Journal of Applied Analysis and Computation
Volume 10, Number 5, October 2020, 2104–2120
Website:http://jaac-online.com
DOI:10.11948/20190330

AN INERTIAL SHRINKING PROJECTION ALGORITHM FOR SPLIT COMMON FIXED POINT PROBLEMS

Zheng Zhou[1,†], Bing Tan[1] and Songxiao Li[1]

Abstract In this paper, the purpose is to introduce and study a new modified shrinking projection algorithm with inertial effects, which solves split common fixed point problems in Banach spaces. The corresponding strong convergence theorems are obtained without the assumption of semi-compactness on mappings. Finally, some numerical examples are presented to illustrate the results in this paper.

Keywords Split common fixed point problem, shrinking projection method, inertial method, strong convergence.

MSC(2010) 47H09, 47H10, 47J25, 47N40.

1. Introduction

To model inverse problems in phase retrievals and medical image reconstruction [2], Censor and Elfving [6] introduced the concept of the split feasibility problem (for short, SFP) in framework of finite-dimensional Hilbert spaces in 1994. It has been founded that the SFP can be used in many areas of applications, such as image restoration, computer tomograph, radiation therapy treatment planning and other areas of mathematical research [3, 5, 7–9, 16, 21].

As a generalization of the SFP in 2009, Censor and Segal [10] introduced the following split common fixed point problem (for short, SCFPP): Let H_1 and H_2 be two Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator, $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be two mappings ($F(T)$ and $F(S)$ denote the fixed point sets of T and S, respectively). The split common fixed point problem for mappings T and S which is to find a point x^* satisfing

$$x^* \in F(T) \text{ and } Ax^* \in F(S). \tag{1.1}$$

The solution set of (1.1) is denoted by Γ, i.e., $\Gamma = \{x^* | x^* \in F(T), Ax^* \in F(S)\}$.

Since then, the SCFPP has been widely studied by many authors in Hilbert spaces (see [11, 12, 22, 23, 31]). Usually in order to achieve strong convergence properties of the SCFPP, we often do this directly by considering the assumption of semi-compactness on the mappings. In addition, there are papers using some algorithms to replace the assumption of the mappings, such as Halpern iterative algorithm, viscosity iterative algorithm, shrinking projection algorithm. Ulteriorly,

[†]The corresponding author. Email: zhouzheng2272@163.com (Z. Zhou)
[1]Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu, China

there are a few studies on the split common fixed point problem in the framework of Banach spaces. For example in recent years, Takahashi and Yao [28] and Takahashi [26, 27] studied the split common fixed point problem in the setting of one Hilbert space and one Banach space, and obtained some weak convergence theorems and strong convergence theorems. Furthermore, in the framework of two Banach spaces, there are many studies established as follows:

In 2015, Tang et al. [29] studied the SCFPP for asymptotically nonexpansive mappings and quasi-strictly pseudo-contractive mappings. Then, the strong convergence theorem was also proved with the condition of semi-compactness on the mappings. In 2016, Shehu et al. [25] studied split feasibility problems and fixed point problems for left Bregman strongly nonexpansive mappings, and showed strong convergence theorems by Halpern iterative method.

Recently, Ma et al. [18] also studied split feasibility problems and fixed point problems in Banach spaces, and obtained the strong convergence theorem by the following shrinking projection iterative algorithm

$$\begin{cases} z_n = J_1^{-1}(J_1x_n + \gamma A^* J_2(P_Q - I)Ax_n), \\ y_n = J_1^{-1}((1-\alpha_n)J_1z_n + \alpha_n J_1 S z_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n), \phi(v, z_n) \le \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, n \ge 1, \end{cases}$$

where E_1 is a 2-uniformly convex and 2-uniformly smooth real Banach space, E_2 is a smooth, strictly convex and reflective Banach space and Q is a nonempty closed convex subset of E_2, $A : E_1 \to E_2$ is a bounded linear operator with the adjoint operator A^*, $S : E_1 \to E_1$ is a closed quasi-ϕ-nonexpansive mapping, $P_Q : E_2 \to Q$ is the metric projection and $\Pi_{C_{n+1}} : E_1 \to C_{n+1}$ is the generalized projection.

In view of the above studies and methods and in order to accelerate better the convergence rate of the iterative algorithms. The inertial effects have been studied recently by many authors in terms of variational inequality problems, inclusion problems, equilibrium problems, etc., see [1, 4, 15, 17, 19, 24] and the references therein. The main characteristic of the inertial method is that the new iterate process is produced by making use of two values of the previous iterative point. In 2001, Alvarez and Attouch [1] studied the problem of approximating the null point of a maximal monotone operator and proposed the following inertial proximal algorithm:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \lambda T)^{-1} y_n, \forall n \ge 1. \end{cases}$$

Then, they obtained the weak convergence of the algorithm.

For these research, the ideas of this paper are as follows: this article introduce a new shrinking projection iterative algorithm with inertial effects to solve problem (1.1) for firmly nonexpansive-like mappings in the framework of p-uniformly convex and uniformly smooth real Banach spaces. Meanwhile, the strong convergence theorems of this algorithm are obtained without assumption of semi-compactness on mappings. As applications, the results are utilized to split fixed point problems and variational inclusion problems, split fixed point problem and equilibrium problems. Furthermore, some numerical examples are used to demonstrate and show the efficiency of our main results. To this end, some basic properties and relevant lemmas will be introduced in next section which will be used in the proof for the convergence analysis of the proposed algorithm.

2. Preliminaries

Throughout this paper, we use notations: $\to$ to denote the strong convergence and $\rightharpoonup$ to denote the weak convergence. The set of all fixed points of T is denoted by $F(T)$.

Let E be a Banach space. A function $\delta_E : [0,2] \to [0,1]$ is called the modulus of convexity of E as follows:

$$\delta_E(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\}.$$

A function $\rho_E : [0,+\infty] \to [0,+\infty]$, which is called the modulus of smoothness of E as follows:

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \le 1, \|y\| \le t\}.$$

Definition 2.1. A Banach space E is said to be

(I) uniformly convex if for any $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x-y\| \ge \varepsilon$, there exists $\eta = \eta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$ such that $\|\frac{x+y}{2}\| \le 1 - \eta$. This is equivalent to $\delta_E(\varepsilon) > 0$, for all $\varepsilon \in (0,2]$.

(II) uniformly smooth if and only if $\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0$.

A Banach space is called p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\varepsilon) > c\varepsilon^p$ for all $\varepsilon \in (0,2]$, where the constant $\frac{1}{c}$ is called the p-uniformly convexity constant. It is obvious that a p-uniformly convex Banach space is uniformly convex. A Banach space is said to be q-uniformly smooth if there exists a constant $C_q > 0$ such that $\rho_E(t) \le C_q t^q$ for all $t > 0$, where C_q is the q-uniformly smoothness constant. In addition, E is a p-uniformly convex and uniformly smooth Banach space if and only if its dual E^* is a q-uniformly smooth and uniformly convex Banach space.

Let E be a Banach space with the dual E^*. The duality mapping $J_E^p : E \to 2^{E^*}$ is defined by $J_E^p(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}$, $p > 1$, $\forall x \in E$.

Definition 2.2. For a Gâteaux differentiable convex function $f : E \to R$, the function

$$\Delta_f(x,y) := f(y) - f(x) - \langle f'(x), y - x\rangle,\ \forall x, y \in E \tag{2.1}$$

is called the Bregman distance of x to y with respect to the function f.

In addition, the duality mapping J_E^p is the derivative of the function $f_p(x) = \frac{1}{p}\|x\|^p$. Then the Bregman distance with respect to f_p can be written as

$$\begin{aligned}\Delta_p(x,y) &= \frac{1}{q}\|x\|^p - \langle J_E^p x, y\rangle + \frac{1}{p}\|y\|^p \\ &= \frac{1}{p}(\|y\|^p - \|x\|^p) + \langle J_E^p x, x - y\rangle \\ &= \frac{1}{q}(\|x\|^p - \|y\|^p) - \langle J_E^p x - J_E^p y, y\rangle.\end{aligned}$$

Definition 2.3. Let C be a nonempty closed convex subset of a Banach space E. The mapping $T : C \to E$ is said to be

(I) left Bregman quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and

$$\Delta_p(Tx, x^*) \leq \Delta_p(x, x^*),\ \forall x \in C,\ x^* \in F(T); \tag{2.2}$$

(II) firmly nonexpansive-like mapping if

$$\langle Tx - Ty, J_E^p(x - Tx) - J_E^p(y - Ty)\rangle \geq 0,\ \forall x, y \in C. \tag{2.3}$$

Obviously, if E is a Hilbert space, the firmly nonexpansive-like mapping reduce to the firmly nonexpansive mapping, i.e., $\langle Tx-Ty, x-y\rangle \geq \|Tx-Ty\|^2, \forall x, y \in C$.

Example 2.1. (1) Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then, the metric projection P_C is a firmly nonexpansive-like mapping.

(2) Let E be a real number space R with Euclidean norm. A mapping $T : [-10, 10] \to [-10, 10]$ is defined by $Tx = \frac{1}{4}x$, $\forall x \in [-10, 10]$. For any $x, y \in [-10, 10]$, we easily get the following result

$$\begin{aligned}\langle Tx - Ty, J_E^p(x - Tx) - J_E^p(y - Ty)\rangle &= \langle \frac{1}{4}x - \frac{1}{4}y, (x - \frac{1}{4}x)^3 - (y - \frac{1}{4}Ty)^3\rangle \\ &= \frac{1}{4} \times (\frac{3}{4})^3(x - y)(x^3 - y^3) \\ &= \frac{1}{4} \times (\frac{3}{4})^3(x - y)^2(x^2 + xy + y^2) \geq 0,\end{aligned}$$

which implies that T is a firmly nonexpansive-like mapping.

Let $T : C \to E$ be a mapping. A point u is said to be an asymptotic fixed point of T if there exists a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$. The set of all asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

Lemma 2.1. *Let E be a smooth, strictly convex and reflexive Banach space, and C be a nonempty closed convex subset of E. $T : C \to E$ is a firmly nonexpansive-like mapping. Then $F(T)$ is a closed convex subset of E and $\widehat{F}(T) = F(T)$.*

Proof. In order to prove that $F(T)$ is a closed convex set, we assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to u$. From the definition of T, we have $\langle x_n - Tu, -J_E^p(u - Tu)\rangle \geq 0$. This inequality is equivalent also to

$$\|u - Tu\|^p \leq \langle x_n - u, J_E^p(Tu - u)\rangle \leq \|x_n - u\|\|u - Tu\|^{p-1}.$$

Then we obtain $\|u - Tu\| = 0$ as $n \to \infty$. This implies $u = Tu$. Hence, $u \in F(T)$ and $F(T)$ is closed.

Next, we show that $F(T)$ is convex. For any $x, y \in F(T)$ and $t \in (0, 1)$, putting $u = tx + (1 - t)y$. From the definition of T, we get $\langle x - Tu, -J_E^p(u - Tu)\rangle \geq 0$ and $\langle y - Tu, -J_E^p(u - Tu)\rangle \geq 0$. Combine the above two inequalities, we have

$$\begin{aligned}\langle tx + (1 - t)y - Tu, -J_E^p(u - Tu)\rangle \geq 0 &\Leftrightarrow \langle u - Tu, -J_E^p(u - Tu)\rangle \geq 0 \\ &\Leftrightarrow \|u - Tu\|^p \leq 0.\end{aligned}$$

This means that $u = Tu$. So, $F(T)$ is closed and convex.

Last, we show that $\widehat{F}(T) = F(T)$. It is obvious that $F(T) \subset \widehat{F}(T)$. Then, we only show that $\widehat{F}(T) \subset F(T)$. For any $z \in \widehat{F}(T)$, there exists a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$. From the definition of T, we have

$$\langle Tx_n - Tz, J_E^p(x_n - Tx_n) - J_E^p(z - Tz)\rangle \geq 0.$$

This is equivalent to

$$\begin{aligned}\langle Tx_n - Tz, J_E^p(x_n - Tx_n)\rangle &\geq \langle Tx_n - Tz, J_E^p(z - Tz)\rangle \\ &= \langle Tx_n - z + z - Tz, J_E^p(z - Tz)\rangle \\ &= \langle Tx_n - z, J_E^p(z - Tz)\rangle + \|z - Tz\|^p.\end{aligned}$$

The inequality can be transformed the following inequality

$$\begin{aligned}\|z - Tz\|^p &\leq \langle z - Tx_n, J_E^p(z - Tz)\rangle + \langle Tx_n - Tz, J_E^p(x_n - Tx_n)\rangle \\ &= \langle z - x_n, J_E^p(z - Tz)\rangle + \langle x_n - Tx_n, J_E^p(z - Tz)\rangle \\ &\quad + \langle Tx_n - x_n, J_E^p(x_n - Tx_n)\rangle + \langle x_n - Tz, J_E^p(x_n - Tx_n)\rangle.\end{aligned}$$

From the setting of $n \to \infty$, we have $\|z - Tz\| = 0$. Hence, $z = Tz$, i.e., $z \in F(T)$. □

Definition 2.4. Let C be a nonempty closed convex subset of a real Banach space E. A mapping $T : C \to C$ is closed (or T has closed graph), that is, if the sequence $\{x_n\}$ in C converges strongly to a point $x \in C$ and $Tx_n \to y$, then $Tx = y$.

Lemma 2.2 ([13]). *Let E be a Banach space and J_E^p be the duality mapping of E. Then, the following statements hold:*

- *(I) $J_E^p(x)$ is nonempty bounded closed and convex, for any $x \in E$;*
- *(II) If E is a reflexive Banach space, then J_E^p is a mapping from E onto E^*;*
- *(III) If E is a smooth Banach space, then J_E^p is single valued;*
- *(IV) If E is a uniformly smooth Banach space, then J_E^p is norm-to-norm uniformly continuous on each bounded subset of E.*

Remark 2.1. By the definition of Δ_p, we easily obtain

$$\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J_E^p x - J_E^p z\rangle,\ \forall x, y, z \in E, \tag{2.4}$$

and

$$\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_E^p x - J_E^p y\rangle,\ \forall x, y, z \in E. \tag{2.5}$$

In addition, it is easy to see from the above that the Bregman distance is not symmetrical, and for p-uniformly convex Banach spaces, we have

$$\tau\|x - y\|^p \leq \Delta_p(x, y) \leq \langle x - y, J_E^p x - J_E^p y\rangle,\ \forall x, y \in E, \tau > 0. \tag{2.6}$$

This indicates that Bregman distance is non-negative.

Definition 2.5. $\Pi_C : E \to C$ is said to be the Bregman projection mapping, that is,

$$\Pi_C x = \operatorname*{argmin}_{y \in C} \Delta_p(x, y),\ \forall x \in E. \tag{2.7}$$

In other words, $\Pi_C x$ corresponds a unique element $x_0 \in C$ such that

$$\Delta_p(x, x_0) = \min_{y \in C} \Delta_p(x, y),\ \forall x \in E.$$

The Bregman projection can also be characterized by the following inequality

$$\langle J_E^p x - J_E^p \Pi_C x, z - \Pi_C x\rangle \leq 0,\ \forall z \in C, \tag{2.8}$$

this is equivalent to

$$\Delta_p(\Pi_C x, z) \leq \Delta_p(x, z) - \Delta_p(x, \Pi_C x),\ \forall z \in C. \tag{2.9}$$

Lemma 2.3 ([30]). *Let E be a q-uniformly smooth Banach space with the q-uniformly smoothness constant $C_q > 0$. For any $x, y \in E$, the following inequality holds:*

$$\|x - y\|^q \leq \|x\|^q - q\langle y, J_E^q x\rangle + C_q\|y\|^q.$$

Lemma 2.4 ([25]). *Let E be a p-uniformly convex and uniformly smooth Banach space and with its dual E^*, J_E^p and $J_{E^*}^q$ are the duality mappings of E and E^*, respectively. For any $\{x_n\} \subset E$, and $\{t_n\} \subset (0,1)$ with $\Sigma_{n=1}^N t_n = 1$, the following inequality holds. $\Delta_p(J_{E^*}^q(\sum_{n=1}^N t_n J_E^p(x_n)), x) \leq \sum_{n=1}^N t_n \Delta_p(x_n, x)$, $\forall x \in E$.*

Lemma 2.5. *Let E be a p-uniformly convex and uniformly smooth real Banach space, and $C_1 = E$. Then, for any sequences $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ in E, the set*

$$C_{n+1} = \{u \in C_n : \Delta_p(y_n, u) \leq \Delta_p(z_n, u) \leq \Delta_p(w_n, u)\}$$

is closed and convex for each $n \geq 1$.

Proof. First, since $C_1 = E$, C_1 is closed and convex. Then we assume that C_n is a closed and convex. For each $u \in C_n$, by the definition of the function Δ_p, we have

$$\Delta_p(y_n, u) \leq \Delta_p(z_n, u) \Leftrightarrow 2\langle J_E^p z_n - J_E^p y_n, u\rangle \leq \frac{1}{q}(\|z_n\|^p - \|y_n\|^p),$$

and

$$\Delta_p(z_n, u) \leq \Delta_p(w_n, u) \Leftrightarrow 2\langle J_E^p w_n - J_E^p z_n, u\rangle \leq \frac{1}{q}(\|w_n\|^p - \|z_n\|^p).$$

Hence, we know that C_{n+1} is closed. In addition, we can easily prove that C_{n+1} is a convex. The proof is completed. □

3. Main Results

In the section, we assume that the following conditions are satisfied:

(1) E_1 and E_2 are two p-uniformly convex and uniformly smooth real Banach spaces;

(2) $A : E_1 \to E_2$ is a bounded linear operator with adjoint operator A^*;

(3) $T : E_1 \to E_1$ is a closed left Bregmen quasi-nonexpansive mapping;

(4) $S : E_2 \to E_2$ is a firmly nonexpansive-like mapping.

In addition, $J_{E_1}^p$ and $J_{E_2}^p$ are the duality mappings of E_1 and E_2, respectively, and $J_{E_1^*}^q$ is the duality mapping of E_1^*. It is worth noting that E_1^* and E_2^* are two q-uniformly smooth and uniformly convex Banach spaces, and $J_{E_1}^p = (J_{E_1^*}^q)^{-1}$, where $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Algorithm. For given initial values x_0, $x_1 \in C_1 = E_1$, the sequence $\{x_n\}$ generated by the following iterative algorithm:

$$\begin{cases} w_n = J_{E_1^*}^q(J_{E_1}^p x_n + \theta_n J_{E_1}^p(x_n - x_{n-1})), \\ z_n = J_{E_1^*}^q(J_{E_1}^p w_n - \gamma_n A^* J_{E_2}^p(I - S)Aw_n), \\ y_n = J_{E_1^*}^q(\alpha_n J_{E_1}^p z_n + (1 - \alpha_n)J_{E_1}^p Tz_n), \\ C_{n+1} = \{u \in C_n : \Delta_p(y_n, u) \leq \Delta_p(z_n, u) \leq \Delta_p(w_n, u)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.1}$$

where $\Pi_{C_{n+1}}$ is a Bregman projection of E_1 onto C_{n+1}, $\{\gamma_n\}$ is a sequence of real number in $(0,(\frac{q}{C_q\|A\|^q})^{\frac{1}{q-1}})$, where $\frac{1}{c}$ is the p-uniformly convexity constant and C_q is the q-uniformly smoothness constant, the sequences of real number $\{\alpha_n\}\subset[a,b]\subset(0,1)$ and $\{\theta_n\}\subset[c,d]\subset(-\infty,+\infty)$.

Lemma 3.1. *Let E_1, E_2, T, S, A, A^* and $J^p_{E_1}$, $J^p_{E_2}$, $J^q_{E_1^*}$ be the same as above. If $\Gamma=\{x^*|x^*\in F(T);Ax^*\in F(S)\}$, then $\Gamma\subseteq C_n$ for any $n\geq 1$.*

Proof. If $\Gamma=\emptyset$, it is obvious that $\Gamma\subseteq C_n$. Conversely, for any $x^*\in\Gamma$, we have $x^*\in F(T)$ and $Ax^*\in F(S)$. According to Lemma 2.4 and the definition of left Bregman quasi-nonexpansive mapping T, we easily obtain

$$\begin{aligned}\Delta_p(y_n,x^*)&=\Delta_p(J^q_{E_1^*}(\alpha_nJ^p_{E_1}z_n+(1-\alpha_n)J^p_{E_1}Tz_n),x^*)\\&\leq\alpha_n\Delta_p(z_n,x^*)+(1-\alpha_n)\Delta_p(Tz_n,x^*)\\&\leq\Delta_p(z_n,x^*).\end{aligned}\tag{3.2}$$

Since E_1 is a p-uniformly convex and uniformly smooth real Banach space, then E_1^* is a q-uniformly smooth and uniformly convex Banach space and $J^p_{E_1}=(J^q_{E_1^*})^{-1}$. From the property of firmly nonexpansive-like mapping S, we easily obtain $\langle J^p_{E_2}(I-S)Aw_n,Ax^*-SAw_n\rangle\leq 0$. Further, we have

$$\begin{aligned}\langle J^p_{E_2}(I-S)Aw_n,Ax^*-Aw_n\rangle&=\langle J^p_{E_2}(I-S)Aw_n,Ax^*-SAw_n+SAw_n-Aw_n\rangle\\&=-\|(I-S)Aw_n\|^p+\langle J^p_{E_2}(I-S)Aw_n,Ax^*-SAw_n\rangle\\&\leq-\|(I-S)Aw_n\|^p.\end{aligned}\tag{3.3}$$

Again from (2.9), (3.3) and Lemma 2.3, we obtain

$$\begin{aligned}\Delta_p(z_n,x^*)&=\Delta_p(J^q_{E_1^*}(J^p_{E_1}w_n-\gamma_nA^*J^p_{E_2}(I-S)Aw_n),x^*)\\&=\frac{1}{q}\|J^q_{E_1^*}(J^p_{E_1}w_n-\gamma_nA^*J^p_{E_2}(I-S)Aw_n)\|^p+\frac{1}{p}\|x^*\|^p\\&\quad-\langle J^p_{E_1}w_n-\gamma_nA^*J^p_{E_2}(I-S)Aw_n,x^*\rangle\\&=\frac{1}{q}\|J^p_{E_1}w_n-\gamma_nA^*J^p_{E_2}(I-S)Aw_n\|^q+\frac{1}{p}\|x^*\|^p\\&\quad-\langle J^p_{E_1}w_n,x^*\rangle+\gamma_n\langle J^p_{E_2}(I-S)Aw_n,Ax^*\rangle\\&\leq\frac{1}{q}\|J^p_{E_1}w_n\|^q+\frac{1}{p}\|x^*\|^p-\langle J^p_{E_1}w_n,x^*\rangle-\gamma_n\langle J^p_{E_2}(I-S)Aw_n,Aw_n\rangle\\&\quad+\gamma_n\langle J^p_{E_2}(I-S)Aw_n,Ax^*\rangle+\frac{C_q(\gamma_n\|A\|)^q}{q}\|J^p_{E_2}(I-S)Aw_n\|^q\\&=\Delta_p(w_n,x^*)+\gamma_n\langle J^p_{E_2}(I-S)Aw_n,Ax^*-Aw_n\rangle\\&\quad+\frac{C_q(\gamma_n\|A\|)^q}{q}\|(I-S)Aw_n\|^p.\end{aligned}$$

Since $\{\gamma_n\}$ is a real number sequence contained in $(0,(\frac{q}{C_q\|A\|^q})^{\frac{1}{q-1}})$, we get

$$\Delta_p(z_n,x^*)\leq\Delta_p(w_n,x^*)+(\frac{C_q(\gamma_n\|A\|)^q}{q}-\gamma_n)\|(I-S)Aw_n\|^p\leq\Delta_p(w_n,x^*).\tag{3.4}$$

From (3.2) and (3.4), we have $x^*\in C_{n+1}$, that is, $\Gamma\subseteq C_n$, $\forall n\geq 1$. □

Theorem 3.1. *Let E_1, E_2, T, S, A, A^* and $J^p_{E_1}$, $J^p_{E_2}$, $J^q_{E_1^*}$ be the same as above. If $\Gamma = \{x^* | x^* \in F(T); Ax^* \in F(S)\} \neq \emptyset$, the sequence $\{x_n\}$ generated by Algorithm (3.1) converges strongly to a point $z = \Pi_\Gamma x_0 \in \Gamma$.*

Proof. By Lemma 2.5 and Lemma 3.1, we know that $\Pi_{C_{n+1}} x_0$ is well defined and $\Gamma \subseteq C_n$. According to Algorithm (3.1), we know $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0$ for each $n \geq 1$. Using $\Gamma \subseteq C_n$ and (2.9), we have

$$\Delta_p(x_0, x_n) = \Delta_p(x_0, \Pi_{C_n} x_0) \leq \Delta_p(x_0, x^*), \ x^* \in \Gamma, \ \forall n \geq 1. \tag{3.5}$$

It implies that $\{\Delta_p(x_0, x_n)\}$ is bounded. Reusing (2.9), we also have

$$\begin{aligned} \Delta_p(x_n, x_{n+1}) = \Delta_p(\Pi_{C_n} x_0, x_{n+1}) &\leq \Delta_p(x_0, x_{n+1}) - \Delta_p(x_0, \Pi_{C_n} x_0) \\ &= \Delta_p(x_0, x_{n+1}) - \Delta_p(x_0, x_n). \end{aligned} \tag{3.6}$$

It follows that $\{\Delta_p(x_0, x_n)\}$ is nondecreasing. Hence, $\lim_{n\to\infty} \Delta_p(x_0, x_n)$ exists, and

$$\lim_{n\to\infty} \Delta_p(x_n, x_{n+1}) = 0. \tag{3.7}$$

It follows from (2.6) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.8}$$

For some positive integers m, n with $m \geq n$, we have $x_m = \Pi_{C_m} x_1 \subseteq C_n$. Using (2.9), we obtain

$$\begin{aligned} \Delta_p(x_n, x_m) = \Delta_p(\Pi_{C_n} x_0, x_m) &\leq \Delta_p(x_0, x_m) - \Delta_p(x_0, \Pi_{C_n} x_0) \\ &= \Delta_p(x_0, x_m) - \Delta_p(x_0, x_n). \end{aligned} \tag{3.9}$$

Since $\lim_{n\to\infty} \Delta_p(x_0, x_n)$ exists, it follows from (3.9) that $\lim_{n\to\infty} \Delta_p(x_n, x_m) = 0$ and $\lim_{n\to\infty} \|x_m - x_n\| = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence. Further, there exists a point $z \in C$ such that $x_n \to z$.

From Algorithm (3.1), Definition 2.2 and Lemma 2.3, we have

$$\begin{aligned} \Delta_p(w_n, x^*) &= \frac{1}{q}\|J^q_{E_1^*}(J^p_{E_1} x_n + \theta_n J^p_{E_1}(x_n - x_{n-1}))\|^p + \frac{1}{p}\|x^*\|^p \\ &\quad - \langle J^p_{E_1} x_n + \theta_n J^p_{E_1}(x_n - x_{n-1}), x^* \rangle \\ &= \frac{1}{q}\|J^p_{E_1} x_n + \theta_n J^p_{E_1}(x_n - x_{n-1})\|^q + \frac{1}{p}\|x^*\|^p \\ &\quad - \langle J^p_{E_1} x_n, x^* \rangle - \theta_n \langle J^p_{E_1}(x_n - x_{n-1}), x^* \rangle \\ &\leq \frac{1}{q}\|J^p_{E_1} x_n\|^q + \frac{1}{p}\|x^*\|^p - \langle J^p_{E_1} x_n, x^* \rangle - \theta_n \langle J^p_{E_1}(x_n - x_{n-1}), x^* \rangle \\ &\quad + \theta_n \langle J^p_{E_1}(x_n - x_{n-1}), x_n \rangle + \frac{C_q(\theta_n)^q}{q}\|J^p_{E_1}(x_n - x_{n-1})\|^q \\ &= \frac{1}{q}\|x_n\|^p + \frac{1}{p}\|x^*\|^p - \langle J^p_{E_1} x_n, x^* \rangle - \theta_n \langle J^p_{E_1}(x_n - x_{n-1}), x^* \rangle \\ &\quad + \theta_n \langle J^p_{E_1}(x_n - x_{n-1}), x_n \rangle + \frac{C_q(\theta_n)^q}{q}\|x_n - x_{n-1}\|^p \\ &= \Delta_p(x_n, x^*) + \theta_n \langle J^p_{E_1}(x_n - x_{n-1}), x_n - x^* \rangle \\ &\quad + \frac{C_q(\theta_n)^q}{q}\|x_n - x_{n-1}\|^p. \end{aligned} \tag{3.10}$$

By virtue of Remark 2.1 and the definition of w_n, we know

$$\begin{aligned}\Delta_p(w_n, x^*) &= \Delta_p(w_n, x_n) + \Delta_p(x_n, x^*) + \langle x_n - x^*, J_{E_1}^p w_n - J_{E_1}^p x_n\rangle \\ &= \Delta_p(w_n, x_n) + \Delta_p(x_n, x^*) + \theta_n\langle x_n - x^*, J_{E_1}^p(x_n - x_{n-1})\rangle.\end{aligned} \tag{3.11}$$

By (3.10) and (3.11), we get $\Delta_p(w_n, x_n) \le \frac{C_q(\theta_n)^q}{q}\|x_n - x_{n-1}\|^p$. Then, using (2.6), (3.8) and the boundedness of the sequence $\{\theta_n\}$, we can obtain

$$\lim_{n\to\infty}\|w_n - x_n\| = 0. \tag{3.12}$$

Using a similar method, we can get

$$\Delta_p(w_n, x_{n+1}) = \Delta_p(w_n, x_n) + \Delta_p(x_n, x_{n+1}) + \langle x_n - x_{n+1}, J_{E_1}^p w_n - J_{E_1}^p x_n\rangle.$$

By setting $n \to \infty$, we have

$$\lim_{n\to\infty}\|w_n - x_{n+1}\| = 0. \tag{3.13}$$

Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subseteq C_n$, we have

$$\Delta_p(y_n, x_{n+1}) \le \Delta_p(z_n, x_{n+1}) \le \Delta_p(w_n, x_{n+1}).$$

According to (3.13), we obtain

$$\lim_{n\to\infty}\Delta_p(y_n, x_{n+1}) = 0,\ \lim_{n\to\infty}\Delta_p(z_n, x_{n+1}) = 0, \tag{3.14}$$

which implies that $\lim_{n\to\infty}\|y_n - x_{n+1}\| = 0,\ \lim_{n\to\infty}\|z_n - x_{n+1}\| = 0$. Hence

$$\|x_n - z_n\| \le \|x_{n+1} - x_n\| + \|x_{n+1} - z_n\| \to 0 \ as\ n \to \infty, \tag{3.15}$$

$$\|y_n - z_n\| \le \|x_{n+1} - y_n\| + \|x_{n+1} - z_n\| \to 0 \ as\ n \to \infty. \tag{3.16}$$

In addition, since E_1 is a p-uniformly convex and uniformly smooth real Banach space, then $J_{E_1}^p$ is uniformly norm-to-norm continuous. It follows from Algorithm (3.1) and real number sequence $\{\alpha_n\}$ in $[a, b] \subset (0, 1)$ that

$$\lim_{n\to\infty}\|J_{E_1}^p Tz_n - J_{E_1}^p z_n\| = \lim_{n\to\infty}\frac{1}{1-\alpha_n}\|J_{E_1}^p y_n - J_{E_1}^p z_n\| = 0,$$

which also implies that $\lim_{n\to\infty}\|Tz_n - z_n\| = 0$. By virtue of (3.12) and $x_n \to z$ we have $z_n \to z$. Using the closedness of T, we obtain $z_n \to z$ and $Tz = z$. From (2.6), (3.4) and (3.15), we have

$$\begin{aligned}(\gamma_n - \frac{C_q(\gamma_n\|A\|)^q}{q})\|(I-S)Aw_n\|^p &\le \Delta_p(w_n, x^*) - \Delta_p(z_n, x^*) \\ &= \frac{1}{q}\|w_n\|^p - \frac{1}{q}\|z_n\|^p - \langle J_{E_1}^p w_n - J_{E_1}^p z_n, x^*\rangle \\ &= \Delta_p(w_n, z_n) + \langle J_{E_1}^p w_n - J_{E_1}^p z_n, z_n - x^*\rangle \\ &\le (\|w_n - z_n\| + \|z_n - x^*\|)\|J_{E_1}^p w_n - J_{E_1}^p z_n\|.\end{aligned}$$

By setting of $n \to \infty$, the right-hand side of last inequality tends to 0. Moreover, $\{\gamma_n\}$ is a real number sequence contained in $(0, (\frac{q}{C_q\|A\|^q})^{\frac{1}{q-1}})$, we have

$$\lim_{n\to\infty} \|(S-I)Aw_n\| = 0.$$

Since A is a bounded linear operator, we have $Aw_n \to Az$. Again according to Definition 2.3 and Lemma 2.1, we get $Az \in F(T)$. Then, from (2.8) and (3.1), we have

$$\langle J^p_{E_1} x_0 - J^p_{E_1} x_n, p - x_n\rangle \le 0, \ \forall p \in \Gamma. \tag{3.17}$$

By setting $n \to \infty$ in (3.17), we obtain

$$\langle J^p_{E_1} x_0 - J^p_{E_1} z, p - z\rangle \le 0, \ \forall p \in \Gamma. \tag{3.18}$$

Again from (2.8), we have $z = \Pi_\Gamma x_0$. Definitively, we obtain that $\{x_n\}$ generated by Algorithm (3.1) strong converge $z = \Pi_\Gamma x_0 \in \Gamma$. The proof is completed. □

Remark 3.1. The significant improvement of the results in this paper is using the shrinking projection algorithm with inertial effects to study the split common fixed problem in the framework of two p-uniformly convex and uniformly smooth Banach spaces, and the iterative sequence generated by Algorithm (3.1) strongly converges to a solution of the split common fixed point problem.

Remark 3.2. The field of study in this paper is the p-uniformly convex and uniformly smooth Banach space, which is more extensive than the Hilbert space [26–28], the uniformly convex and 2-uniformly smooth Banach space [29] and the 2-uniformly convex and 2-uniformly smooth real Banach space [25]. The split common fixed point problem of firmly nonexpansive-like mappings in Theorem 3.1 is more general than the split feasibility problem and fixed point problem in [18, 25].

As a corollary of Theorem 3.1, when E_1 and E_2 reduce to Hilbert spaces, the function Δ_p is equal to $\Delta_p(x,y) = \frac{1}{2}\|x-y\|^2$ and the Bregman projection Π_C is equivalent to the metric projection P_C. Then, we obtain the following corollary.

Corollary 3.1. *Let H_1 and H_2 be two real Hilbert spaces. Let $T : H_1 \to H_1$ be a closed quasi-nonexpansive mapping, $S : H_2 \to H_2$ be a firmly nonexpansive mapping, and $A : H_1 \to H_2$ be a bounded linear operator with adjoint operator A^*. For given initial values $x_0, x_1 \in C_1 = H_1$, the sequence $\{x_n\}$ generated by the following iterative algorithm:*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = w_n - \gamma_n A^*(I-S)Aw_n, \\ y_n = \alpha_n z_n + (1-\alpha_n)Tz_n, \\ C_{n+1} = \{u \in C_n : \|y_n - u\| \le \|z_n - u\| \le \|w_n - u\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases} \tag{3.19}$$

where $P_{C_{n+1}}$ is a metric projection of H_1 onto C_{n+1}, the sequences of real numbers $\{\gamma_n\} \subset (0, \frac{2}{\|A\|^2})$, $\{\alpha_n\} \subset [a,b] \subset (0,1)$ and $\{\theta_n\} \subset [c,d] \subset (-\infty,+\infty)$. If $\Gamma = \{x^* | x^* \in F(T), Ax^* \in F(S)\} \neq \emptyset$, the sequence $\{x_n\}$ generated by (3.19) converges strongly to a point $z = P_\Gamma x_0 \in \Gamma$.

4. Applications

4.1. Split fixed point problems and variational inclusion problems

Let H be a real Hilbert space and $B : H \to 2^H$ be a set-valued mapping with domain $D(B) := \{x \in H : B(x) \neq \emptyset\}$. An operator $B : H \to 2^H$ is called monotone if $\langle u - v, x - y\rangle \geq 0$, $\forall u \in Bx,\ v \in By$. Further, B is called maximal monotone if its graph $G(B) = \{(x, y) : x \in D(B),\ y \in D(B)\}$ is not properly contained in the graph of any other monotone operator.

The problem of zero points of maximal monotone operator is

$$\text{finding } x^* \in H, \text{ such that } 0 \in B(x^*),$$

where $B : H \to 2^H$ is a set-valued maximal monotone operator. Martinet [20] first proposed proximal point algorithm to solve the problem of a zero point of maximal monotone operator.

Lemma 4.1 ([23])**.** *Let H be a real Hilbert space. Let $B : H \to 2^H$ be a maximal monotone operator and $\mu > 0$, and its associated resolvent of order μ, defined by $J_\mu^B = (I + \mu A)^{-1}$, where I denotes the identity mapping. Then, the following properties are true:*

(I) For each $\mu > 0$, J_μ^B is a sing-valued and firmly nonexpansive mapping;

(II) $D(J_\mu^B) = H$ and $F(J_\mu^B) = B^{-1}(0) := \{x \in D(B),\ 0 \in Bx\}$.

Definition 4.1. Let H_1 and H_2 be two Hilbert spaces, and $A : H_1 \to H_2$ be a bounded linear operator. Let $B : H_1 \to 2^{H_1}$ and $K : H_2 \to 2^{H_2}$ be two set-valued mappings, $T : H_1 \to H_1$ be a mapping. The split fixed point problem and variational inclusion problem is to find a point x^* such that

$$x^* \in F(T) \cap K^{-1}(0),\ Ax^* \in B^{-1}(0).$$

For this problem, we propose the following theorem by the result in Theorem 3.1.

Theorem 4.1. *Let H_1 and H_2 be two Hilbert spaces, and $A : H_1 \to H_2$ be a bounded linear operator with adjoint operator A^*. Let $B : H_1 \to 2^{H_1}$ and $K : H_2 \to 2^{H_2}$ be two maximal monotone operators and $\mu > 0$, $T : H_1 \to H_1$ be a closed quasi-nonexpansive mapping. For given initial values $x_0,\ x_1 \in C_1 = H_1$, the sequence $\{x_n\}$ generated by the following iterative algorithm:*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = w_n - \gamma_n A^*(I - J_\mu^B)Aw_n, \\ y_n = \alpha_n z_n + (1 - \alpha_n)TJ_\mu^K z_n, \\ C_{n+1} = \{u \in C_n : \|y_n - u\| \leq \|z_n - u\| \leq \|w_n - u\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases} \tag{4.1}$$

where $P_{C_{n+1}}$ is a metric projection of H_1 to C_{n+1}, the sequences of real numbers $\{\gamma_n\} \subset (0, \frac{2}{\|A\|^2})$, $\{\alpha_n\} \subset [a, b] \subset (0, 1)$ and $\{\theta_n\} \subset [c, d] \subset (-\infty, +\infty)$. If $\Gamma = \{x^* | x^* \in F(T) \cap K^{-1}(0), Ax^* \in B^{-1}(0)\} \neq \emptyset$, the sequence $\{x_n\}$ generated by iterative algorithm (4.1) converges strongly to a point $z = P_\Gamma x_0 \in \Gamma$.

Proof. It is obvious that TJ_{μ}^{K} is closed quasi-nonexpansive mapping from the property of T and Lemma 4.1. Hence, the strong convergence theorem of iterative algorithm (4.1) is obviously proved. $\square$

4.2. Split fixed point problems and equilibrium problems

Let C be a nonempty closed and convex subset of a Hilbert space H. Let bifunction $F : C \times C \to R$ satisfy the following conditions:

(A1) $F(x, x) = 0$, $\forall x \in C$;

(A2) $F(x, y) + F(y, x) \leq 0$, $\forall x, y \in C$;

(A3) $\lim_{n\to\infty} F(tz + (1-t)x, y) \leq F(x, y)$, $\forall x, y, z \in C$;

(A4) For each $x \in C$, the function $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Then, the so-called equilibrium problem for F is to find a point $x^* \in C$ such that $F(x^*, x) \geq 0$, $\forall x \in C$, and the set of solutions of equilibrium problem is denoted by $EP(F)$.

Lemma 4.2 ([14]). *Let C be a nonempty closed convex subset of a Hilbert space H, and let $F : C \times C \to R$ be a bifunction satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then there exists a point $z \in C$ such that $F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$, $\forall y \in C$.*

Lemma 4.3 ([14]). *Assume that $F : C \times C \to R$ be a bifunction satisfying (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to H$ as follows: $T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$, $\forall x \in H$. Then*

(1) T_r^F is single-valued;

(2) $\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle$;

(3) $F(T_r^F) = EP(F)$ is nonempty, closed and convex.

Definition 4.2. Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ be a bounded linear operator, $T : C \to C$ be a mapping, $F : C \times C \to R$ be a bifunction satisfying (A1)-(A4). The split fixed point problem and equilibrium problem is to find a point x^* such that

$$x^* \in F(T),\ Ax^* \in EP(F).$$

Theorem 4.2. *Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $T : C \to C$ be a closed quasi-nonexpansive mapping, $F : C \times C \to R$ be a bifunction satisfying (A1)-(A4), and $A : H_1 \to H_2$ be a bounded linear operator with adjoint operator A^*. For given initial values $x_0,\ x_1 \in C_1 = C$, the sequence $\{x_n\}$ generated by the following iterative algorithm:*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = P_C(w_n - \gamma_n A^*(I - T_r^F)Aw_n), \\ y_n = \alpha_n z_n + (1 - \alpha_n)Tz_n, \\ C_{n+1} = \{u \in C_n : \|y_n - u\| \leq \|z_n - u\| \leq \|w_n - u\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases} \tag{4.2}$$

where $P_{C_{n+1}}$ is a metric projection of H_1 onto C_{n+1}, the sequences of real numbers $\{\gamma_n\} \subset (0, \frac{2}{\|A\|^2})$, $\{\alpha_n\} \subset [a,b] \subset (0,1)$ and $\{\theta_n\} \subset [c,d] \subset (-\infty,+\infty)$. If $\Gamma = \{x^*|x^* \in F(T),\ Ax^* \in EP(F)\} \neq \emptyset$, the sequence $\{x_n\}$ generated by iterative algorithm (4.2) *converges strongly to a point $z = P_\Gamma x_0 \in \Gamma$.*

5. Numerical examples

In this section, we come up with some numerical examples to demonstrate the effectiveness and realization of convergence of Theorem 3.1. All codes were written in Matlab R2018b, and ran on a Lenovo ideapad 720S with 1.6 GHz Intel Core i5 processor and 8GB of RAM. Using numerical experiments, we will compare the convergence speed of our algorithm with the algorithm in [18]. Ma et al. [18] proved strong convergence theorems of split feasibility problems and fixed point problems of quasi-ϕ-nonexpansive mapping in Banach spaces and introduced the following algorithm.

Theorem 5.1 ([18])**.** *Let E_1 be a 2-uniformly convex and 2-uniformly smooth real Banach space with best smoothness constant $k > 0$, E_2 be a smooth, strictly convex, and reflective Banach space. Let $T : E_1 \to E_1$ be a closed quasi-ϕ-nonexpansive mapping with $F(T) \neq \emptyset$, $A : E_1 \to E_1$ be a bounded linear operator with adjoint A^*, and Q be a nonempty, closed, and convex subset of E_2. Let $x_1 \in E_1$ and $C_1 = E_1$, and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_1^{-1}(J_1 x_n - \gamma A^* J_2(I - P_Q)Ax_n), \\ y_n = J_1^{-1}(\alpha_n J_1 z_n + (1-\alpha_n)J_1 T z_n), \\ C_{n+1} = \{u \in C_n : \phi(u, y_n) \leq \phi(u, x_n),\ \phi(u, z_n) \leq \phi(u, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where P_Q is the metric projection of E_2 onto Q and $\Pi_{C_{n+1}}$ is the generalized projection of E_1 onto C_{n+1}, $\{\alpha_n\}$ is a sequence in $(0,\delta]$, $\delta < 1$, and γ is a positive constant satisfying $0 < \gamma < \frac{1}{\|A\|^2 k^2}$. If $\Gamma = \{x^*|x^* \in F(T), Ax^* \in Q\} \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to a point $z = \Pi_\Gamma x_1$.

To make sure the initial conditions of Theorem 5.1 and Theorem 3.1 are consistent, the initial conditions are given as follows:

Let $E_1 = R$ and $E_2 = R \times R$ with the Euclidean norm, and $C = [0,+\infty)$ and $Q = [0,+\infty) \times (-\infty, 0]$. Let $A : E_1 \to E_2$ be a bounded linear operator and defined as $Ax = (\frac{x}{2}, \frac{x}{3})$, $\forall x \in E_1$ with its adjoint $A^*(u,v) = \frac{u}{2} + \frac{v}{3}$, $\forall (u,v) \in E_2$.

Example 5.1 (Ma et al.)**.** Let $T : C \to C$ be defined as $Tx = \frac{1}{4}x$, $\forall x \in C$, and $P_Q : E_2 \to Q$ be a metric projection. In addition, we choose parameters $\gamma = 1$, $\alpha_n = \frac{1}{7}$. For given initial value $x_1 \in C_1 = C$, the iterative algorithm in Theorem 5.1 can be simplified as

$$\begin{cases} Ax_n = (\dfrac{x_n}{2}, \dfrac{x_n}{3}), \\ z_n = x_n - A^*(I - P_Q)Ax_n, \\ y_n = \dfrac{1}{7}z_n + (1 - \dfrac{1}{7})Tz_n, \\ C_{n+1} = \{u \in C_n : |y_n - u| \leq |x_n - u|, |z_n - u| \leq |x_n - u|\}, \\ x_{n+1} = P_{C_{n+1}} x_1. \end{cases}$$

Then, the sequence $\{x_n\}$ converges strongly 0.

Example 5.2 (Our algorithm with inertial effects in Theorem 3.1)**.** Let $T : C \to C$ be defined as $Tx = \frac{1}{4}x$, $\forall x \in C$, and $S = P_Q : E_2 \to Q$ be a metric projection. In addition, we choose parameters $\gamma_n = 1$, $\alpha_n = \frac{1}{7}$ and $\theta_n = \frac{1}{2}$. For given initial values $x_0 = x_1 \in C_1 = C$, the iterative algorithm in Theorem 3.1 can be simplified as

$$\begin{cases} w_n = x_n + \dfrac{1}{2}(x_n - x_{n-1}), \\ z_n = w_n - A^*(I - P_Q)Aw_n, \\ y_n = \dfrac{1}{7}z_n + (1 - \dfrac{1}{7})Tz_n, \\ C_{n+1} = \{u \in C_n : |y_n - u| \le |z_n - u| \le |w_n - u|\}, \\ x_{n+1} = P_{C_{n+1}}x_0. \end{cases}$$

Then, the sequence $\{x_n\}$ converges strongly to 0.

Next, taking initial values $x_1 = 6$, $x_1 = 3$ in Example 5.1 and $x_0 = x_1 = 6$, $x_0 = x_1 = 3$ in Example 5.2. We get the following Table 1 and Figure 1 to demonstrate rate of convergence of the algorithms in Theorem 5.1 and Theorem 3.1.

Table 1. Numerical results of Theorem 5.1 and Theorem 3.1

n	Ma et al.		Algorithm 3.1	
	$x_1 = 6$	$x_1 = 3$	$x_0 = x_1 = 6$	$x_0 = x_1 = 3$
0	/	/	6	3
1	6	3	6	3
2	3.952380952380953	1.976190476190476	3.619047619047619	1.809523809523809
3	2.603552532123961	1.301776266061980	1.895691609977324	0.947845804988662
4	1.715038572748323	0.857519286374162	0.935536119209588	0.467768059604794
5	1.129747631254848	0.564873815627424	0.448463346033803	0.224231673016901
6	0.744198836461527	0.372099418230763	0.211743715446802	0.105871857723401
⋮	⋮	⋮	⋮	⋮
21	0.001420045289491	0.000710022644746	0.000002127155711	0.000001063577855
22	0.000935426658951	0.000467713329475	0.000000986132411	0.000000493066206
23	0.000616193751531	0.000308096875766	0.000000457162771	0.000000228581385
24	0.000405905407755	0.000202952703877	0.000000211936762	0.000000105968381
25	0.000267382133680	0.000133691066840	0.000000098252052	0.000000049126026

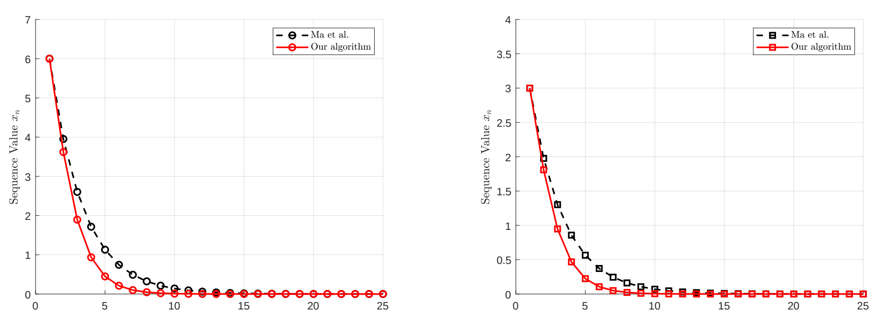

Figure 1. Analysis of convergence speed of Theorem 5.1 and Theorem 3.1

It is worth noting that γ_n, α_n and θ_n are only constant step-size and initial point $x_0 = x_1$(there is no using an inertial effect in the first iterative process) in Example 5.2. To change this, we consider the following four cases of the step-size parameters γ_n, α_n and θ_n, with the initial points $x_0 = 8$, $x_1 = 6$.

Case 1: $\gamma_n = 1$, $\alpha_n = \frac{1}{7}$ and $\theta_n = \frac{1}{2}$;

Case 2: $\gamma_n = \frac{2n+3}{2n}$, $\alpha_n = \frac{1}{7}$ and $\theta_n = \frac{1}{2}$;

Case 3: $\gamma_n = \frac{2n+3}{2n}$, $\alpha_n = \frac{n}{7n+5}$ and $\theta_n = \frac{1}{2}$;

Case 4: $\gamma_n = \frac{2n+3}{2n}$, $\alpha_n = \frac{n}{7n+5}$ and $\theta_n = \frac{2n+1}{10n+2}$.

Table 2. Numerical results of Case1-4

n	Case 1	Case 2	Case 3	Case 4
0	8	8	8	8
1	6	6	6	6
2	3.377777777777778	2.744444444444444	2.654166666666667	2.606770833333333
3	1.721058201058201	1.144272486772487	1.062511878654971	0.982533042700857
4	0.838240362811792	0.466087018140590	0.414427676387483	0.349299329383015
5	0.399106638351990	0.189975923807574	0.161779387393689	0.121333554378606
6	0.187756126213986	0.078232012035175	0.063949280327919	0.042025571691499
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
21	0.000001877748043	0.000000362109760	0.000000229101124	0.000000057064087
22	0.000000870508717	0.000000164098265	0.000000103109472	0.000000025326458
23	0.000000403560514	0.000000074452809	0.000000046477831	0.000000011270697
24	0.000000187087193	0.000000033815734	0.000000020979640	0.000000005027255
25	0.000000086732002	0.000000015373490	0.000000009481898	0.000000002246927

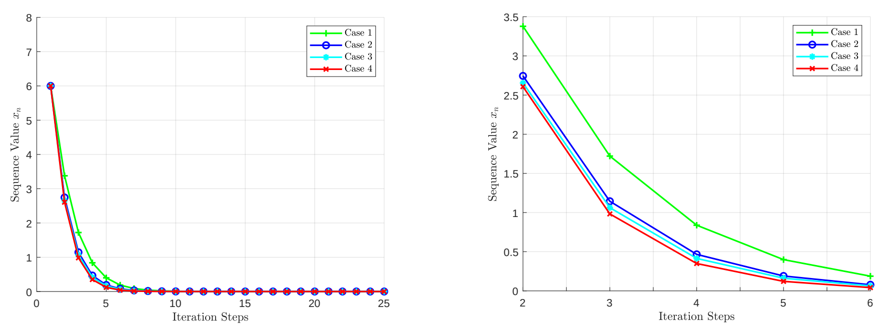

Figure 2. Analysis of convergence speed of Case1-4

Remark 5.1. (I) From Figure 1 and Table 1, we found that the convergence rate of the algorithm in Theorem 3.1 is faster than the algorithm in Theorem 5.1. This also shows the efficiency of our proposed Algorithm (3.1) in this paper.

(II) From Figure 2 and Table 2, we know that the convergence speed of the algorithm in Theorem 3.1 is improved under suitable condition of step-size.

References

[1] F. Alvarez and H. Attouch, *An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping*, Set-Valued Anal., 2001, 9(1–2), 3–11.

[2] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Problems, 2002, 18(2), 441–453.

[3] C. Byrne, Y. Censor, A. Gibali and S. Reich, *The split common null point problem*, J. Nonlinear Convex Anal., 2012, 13(4), 759-775.

[4] L. Ceng, A. Petruşel, C. Wen and J. C. Yao, *Inertial-like subgradient extragradient methods for variational inequalities and fixed points of asymptotically nonexpansive and strictly pseudocontractive mappings*, Mathematics, 2019, 7(9), 860.

[5] Y. Censor, T. Bortfeld, B. Martin and A. Trofimov, *A unified approach for inversion problems in intensity-modulated radiation therapy*, Phys. Med. Biol., 2006, 51(10), 2353–2365.

[6] Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms, 1994, 8(2), 221–239.

[7] Y. Censor, T. Elfving, N. Kopf and T. Bortfeld, *The multiple-sets split feasibility problem and its applications for inverse problems*, Inverse Problems, 2005, 21(6), 2071.

[8] Y. Censor, A. Gibali and S. Reich, *Algorithms for the split variational inequality problem*, Numer. Algorithms, 2012, 59(2), 301–323.

[9] Y. Censor, A. Motova and A. Segal, *Perturbed projections and subgradient projections for the multiple-sets split feasibility problem*, J. Math. Anal. Appl., 2007, 327(2), 1244–1256.

[10] Y. Censor and A. Segal, *The split common fixed point problem for directed operators*, J. Convex Anal., 2009, 16(2), 587–600.

[11] S. Y. Cho, *Strong convergence analysis of a hybrid algorithm for nonlinear operators in a Banach space*, J. Appl. Anal. Comput., 2018, 8(1), 19–31.

[12] S. Y. Cho, B. A. Bin Dehaish and X. Qin, *Weak convergence of a splitting algorithm in Hilbert spaces*, J. Appl. Anal. Comput., 2017, 7(2), 427–438.

[13] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Springer Science & Business Media, 2012.

[14] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., 2005, 6(1), 117–136.

[15] V. H. Dang, *New inertial algorithm for a class of equilibrium problems*, Numer. Algorithms, 2019, 80(4), 1413–1436.

[16] L. Liu, *A hybrid steepest descent method for solving split feasibility problems involving nonexpansive mappings*, J. Nonlinear Convex Anal., 2019, 20(3), 471–488.

[17] L. Liu, X. Qin and R. P. Agarwal, *Iterative methods for fixed points and zero points of nonlinear mappings with applications*, Optimization, 2019, 1–21.

[18] Z. Ma, L. Wang and S. Chang, *On the split feasibility problem and fixed point problem of quasi-ϕ-nonexpansive mapping in Banach spaces*, Numer. Algorithms, 2019, 80(4), 1203–1218.

[19] P. E. Maingé and A. Moudafi, *Convergence of new inertial proximal methods for DC programming*, SIAM J. Optim., 2008, 19(1), 397–413.

[20] B. Martinet, *Brève communication. Régularisation d'inéquations variationnelles par approximations successives*, Revue française d'informatique et de recherche opérationnelle. Série rouge, 1970, 4(R3), 154–158.

[21] X. Qin and N. T. An, *Smoothing algorithms for computing the projection onto a Minkowski sum of convex sets*, Comput. Optim. Appl., 2019, 74(3), 821–850.

[22] X. Qin, A. Petrusel and J. C. Yao, *CQ iterative algorithms for fixed points of nonexpansive mappings and split feasibility problems in Hilbert spaces*, J. Nonlinear Convex Anal., 2018, 19(1), 157-165.

[23] X. Qin and J. C. Yao, *A viscosity iterative method for a split feasibility problem*, J. Nonlinear Convex Anal., 2019, 20(8), 1497-1506.

[24] Y. Shehu, O. Iyiola and E. Akaligwo, *Modified inertial methods for finding common solutions to variational inequality problems*, Fixed Point Theory, 2019, 20(2), 683-702.

[25] Y. Shehu, F. Ogbuisi and O. Iyiola, *Convergence analysis of an iterative algorithm for fixed point problems and split feasibility problems in certain Banach spaces*, Optimization, 2016, 65(2), 299–323.

[26] W. Takahashi, *Mann and Halpern iterations for the split common fixed point problem in Banach spaces*, Linear Nonlinear Anal., 2017, 3, 1–18.

[27] W. Takahashi, *Weak and strong convergence theorems for new demimetric mappings and the split common fixed point problem in Banach spaces*, Numer. Funct. Anal. Optim., 2018, 39(10), 1011–1033.

[28] W. Takahashi and J. C. Yao, *Strong convergence theorems by hybrid methods for the split common null point problem in Banach spaces*, Fixed Point Theory Appl., 2015, 2015(1), 87.

[29] J. Tang, S. Chang, L. Wang and X. Wang, *On the split common fixed point problem for strict pseudocontractive and asymptotically nonexpansive mappings in Banach spaces*, J. Inequal. Appl., 2015, 2015(1), 305.

[30] H. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., 1991, 16(12), 1127–1138.

[31] X. Zhang, L. Wang, Z. Ma and L. Qin, *The strong convergence theorems for split common fixed point problem of asymptotically nonexpansive mappings in Hilbert spaces*, J. Inequal. Appl., 2015, 2015(1), 1.