{t} = 0 \).

A Banach space is called \( p \)-uniformly convex if there exists a constant \( c > 0 \) such that \( \delta_E(\varepsilon) > c\varepsilon^p \) for all \( \varepsilon \in (0, 2] \), where the constant \( \frac{1}{c} \) is called the \( p \)-
uniformly convexity constant. It is obvious that a \( p \)-uniformly convex Banach space
is uniformly convex. A Banach space is said to be \( q \)-uniformly smooth if there exists
a constant \( C_q > 0 \) such that \( \rho_E(t) \leq C_q t^q \) for all \( t > 0 \), where \( C_q \) is the \( q \)-uniformly
smoothness constant. In addition, E is a \( p \)-uniformly convex and uniformly smooth
Banach space if and only if its dual \( E^* \) is a \( q \)-uniformly smooth and uniformly convex
Banach space.

Let \( E \) be a Banach space with the dual \( E^* \). The duality mapping \( J_E^p : E \to 2^{E^*} \)
is defined by \( J_E^p(x) = \{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p - 1} \}, \ p > 1, \forall x \in E \).

**Definition 2.2.** For a Gâteaux differentiable convex function \( f : E \to R \), the function

\[
\Delta_f(x, y) := f(y) - f(x) - \langle f'(x), y - x \rangle, \ \forall x, y \in E
\]

is called the Bregman distance of \( x \) to \( y \) with respect to the function \( f \).

In addition, the duality mapping \( J_E^p \) is the derivative of the function \( f_p(x) = \frac{1}{p} \|x\|^p \). Then the Bregman distance with respect to \( f_p \) can be written as

\[
\Delta_p(x, y) = \frac{1}{q} \|x\|^p - \langle J_E^p(x, y) \rangle + \frac{1}{p} \|y\|^p
= \frac{1}{p} (\|y\|^p - \|x\|^p) + \langle J_E^p(x, y), x - y \rangle
= \frac{1}{q} (\|x\|^p - \|y\|^p) - \langle J_E^p(x, y) - J_E^p(y, y) \rangle.
\]

**Definition 2.3.** Let \( C \) be a nonempty closed convex subset of a Banach space \( E \). The mapping \( T : C \to E \) is said to be
(I) left Bregman quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and
\[
\Delta_p(Tx, x^*) \leq \Delta_p(x, x^*), \forall x \in C, x^* \in F(T);
\] (2.2)

(II) firmly nonexpansive-like mapping if
\[
\langle Tx - Ty, J_E^p(x - Tx) - J_E^p(y - Ty) \rangle \geq 0, \forall x, y \in C.
\] (2.3)

Obviously, if $E$ is a Hilbert space, the firmly nonexpansive-like mapping reduces to the firmly nonexpansive mapping, i.e., $\langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|^2, \forall x, y \in C.$

**Example 2.1.** (1) Let $E$ be a smooth, strictly convex and reflexive Banach space and $C$ be a nonempty closed convex subset of $E$. Then, the metric projection $P_C$ is a firmly nonexpansive-like mapping.

(2) Let $E$ be a real number space $R$ with Euclidean norm. A mapping $T : [-10, 10] \to [-10, 10]$ is defined by $Tx = \frac{1}{4}x$, $\forall x \in [-10, 10].$ For any $x, y \in [-10, 10]$, we easily get the following result
\[
\langle Tx - Ty, J_E^p(x - Tx) - J_E^p(y - Ty) \rangle = \left\langle \frac{1}{4}x - \frac{1}{4}y, (x - \frac{1}{4})^3 - (y - \frac{1}{4})^3 \right\rangle
\]
\[
= \frac{1}{4} \times \left( \frac{3}{4} \right)^3 (x - y)^2 (x^3 - y^3)
\]
\[
= \frac{1}{4} \times \left( \frac{3}{4} \right)^3 (x - y)^2 (x^2 + xy + y^2) \geq 0,
\]

which implies that $T$ is a firmly nonexpansive-like mapping.

Let $T : C \to E$ be a mapping. A point $u$ is said to be an asymptotic fixed point of $T$ if there exists a sequence $\{x_n\}$ in $C$ such that $x_n \to u$ and $x_n - Tx_n \to 0$. The set of all asymptotic fixed points of $T$ is denoted by $\tilde{F}(T)$.

**Lemma 2.1.** Let $E$ be a smooth, strictly convex and reflexive Banach space, and $C$ be a nonempty closed convex subset of $E$. $T : C \to E$ is a firmly nonexpansive-like mapping. Then $F(T)$ is a closed convex subset of $E$ and $\tilde{F}(T) = F(T)$.

**Proof.** In order to prove that $F(T)$ is a closed convex set, we assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to u$. From the definition of $T$, we have $\langle x_n - Tu, -J_E^p(u - Tu) \rangle \geq 0.$ This inequality is equivalent also to
\[
\|u - Tu\|^p \leq \langle x_n - u, J_E^p(Tu - u) \rangle \leq \|x_n - u\| \|u - Tu\|^{p-1}.
\]

Then we obtain $\|u - Tu\| = 0$ as $n \to \infty$. This implies $u = Tu$. Hence, $u \in F(T)$ and $F(T)$ is closed.

Next, we show that $F(T)$ is convex. For any $x, y \in F(T)$ and $t \in (0, 1)$, putting $u = tx + (1-t)y$. From the definition of $T$, we get $\langle x - Tu, -J_E^p(u - Tu) \rangle \geq 0$ and $\langle y - Tu, -J_E^p(u - Tu) \rangle \geq 0$. Combine the above two inequalities, we have
\[
\langle tx + (1-t)y - Tu, -J_E^p(u - Tu) \rangle \geq 0 \Leftrightarrow \langle u - Tu, -J_E^p(u - Tu) \rangle \geq 0
\]
\[
\Leftrightarrow \|u - Tu\|^p \leq 0.
\]

This means that $u = Tu$. So, $F(T)$ is closed and convex.

Lastly, we show that $\tilde{F}(T) = F(T)$. It is obvious that $F(T) \subset \tilde{F}(T)$. Then, we only show that $\tilde{F}(T) \subset F(T)$. For any $z \in \tilde{F}(T)$, there exists a sequence $\{x_n\}$ in $C$ such that $x_n \to z$ and $x_n - Tx_n \to 0$. From the definition of $T$, we have
\[
\langle Tx_n - Tz, J_E^p(x_n - Tx_n) - J_E^p(z - Tz) \rangle \geq 0.
\]
This is equivalent to
\[ \langle Tx_n - Tz, J_E^p(x_n - Tx_n) \rangle \geq \langle Tx_n - Tz, J_E^p(z - Tz) \rangle = \langle Tx_n - z + z - Tz, J_E^p(z - Tz) \rangle = \langle Tx_n - z, J_E^p(z - Tz) \rangle + \|z - Tz\|^p. \]

The inequality can be transformed the following inequality
\[ \|z - Tz\|^p \leq \langle z - Tx_n, J_E^p(z - Tz) \rangle + \langle Tx_n - Tz, J_E^p(x_n - Tx_n) \rangle = \langle z - x_n, J_E^p(z - Tz) \rangle + \langle x_n - Tx_n, J_E^p(z - Tz) \rangle + \langle Tx_n - x_n, J_E^p(x_n - Tx_n) \rangle + \langle x_n - Tz, J_E^p(x_n - Tz) \rangle. \]

From the setting of \( n \to \infty \), we have \( \|z - Tz\| = 0 \). Hence, \( z = Tz \), i.e., \( z \in F(T) \).

**Definition 2.4.** Let \( C \) be a nonempty closed convex subset of a real Banach space \( E \). A mapping \( T : C \to C \) is closed (or \( T \) has closed graph), that is, if the sequence \( \{x_n\} \) in \( C \) converges strongly to a point \( x \in C \) and \( Tx_n \to y \), then \( Tx = y \).

**Lemma 2.2 ([13]).** Let \( E \) be a Banach space and \( J_E^p \) be the duality mapping of \( E \). Then, the following statements hold:

1. \( J_E^p(x) \) is nonempty bounded closed and convex, for any \( x \in E \);
2. \( J_E^p \) is a mapping from \( E \) onto \( E^* \);
3. \( J_E^p \) is a smooth Banach space, then \( J_E^p \) is single valued;
4. \( J_E^p \) is a uniformly smooth Banach space, then \( J_E^p \) is norm-to-norm uniformly continuous on each bounded subset of \( E \).

**Remark 2.1.** By the definition of \( \Delta_p \), we easily obtain
\[ \Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J_E^p x - J_E^p z \rangle, \forall x, y, z \in E, \] (2.4)
and
\[ \Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_E^p x - J_E^p y \rangle, \forall x, y, z \in E. \] (2.5)

In addition, it is easy to see from the above that the Bregman distance is not symmetrical, and for \( p \)-uniformly convex Banach spaces, we have
\[ \tau \|x - y\|^p \leq \Delta_p(x, y) \leq \langle x - y, J_E^p x - J_E^p y \rangle, \forall x, y \in E, \tau > 0. \] (2.6)

This indicates that Bregman distance is non-negative.

**Definition 2.5.** \( \Pi_C : E \to C \) is said to be the Bregman projection mapping, that is,
\[ \Pi_C x = \arg\min_{y \in C} \Delta_p(x, y), \forall x \in E. \] (2.7)

In other words, \( \Pi_C x \) corresponds a unique element \( x_0 \in C \) such that
\[ \Delta_p(x, x_0) = \min_{y \in C} \Delta_p(x, y), \forall x \in E. \]

The Bregman projection can also be characterized by the following inequality
\[ \langle J_E^p x - J_E^p \Pi_C x, z - \Pi_C x \rangle \leq 0, \forall z \in C, \] (2.8)
this is equivalent to
\[ \Delta_p(\Pi_C x, z) \leq \Delta_p(x, z) - \Delta_p(x, \Pi_C x), \forall z \in C. \] (2.9)
Lemma 2.3 (\cite{30}). Let $E$ be a $q$-uniformly smooth Banach space with the $q$-uniformly smoothness constant $C_q > 0$. For any $x,y \in E$, the following inequality holds:
\[
\|x - y\|^q \leq \|x\|^q - q\langle y, J_E^q x \rangle + C_q \|y\|^q.
\]
Lemma 2.4 (\cite{25}). Let $E$ be a $p$-uniformly convex and uniformly smooth Banach space and with its dual $E^*$, $J_{E^*}$ and $J_{E^*}^q$ are the duality mappings of $E$ and $E^*$, respectively. For any $\{x_n\} \subset E$, and $\{t_n\} \subset (0, 1)$ with $\sum_{n=1}^N t_n = 1$, the following inequality holds. $\Delta_p(J_{E^*}^{|E^*|}(\sum_{n=1}^N t_n J_{E^*}(x_n)), x) \leq \sum_{n=1}^N t_n \Delta_p(x, x), \forall x \in E$.
Lemma 2.5. Let $E$ be a $p$-uniformly convex and uniformly smooth real Banach space, and $C_1 = E$. Then, for any sequences $\{y_n\}, \{z_n\}$ and $\{w_n\}$ in $E$, the set $C_{n+1} = \{u \in C_n : \Delta_p(y_n, u) \leq \Delta_p(z_n, u) \leq \Delta_p(w_n, u)\}$ is closed and convex for each $n \geq 1$.
\textbf{Proof.} First, since $C_1 = E$, $C_1$ is closed and convex. Then we assume that $C_n$ is a closed and convex. For each $u \in C_n$, by the definition of the function $\Delta_p$, we have
\[
\Delta_p(y_n, u) \leq \Delta_p(z_n, u) \Leftrightarrow 2\langle J_{E^*}^q z_n - J_{E^*}^q y_n, u \rangle \leq \frac{1}{q}(\|z_n\|^p - \|y_n\|^p),
\]
and
\[
\Delta_p(z_n, u) \leq \Delta_p(w_n, u) \Leftrightarrow 2\langle J_{E^*}^p w_n - J_{E^*}^q z_n, u \rangle \leq \frac{1}{q}(\|w_n\|^p - \|z_n\|^p).
\]
Hence, we know that $C_{n+1}$ is closed. In addition, we can easily prove that $C_{n+1}$ is a convex. The proof is completed. 

\section{Main Results}
In the section, we assume that the following conditions are satisfied:

(1) $E_1$ and $E_2$ are two $p$-uniformly convex and uniformly smooth real Banach spaces;
(2) $A : E_1 \rightarrow E_2$ is a bounded linear operator with adjoint operator $A^*$;
(3) $T : E_1 \rightarrow E_1$ is a closed left Bregman quasi-nonexpansive mapping;
(4) $S : E_2 \rightarrow E_2$ is a firmly nonexpansive-like mapping.

In addition, $J_{E_2}^p$ and $J_{E_1}^p$ are the duality mappings of $E_1$ and $E_2$, respectively, and $J_{E_1}^{q}$ is the duality mapping of $E_1^*$. It is worth noting that $E_1^*$ and $E_2^*$ are two $q$-uniformly smooth and uniformly convex Banach spaces, and $J_{E_1}^p = (J_{E_1}^q)^{-1}$, where $1 < q \leq 2$, $p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

\textbf{Algorithm.} For given initial values $x_0, x_1 \in E_1$, the sequence $\{x_n\}$ generated by the following iterative algorithm:

\[
\begin{align*}
\{w_n &= J_{E_2}^p(J_{E_1}^p x_n + \theta_n J_{E_1}^p (x_n - x_{n-1})), \\
\{z_n &= J_{E_1}^q(J_{E_1}^p w_n - \gamma_n A^* J_{E_2}^p (I - S) A w_n), \\
\{y_n &= J_{E_1}^q(\alpha_n J_{E_1}^p z_n + (1 - \alpha_n) J_{E_1}^p T z_n), \\
C_{n+1} &= \{u \in C_n : \Delta_p(y_n, u) \leq \Delta_p(z_n, u) \leq \Delta_p(w_n, u)\}, \\
x_{n+1} &= \Pi_{C_{n+1}} x_0,
\end{align*}
\]

(3.1)
where $\Pi_{C_{n+1}}$ is a Bregman projection of $E_1$ onto $C_{n+1}$, $\{\gamma_n\}$ is a sequence of real number in \((0, (\frac{q}{C_n\|A\|})^{\frac{1}{p-1}})\), where $\frac{1}{c}$ is the $p$-uniformly convexity constant and $C_q$ is the $q$-uniformly smoothness constant, the sequences of real number $\{\alpha_n\} \subset [a, b] \subset (0, 1)$ and $\{\theta_n\} \subset [c, d] \subset (-\infty, +\infty)$.

Lemma 3.1. Let $E_1, E_2, T, S, A, A^*$ and $J^P_{E_1}, J^P_{E_2}, J^P_{E_1}$ be the same as above. If $\Gamma = \{x^*|x^* \in F(T); Ax^* \in F(S)\}$, then $\Gamma \subseteq C_n$ for any $n \geq 1$.

Proof. If $\Gamma = \emptyset$, it is obvious that $\Gamma \subseteq C_n$. Conversely, for any $x^* \in \Gamma$, we have $x^* \in F(T)$ and $Ax^* \in F(S)$. According to Lemma 2.4 and the definition of left Bregman quasi-nonexpansive mapping $T$, we easily obtain
\[
\Delta_p(y_n, x^*) = \Delta_p(J^{P_{E_1}}(\alpha_n J^{P_{E_1}}z_n + (1 - \alpha_n)J^{P_{E_1}}Tz_n), x^*) \\
\leq \alpha_n \Delta_p(z_n, x^*) + (1 - \alpha_n)\Delta_p(Tz_n, x^*) \\
\leq \Delta_p(z_n, x^*). 
\]

Since $E_1$ is a $p$-uniformly convex and uniformly smooth real Banach space, then $E_1^*$ is a $q$-uniformly smooth and uniformly convex Banach space and $J^{P_{E_1}} = (J^{P_{E_1}})^{-1}$. From the property of firmly nonexpansive-like mapping $S$, we easily obtain $\langle J^{P_{E_1}}(I - S)Aw_n, Ax^* - SAw_n \rangle \leq 0$. Further, we have
\[
\langle J^{P_{E_2}}(I - S)Aw_n, Ax^* - Aw_n \rangle = \langle J^{P_{E_2}}(I - S)Aw_n, Ax^* - SAw_n + SAw_n - Aw_n \rangle \\
= -\|\| (I - S)Aw_n \| \| + \langle J^{P_{E_2}}(I - S)Aw_n, Ax^* - SAw_n \rangle \\
\leq -\| (I - S)Aw_n \| \| . 
\]

Again from (2.9), (3.3) and Lemma 2.3, we obtain
\[
\Delta_p(z_n, x^*) = \Delta_p(J^{P_{E_1}}(J^{P_{E_1}}w_n - \gamma_n A^* J^{P_{E_2}}(I - S)Aw_n), x^*) \\
= \frac{1}{q}\| J^{P_{E_1}}(J^{P_{E_1}}w_n - \gamma_n A^* J^{P_{E_2}}(I - S)Aw_n) \| \| + \frac{1}{p}\| x^* \| \| \\
- \langle J^{P_{E_2}}w_n - \gamma_n A^* J^{P_{E_2}}(I - S)Aw_n, x^* \rangle \\
= \frac{1}{q}\| J^{P_{E_1}}w_n - \gamma_n A^* J^{P_{E_2}}(I - S)Aw_n \| \| + \frac{1}{p}\| x^* \| \| \\
- \langle J^{P_{E_2}}w_n, x^* \rangle + \gamma_n \langle J^{P_{E_2}}(I - S)Aw_n, Ax^* \rangle \\
\leq \frac{1}{q}\| J^{P_{E_1}}w_n \| \| + \frac{1}{p}\| x^* \| \| - \langle J^{P_{E_2}}w_n, x^* \rangle - \gamma_n \langle J^{P_{E_2}}(I - S)Aw_n, Ax^* \rangle \\
+ \gamma_n \langle J^{P_{E_2}}(I - S)Aw_n, Ax^* - Aw_n \rangle \\
+ \frac{C_q(\gamma_n \| A \|)^q}{q}\| J^{P_{E_2}}(I - S)Aw_n \| \| \\
= \Delta_p(w_n, x^*) + \gamma_n \langle J^{P_{E_2}}(I - S)Aw_n, Ax^* - Aw_n \rangle \\
+ \frac{C_q(\gamma_n \| A \|)^q}{q}\| (I - S)Aw_n \| \| . 
\]

Since $\{\gamma_n\}$ is a real number sequence contained in \((0, (\frac{q}{C_n\|A\|})^{\frac{1}{p-1}})\), we get
\[
\Delta_p(z_n, x^*) \leq \Delta_p(w_n, x^*) + \left(\frac{C_q(\gamma_n \| A \|)^q}{q} - \gamma_n\right)\| (I - S)Aw_n \| \| \leq \Delta_p(w_n, x^*). 
\]

From (3.2) and (3.4), we have $x^* \in C_{n+1}$, that is, $\Gamma \subseteq C_n$, $\forall n \geq 1$. \( \square \)
Theorem 3.1. Let $E_1, E_2, T, S, A, A^*$ and $J^p_{E_1}, J^p_{E_2}, J^p_{E_1}$ be the same as above. If $\Gamma = \{x^*|x^* \in F(T); Ax^* \in F(S)\} \neq \emptyset$, the sequence $\{x_n\}$ generated by Algorithm (3.1) converges strongly to a point $z = \Pi_{F} x_0 \in \Gamma$.

Proof. By Lemma 2.5 and Lemma 3.1, we know that $\Pi_{C_{n+1}} x_0$ is well defined and $\Gamma \subseteq C_n$. According to Algorithm (3.1), we know $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0$ for each $n \geq 1$. Using $\Gamma \subseteq C_n$ and (2.9), we have
\[
\Delta_p(x_0, x_n) = \Delta_p(x_0, \Pi_{C_n} x_0) \leq \Delta_p(x_0, x^*), \quad x^* \in \Gamma, \quad \forall n \geq 1. \tag{3.5}
\]
It implies that $\{\Delta_p(x_0, x_n)\}$ is bounded. Reusing (2.9), we also have
\[
\Delta_p(x_n, x_{n+1}) = \Delta_p(\Pi_{C_n} x_0, x_{n+1}) \leq \Delta_p(x_0, x_{n+1}) - \Delta_p(x_0, \Pi_{C_n} x_0)
= \Delta_p(x_0, x_{n+1}) - \Delta_p(x_0, x_n). \tag{3.6}
\]
It follows that $\{\Delta_p(x_0, x_n)\}$ is nondecreasing. Hence, $\lim_{n \to \infty} \Delta_p(x_0, x_n)$ exists, and
\[
\lim_{n \to \infty} \Delta_p(x_n, x_{n+1}) = 0. \tag{3.7}
\]
It follows from (2.6) that
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.8}
\]
For some positive integers $m$, $n$ with $m \geq n$, we have $x_m = \Pi_{C_m} x_1 \subseteq C_n$. Using (2.9), we obtain
\[
\Delta_p(x_n, x_m) = \Delta_p(\Pi_{C_m} x_1, x_m) \leq \Delta_p(x_0, x_m) - \Delta_p(x_0, \Pi_{C_n} x_0)
= \Delta_p(x_0, x_m) - \Delta_p(x_0, x_n). \tag{3.9}
\]
Since $\lim_{n \to \infty} \Delta_p(x_0, x_n)$ exists, it follows from (3.9) that $\lim_{n \to \infty} \Delta_p(x_n, x_m) = 0$ and $\lim_{n \to \infty} \|x_m - x_n\| = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence. Further, there exists a point $z \in C$ such that $x_n \to z$.

From Algorithm (3.1), Definition 2.2 and Lemma 2.3, we have
\[
\Delta_p(w_n, x^*) = \frac{1}{q} \|J^q_{E_1}(J^p_{E_1} x_n + \theta_n J^p_{E_1} (x_n - x_{n-1}))\|^p + \frac{1}{p} \|x^*\|^p
- \langle J^p_{E_1} x_n + \theta_n J^p_{E_1} (x_n - x_{n-1}), x^* \rangle
\]
\[
= \frac{1}{q} \|J^q_{E_1} x_n + \theta_n J^p_{E_1} (x_n - x_{n-1})\|^q + \frac{1}{p} \|x^*\|^p
- \langle J^p_{E_1} x_n, x^* \rangle - \theta_n \langle J^p_{E_1} (x_n - x_{n-1}), x^* \rangle
\]
\[
\leq \frac{1}{q} \|J^q_{E_1} x_n\|^q + \frac{1}{p} \|x^*\|^p - \langle J^p_{E_1} x_n, x^* \rangle - \theta_n \langle J^p_{E_1} (x_n - x_{n-1}), x^* \rangle + \theta_n \langle J^p_{E_1} (x_n - x_{n-1}), x_n \rangle + \frac{C_q(\theta_n)^q}{q} \|J^p_{E_1} (x_n - x_{n-1})\|^q
\]
\[
= \frac{1}{q} \|x_n\|^q + \frac{1}{p} \|x^*\|^p - \langle J^p_{E_1} x_n, x^* \rangle - \theta_n \langle J^p_{E_1} (x_n - x_{n-1}), x^* \rangle + \theta_n \langle J^p_{E_1} (x_n - x_{n-1}), x_n \rangle + \frac{C_q(\theta_n)^q}{q} \|x_n - x_{n-1}\|^p
\]
\[
= \Delta_p(x_n, x^*) + \theta_n \langle J^p_{E_1} (x_n - x_{n-1}), x_n - x^* \rangle + \frac{C_q(\theta_n)^q}{q} \|x_n - x_{n-1}\|^p. \tag{3.10}
\]
By virtue of Remark 2.1 and the definition of $w_n$, we know
\[
\Delta_p(w_n, x^*) = \Delta_p(w_n, x_n) + \Delta_p(x_n, x^*) + \langle x_n - x^*, J^p_{E_1} w_n - J^p_{E_1} x_n \rangle \\
= \Delta_p(w_n, x_n) + \Delta_p(x_n, x^*) + \theta_n(x_n - x^*, J^p_{E_1}(x_n - x_n - 1)).
\] (3.11)

By (3.10) and (3.11), we get \(\Delta_p(w_n, x_n) \leq \frac{C_4(\theta_n)^q}{q} \|x_n - x_n - 1\|^p\). Then, using (2.6), (3.8) and the boundedness of the sequence \(\{\theta_n\}\), we can obtain
\[
\lim_{n \to \infty} \|w_n - x_n\| = 0.
\] (3.12)

Using a similar method, we can get
\[
\Delta_p(w_n, x_{n+1}) = \Delta_p(w_n, x_n) + \Delta_p(x_n, x_{n+1}) + \langle x_n - x_{n+1}, J^p_{E_1} w_n - J^p_{E_1} x_n \rangle.
\]

By setting \(n \to \infty\), we have
\[
\lim_{n \to \infty} \|w_n - x_{n+1}\| = 0.
\] (3.13)

Since \(x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subseteq C_n\), we have
\[
\Delta_p(y_n, x_{n+1}) \leq \Delta_p(z_n, x_{n+1}) \leq \Delta_p(w_n, x_{n+1}).
\]

According to (3.13), we obtain
\[
\lim_{n \to \infty} \Delta_p(y_n, x_{n+1}) = 0, \quad \lim_{n \to \infty} \Delta_p(z_n, x_{n+1}) = 0,
\] (3.14)

which implies that \(\lim_{n \to \infty} \|y_n - x_{n+1}\| = 0, \quad \lim_{n \to \infty} \|z_n - x_{n+1}\| = 0\). Hence
\[
\|x_n - z_n\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - z_n\| \to 0 \text{ as } n \to \infty,
\] (3.15)

\[
\|y_n - z_n\| \leq \|x_{n+1} - y_n\| + \|x_{n+1} - z_n\| \to 0 \text{ as } n \to \infty.
\] (3.16)

In addition, since \(E_1\) is a \(p\)-uniformly convex and uniformly smooth real Banach space, then \(J^p_{E_1}\) is uniformly norm-to-norm continuous. It follows from Algorithm (3.1) and real number sequence \(\{\alpha_n\}\) in \([a, b]\) that
\[
\lim_{n \to \infty} \|J^p_{E_1} T z_n - J^p_{E_1} z_n\| = \lim_{n \to \infty} \frac{1}{1 - \alpha_n} \|J^p_{E_1} y_n - J^p_{E_1} z_n\| = 0,
\]

which also implies that \(\lim_{n \to \infty} \|T z_n - z_n\| = 0\). By virtue of (3.12) and \(x_n \to z\) we have \(z_n \to z\). Using the closedness of \(T\), we obtain \(z_n \to z\) and \(T z = z\). From (2.6), (3.4) and (3.15), we have
\[
(\gamma_n - \frac{C_4(\gamma_n||A||)^q}{q}) (I - S) A w_n \|^p \leq \Delta_p(w_n, x^*) - \Delta_p(z_n, x^*)
\]
\[
= \frac{1}{q} \|w_n\|^p - \frac{1}{q} \|z_n\|^p - \langle J^p_{E_1} w_n - J^p_{E_1} z_n, x^* \rangle
\]
\[
= \Delta_p(w_n, z_n) + \langle J^p_{E_1} w_n - J^p_{E_1} z_n, z_n - x^* \rangle
\]
\[
\leq (\|w_n - z_n\| + \|z_n - x^*\|) \|J^p_{E_1} w_n - J^p_{E_1} z_n\|. 
\]
By setting of $n \to \infty$, the right-hand side of last inequality tends to 0. Moreover, 
\{\gamma_n\} is a real number sequence contained in $(0, \frac{q}{\sqrt{\|A\|^2}})$, we have 
\[
\lim_{n \to \infty} \|(S - I)Aw_n\| = 0.
\]
Since $A$ is a bounded linear operator, we have $Aw_n \to Az$. Again according to 
Definition 2.3 and Lemma 2.1, we get $Az \in F(T)$. Then, from (2.8) and (3.1), we have 
\[
(J_{E_1}^p x_0 - J_{E_1}^p x_n, p - x_n) \leq 0, \quad \forall p \in \Gamma.
\] (3.17)
By setting $n \to \infty$ in (3.17), we obtain 
\[
(J_{E_1}^p x_0 - J_{E_1}^p z, p - z) \leq 0, \quad \forall p \in \Gamma.
\] (3.18)
Again from (2.8), we have $z = \Pi_\Gamma x_0$. Definitively, we obtain that 
\{x_n\} generated by Algorithm (3.1) strongly converge $z = \Pi_\Gamma x_0 \in \Gamma$. The proof is completed. \hfill $\Box$

**Remark 3.1.** The significant improvement of the results in this paper is using the 
shrinking projection algorithm with inertial effects to study the split common fixed 
problem in the framework of two $p$-uniformly convex and uniformly smooth Banach spaces, 
and the iterative sequence generated by Algorithm (3.1) strongly converges 
to a solution of the split common fixed point problem.

**Remark 3.2.** The field of study in this paper is the $p$-uniformly convex and uniformly 
smooth Banach space, which is more extensive than the Hilbert space [26–28], 
the uniformly convex and 2-uniformly smooth Banach space [29] and the 2-uniformly 
convex and 2-uniformly smooth real Banach space [25]. The split common fixed 
point problem of firmly nonexpansive-like mappings in Theorem 3.1 is more general 
than the split feasibility problem and fixed point problem in [18, 25].

As a corollary of Theorem 3.1, when $E_1$ and $E_2$ reduce to Hilbert spaces, the 
function $\Delta_p$ is equal to $\Delta_p(x,y) = \frac{1}{2}\|x - y\|^2$ and the Bregman projection $\Pi_C$ 
is equivalent to the metric projection $P_C$. Then, we obtain the following corollary.

**Corollary 3.1.** Let $H_1$ and $H_2$ be two real Hilbert spaces. Let $T : H_1 \to H_1$ 
be a closed quasi-nonexpansive mapping, $S : H_2 \to H_2$ be a firmly nonexpansive 
mapping, and $A : H_1 \to H_2$ be a bounded linear operator with adjoint operator $A^*$. For 
given initial values $x_0, x_1 \in C_1 = H_1$, the sequence $\{x_n\}$ generated by the 
following iterative algorithm:

\[
\begin{align*}
     w_n &= x_n + \theta_n(x_n - x_{n-1}), \\
     z_n &= w_n - \gamma_n A^*(I - S)Aw_n, \\
     y_n &= \alpha_n z_n + (1 - \alpha_n)Tz_n, \\
     C_{n+1} &= \{u \in C_n : \|y_n - u\| \leq \|z_n - u\| \leq \|w_n - u\|\}, \\
     x_{n+1} &= P_{C_{n+1}} x_0,
\end{align*}
\] (3.19)

where $P_{C_{n+1}}$ is a metric projection of $H_1$ onto $C_{n+1}$, the sequences of real numbers 
$\{\gamma_n\} \subset (0, \frac{2}{\sqrt{\|A\|^2}})$, $\{\alpha_n\} \subset [a, b] \subset (0, 1)$ and $\{\theta_n\} \subset [c, d] \subset (-\infty, +\infty)$. If $\Gamma = \{x^* : x^* \in F(T), Ax^* \in F(S)\} \neq \emptyset$, the sequence $\{x_n\}$ generated by (3.19) converges 
strongly to a point $z = P_\Gamma x_0 \in \Gamma$. 
4. Applications

4.1. Split fixed point problems and variational inclusion problems

Let $H$ be a real Hilbert space and $B : H \to 2^H$ be a set-valued mapping with domain $D(B) := \{ x \in H : B(x) \neq \emptyset \}$. An operator $B : H \to 2^H$ is called monotone if $\langle u - v, x - y \rangle \geq 0$, $\forall u \in Bx$, $v \in By$. Further, $B$ is called maximal monotone if its graph $G(B) = \{(x, y) : x \in D(B), y \in D(B)\}$ is not properly contained in the graph of any other monotone operator.

The problem of zero points of maximal monotone operator is

$$x^* \in H, \text{ such that } 0 \in B(x^*),$$

where $B : H \to 2^H$ is a set-valued maximal monotone operator. Martinet [20] first proposed proximal point algorithm to solve the problem of a zero point of maximal monotone operator.

Lemma 4.1 ([23]). Let $H$ be a real Hilbert space. Let $B : H \to 2^H$ be a maximal monotone operator and $\mu > 0$, and its associated resolvent of order $\mu$, defined by $J_\mu^B = (I + \mu A)^{-1}$, where $I$ denotes the identity mapping. Then, the following properties are true:

(I) For each $\mu > 0$, $J_\mu^B$ is a single-valued and firmly nonexpansive mapping;

(II) $D(J_\mu^B) = H$ and $F(J_\mu^B) = B^{-1}(0) := \{ x \in D(B), 0 \in Bx \}$.

Definition 4.1. Let $H_1$ and $H_2$ be two Hilbert spaces, and $A : H_1 \to H_2$ be a bounded linear operator. Let $B : H_1 \to 2^{H_1}$ and $K : H_2 \to 2^{H_2}$ be two set-valued mappings, $T : H_1 \to H_1$ be a mapping. The split fixed point problem and variational inclusion problem is to find a point $x^*$ such that

$$x^* \in F(T) \cap K^{-1}(0), Ax^* \in B^{-1}(0).$$

For this problem, we propose the following theorem by the result in Theorem 3.1.

Theorem 4.1. Let $H_1$ and $H_2$ be two Hilbert spaces, and $A : H_1 \to H_2$ be a bounded linear operator with adjoint operator $A^*$. Let $B : H_1 \to 2^{H_1}$ and $K : H_2 \to 2^{H_2}$ be two maximal monotone operators and $\mu > 0$, $T : H_1 \to H_1$ be a closed quasi-nonexpansive mapping. For given initial values $x_0, x_1 \in C_1 = H_1$, the sequence $\{x_n\}$ generated by the following iterative algorithm:

\[\begin{align*}
  w_n &= x_n + \theta_n(x_n - x_{n-1}), \\
  z_n &= w_n - \gamma_n A^*(I - J_\mu^B)Aw_n, \\
  y_n &= \alpha_n z_n + (1 - \alpha_n)TJ_\mu^K z_n, \\
  C_{n+1} &= \{ u \in C_n : \| y_n - u \| \leq \| z_n - u \| \leq \| w_n - u \| \}, \\
  x_{n+1} &= P_{C_{n+1}}x_0,
\end{align*}\]

where $P_{C_{n+1}}$ is a metric projection of $H_1$ to $C_{n+1}$, the sequences of real numbers $\{\gamma_n\} \subset (0, \frac{2}{\|A\|^2})$, $\{\alpha_n\} \subset [a, b] \subset (0, 1)$ and $\{\theta_n\} \subset [c, d] \subset (-\infty, +\infty)$. If $\Gamma = \{ x^* | x^* \in F(T) \cap K^{-1}(0), Ax^* \in B^{-1}(0) \} \neq \emptyset$, the sequence $\{x_n\}$ generated by iterative algorithm (4.1) converges strongly to a point $z = P_{\Gamma}x_0 \in \Gamma$. 
Proof. It is obvious that $TJ^K_\mu$ is closed quasi-nonexpansive mapping from the property of $T$ and Lemma 4.1. Hence, the strong convergence theorem of iterative algorithm (4.1) is obviously proved.

4.2. Split fixed point problems and equilibrium problems

Let $C$ be a nonempty closed and convex subset of a Hilbert space $H$. Let bifunction $F : C \times C \rightarrow R$ satisfy the following conditions:

(A1) $F(x, x) = 0$, $\forall x \in C$;

(A2) $F(x, y) + F(y, x) \leq 0$, $\forall x, y \in C$;

(A3) $\lim_{t \to 0} F(tx + (1-t)x, y) \leq F(x, y)$, $\forall x, y, z \in C$;

(A4) For each $x \in C$, the function $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Then, the so-called equilibrium problem for $F$ is to find a point $x^* \in C$ such that $F(x^*, x) \geq 0$, $\forall x \in C$, and the set of solutions of equilibrium problem is denoted by $EP(F)$.

Lemma 4.2 ([14]). Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, and let $F : C \times C \rightarrow R$ be a bifunction satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then there exists a point $z \in C$ such that $F(z, y) + \frac{1}{2t}(y - z, z - x) \geq 0$, $\forall y \in C$.

Lemma 4.3 ([14]). Assume that $F : C \times C \rightarrow R$ be a bifunction satisfying (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \rightarrow H$ as follows: $T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{2t}(y - z, z - x) \geq 0, \forall y \in C\}$, $\forall x \in H$. Then

(1) $T_r^F$ is single-valued;

(2) $\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y \rangle$;

(3) $F(T_r^F) = EP(F)$ is nonempty, closed and convex.

Definition 4.2. Let $C$ and $Q$ be two nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively, and $A : H_1 \rightarrow H_2$ be a bounded linear operator, $T : C \rightarrow C$ be a mapping, $F : C \times C \rightarrow R$ be a bifunction satisfying (A1)-(A4). The split fixed point problem and equilibrium problem is to find a point $x^*$ such that

$x^* \in F(T)$, $Ax^* \in EP(F)$.

Theorem 4.2. Let $C$ and $Q$ be two nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively, $T : C \rightarrow C$ be a closed quasi-nonexpansive mapping, $F : C \times C \rightarrow R$ be a bifunction satisfying (A1)-(A4), and $A : H_1 \rightarrow H_2$ be a bounded linear operator with adjoint operator $A^*$. For given initial values $x_0$, $x_1 \in C_1 = C$, the sequence $\{x_n\}$ generated by the following iterative algorithm:

\[
\begin{align*}
  w_n &= x_n + \theta_n(x_n - x_{n-1}), \\
  z_n &= P_C(w_n - \gamma_n A^*(I - T_r^F)Aw_n), \\
  y_n &= \alpha_n z_n + (1 - \alpha_n)Tz_n, \\
  C_{n+1} &= \{u \in C_n : \|y_n - u\| \leq \|z_n - u\| \leq \|w_n - u\|\}, \\
  x_{n+1} &= P_{C_{n+1}}x_0,
\end{align*}
\]

(4.2)
where $P_{C_{n+1}}$ is a metric projection of $H_1$ onto $C_{n+1}$, the sequences of real numbers $\{\gamma_n\} \subset (0, \frac{1}{\|A\|^2})$, $\{\alpha_n\} \subset [0, b] \subset (0, 1)$ and $\{\theta_n\} \subset [c, d] \subset (-\infty, +\infty)$. If $\Gamma = \{x^* | x^* \in F(T), Ax^* \in EP(F)\} \neq \emptyset$, the sequence $\{x_n\}$ generated by iterative algorithm (4.2) converges strongly to a point $z = P_{\Gamma}x_0 \in \Gamma$.

5. Numerical examples

In this section, we come up with some numerical examples to demonstrate the effectiveness and realization of convergence of Theorem 3.1. All codes were written in Matlab R2018b, and ran on a Lenovo ideapad 720S with 1.6 GHz Intel Core i5 processor and 8GB of RAM. Using numerical experiments, we will compare the convergence speed of our algorithm with the algorithm in [18]. Ma et al. [18] proved strong convergence theorems of split feasibility problems and fixed point problems of quasi-$\phi$-nonexpansive mapping in Banach spaces and introduced the following algorithm.

**Theorem 5.1** ([18]). Let $E_1$ be a 2-uniformly convex and 2-uniformly smooth real Banach space with best smoothness constant $k > 0$, $E_2$ be a smooth, strictly convex, and reflective Banach space. Let $T : E_1 \rightarrow E_1$ be a closed quasi-$\phi$-nonexpansive mapping with $F(T) \neq \emptyset$, $A : E_1 \rightarrow E_1$ be a bounded linear operator with adjoint $A^*$, and $Q$ be a nonempty, closed, and convex subset of $E_2$. Let $x_0 \in E_1$ and $C_1 = E_1$, and $\{x_n\}$ be a sequence generated by

$$
\begin{align*}
    z_n &= J_1^{-1}(J_1x_n - \gamma A^*J_2(I - P_Q)Ax_n), \\
    y_n &= J_1^{-1}(\alpha_n J_1z_n + (1 - \alpha_n)J_1Tz_n), \\
    C_{n+1} &= \{u \in C_n : \phi(u, y_n) \leq \phi(u, x_n), \phi(u, z_n) \leq \phi(u, x_n)\}, \\
    x_{n+1} &= \Pi_{C_{n+1}}x_1,
\end{align*}
$$

where $P_Q$ is the metric projection of $E_2$ onto $Q$ and $\Pi_{C_{n+1}}$ is the generalized projection of $E_1$ onto $C_{n+1}$, $\{\alpha_n\}$ is a sequence in $(0, \delta]$, $\delta < 1$, and $\gamma$ is a positive constant satisfying $0 < \gamma < \frac{1}{\|A\|^2}$. If $\Gamma = \{x^* | x^* \in F(T), Ax^* \in Q\} \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to a point $z = \Pi_{\Gamma}x_1$.

To make sure the initial conditions of Theorem 5.1 and Theorem 3.1 are consistent, the initial conditions are given as follows:

Let $E_1 = R$ and $E_2 = R \times R$ with the Euclidean norm, and $C = [0, +\infty)$ and $Q = [0, +\infty) \times (-\infty, 0]$. Let $A : E_1 \rightarrow E_2$ be a bounded linear operator and defined as $Ax = (\frac{x_1}{2}, \frac{x_2}{3}), \forall x \in E_1$ with its adjoint $A^*(u, v) = \frac{u}{2} + \frac{v}{3}, \forall (u, v) \in E_2$.

**Example 5.1** (Ma et al.). Let $T : C \rightarrow C$ be defined as $Tx = \frac{1}{3}x, \forall x \in C$, and $P_Q : E_2 \rightarrow Q$ be a metric projection. In addition, we choose parameters $\gamma = 1$, $\alpha_n = \frac{1}{n}$. For given initial value $x_1 \in C_1 = C$, the iterative algorithm in Theorem 5.1 can be simplified as

$$
\begin{align*}
    Ax_n &= (\frac{x_n}{2}, \frac{x_n}{3}), \\
    z_n &= x_n - A^*(I - P_Q)Ax_n, \\
    y_n &= \frac{1}{4}z_n + (1 - \frac{1}{4})Tz_n, \\
    C_{n+1} &= \{u \in C_n : |y_n - u| \leq |x_n - u|, |z_n - u| \leq |x_n - u|\}, \\
    x_{n+1} &= P_{C_{n+1}}x_1.
\end{align*}
$$
Then, the sequence \( \{x_n\} \) converges strongly to 0.

**Example 5.2** (Our algorithm with inertial effects in Theorem 3.1). Let \( T : C \to C \) be defined as \( Tx = \frac{1}{4}x, \; \forall x \in C \), and \( S = P_Q : E_2 \to Q \) be a metric projection. In addition, we choose parameters \( \gamma_n = 1, \; \alpha_n = \frac{1}{7} \) and \( \theta_n = \frac{1}{2} \). For given initial values \( x_0 = x_1 \in C_1 = C \), the iterative algorithm in Theorem 3.1 can be simplified as

\[
\begin{align*}
w_n &= x_n + \frac{1}{2} (x_n - x_{n-1}), \\
z_n &= w_n - A^*(I - P_Q)Aw_n, \\
y_n &= \frac{1}{t} z_n + \left(1 - \frac{1}{t}\right) Tz_n, \\
C_{n+1} &= \{ u \in C : |y_n - u| \leq |z_n - u| \leq |w_n - u| \}, \\
x_{n+1} &= P_{C_{n+1}} x_0.
\end{align*}
\]

Then, the sequence \( \{x_n\} \) converges strongly to 0.

Next, taking initial values \( x_1 = 6, x_1 = 3 \) in Example 5.1 and \( x_0 = x_1 = 6, x_0 = x_1 = 3 \) in Example 5.2. We get the following Table 1 and Figure 1 to demonstrate rate of convergence of the algorithms in Theorem 5.1 and Theorem 3.1.

<table>
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**Figure 1.** Analysis of convergence speed of Theorem 5.1 and Theorem 3.1
It is worth noting that $\gamma_n$, $\alpha_n$ and $\theta_n$ are only constant step-size and initial point $x_0 = x_1$ (there is no using an inertial effect in the first iterative process) in Example 5.2. To change this, we consider the following four cases of the step-size parameters $\gamma_n$, $\alpha_n$ and $\theta_n$, with the initial points $x_0 = 8$, $x_1 = 6$.

**Case 1:** $\gamma_n = 1$, $\alpha_n = \frac{1}{7}$ and $\theta_n = \frac{1}{2}$;

**Case 2:** $\gamma_n = \frac{2n+3}{2n}$, $\alpha_n = \frac{1}{7}$ and $\theta_n = \frac{1}{2}$;

**Case 3:** $\gamma_n = \frac{2n+3}{2n}$, $\alpha_n = \frac{n}{7n+5}$ and $\theta_n = \frac{1}{2}$;

**Case 4:** $\gamma_n = \frac{2n+3}{2n}$, $\alpha_n = \frac{n}{7n+5}$ and $\theta_n = \frac{2n+1}{10n+2}$.

**Table 2.** Numerical results of Case1-4

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<tr>
<th>n</th>
<th>Case 1</th>
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<th>Case 3</th>
<th>Case 4</th>
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**Figure 2.** Analysis of convergence speed of Case1-4

**Remark 5.1.** (I) From Figure 1 and Table 1, we found that the convergence rate of the algorithm in Theorem 3.1 is faster than the algorithm in Theorem 5.1. This also shows the efficiency of our proposed Algorithm (3.1) in this paper.

(II) From Figure 2 and Table 2, we know that the convergence speed of the algorithm in Theorem 3.1 is improved under suitable condition of step-size.
References


