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# ADAPTIVE INERTIAL ALGORITHMS FOR SOLVING SPLIT MONOTONE VARIATIONAL INCLUSION AND FIXED POINT PROBLEMS OF DEMICONTRACTIVE MAPPINGS

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Abstract. The purpose of this paper is to study the common solution of the split monotone variational inclusion and fixed point problem of demicontractive mappings in infinite-dimensional Hilbert spaces. Two inertial algorithms are introduced for approximating the solution of such problems and their strong convergence is guaranteed under the choice of an adaptive step size that does not need to estimate the operator norm. The performance and efficiency of our algorithms are also verified and explained by some practical applications including signal recovery.

## 1. Introduction

In recent years, the split feasibility problem proposed by Censor and Elfving [6] has become an important form of nonlinear analysis and is often used in signal recovery, image restoration, computed tomography, etc. Moreover, as one of its generalizations, the split monotone variational inclusion problem (for short, SMVIP) was proposed by Moudafi [17], which covers the split variational inclusion problem, the split variational inequality problem and the split feasibility problem. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be set-valued maximal monotone mappings. Let $f_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $f_2 : \mathcal{H}_2 \to \mathcal{H}_2$ be single-valued mappings, $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. The split monotone variational inclusion problem is to find a point $x^* \in \mathcal{H}_1$ such that

$$\text{(SMVIP)} \qquad 0 \in f_1(x^*) + B_1(x^*) \text{ and } 0 \in f_2(Ax^*) + B_2(Ax^*).$$

It is well known that $0 \in f_1(x^*) + B_1(x^*)$ if and only if $x^* = J_\gamma^{B_1}(I - \gamma f_1)x^*$, for $\gamma > 0$, i.e., $x^* \in Fix(J_\gamma^{B_1}(I - \gamma f_1))$, where $Fix(J_\gamma^{B_1}(I - \gamma f_1))$ represents the set of fixed points of $J_\gamma^{B_1}(I - \gamma f_1)$. That is to say, the SMVIP can be characterized as the following fixed point form:

$$x^* \in Fix(J_\gamma^{B_1}(I - \gamma f_1)) \text{ and } Ax^* \in Fix(J_\gamma^{B_2}(I - \gamma f_2)),$$

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where $J_\gamma^{B_i}$ is the resolvent operator of $B_i$ and is defined by $J_\gamma^{B_i} = (I + \gamma B_i)^{-1}$ for $i = 1, 2$. Inspired by the CQ algorithm in [3], the following weakly convergent algorithm was introduced by Moudafi [17] to approximate a solution of the SMVIP:

$$x_{n+1} = J_\gamma^{B_1}(I - \gamma f_1)\left(x_n - \lambda A^*(I - J_\gamma^{B_2}(I - \gamma f_2))Ax_n\right), \ n \geq 1, \tag{1.1}$$

where $A^*$ is the adjoint of $A$ and $\lambda \in (0, 1/L)$ with the spectral radius $L$ of $A^*A$. Two important special cases need to be emphasized. One is that $f_1 \equiv f_2 \equiv 0$, the SMVIP is equivalent to the split variational inclusion problem. Furthermore, Algorithm (1.1) is transformed into the algorithm proposed by Byrne [5] to solve the split variational inclusion problem. The other is that $B_1 = N_C$ and $B_2 = N_Q$, where $C$ and $Q$ are nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, $N_C$ and $N_Q$ are normal cones of $C$ and $Q$, respectively, the SMVIP degenerates into the split variational inequality problem. For more study on these problems, see [2, 5, 7, 17, 21, 26].

More specifically, when $f_1 \equiv f_2 \equiv 0$, $B_1 = N_C$ and $B_2 = N_Q$, the SMVIP is equivalent to the split feasibility problem (for short, SFP). To solve such problems, Byrne [3] proposed an important algorithm, namely CQ algorithm, by the idea of the fixed point algorithm. In addition, from the perspective of the optimization problem, the SFP can be regarded as a constrained optimization problem. In the meantime, López et al. [13] suggested a modified CQ algorithm with an adaptive step size, which makes it easy to implement in practical applications where it is impossible or inconvenient to estimate the operator norm. Furthermore, some new iterative algorithms are considered to solve the SFP in Yao et al. [23] and Zhou et al. [27], and strong convergence of their algorithms are guaranteed.

Recall that the mapping $T : \mathcal{H}_1 \to \mathcal{H}_1$ is said to be

(1) $k$-strictly pseudo-contractive if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2, \ \forall x, y \in \mathcal{H}_1.$$

(2) $k$-demicontractive if there exists $k \in (-\infty, 1)$ and $Fix(T) \neq \emptyset$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|Tx - x\|^2, \ \forall p \in Fix(T), \ \forall x \in \mathcal{H}_1,$$

or equivalently

$$\langle x - p, x - Tx\rangle \geq \frac{1 - k}{2}\|x - Tx\|^2, \ \forall p \in Fix(T), \ \forall x \in \mathcal{H}_1.$$

Obviously, when the fixed point set is nonempty, the class of $k$-strictly pseudo-contractive mappings is contained in the class of $k$-demicontractive mappings. Due to the wide application background of fixed point problems, these mappings have also been extensively studied and considered in many mathematical problems, for more detail, see [19, 23, 27, 28] and the references therein. Simultaneously, the improvement of the convergence speed of the algorithm on the above problems and other interesting problems are often studied by adding inertial method in [1, 11, 14, 18, 27, 28].

Motivated by [13, 17, 19, 23, 27], we will study the common solution of split monotone variational inclusion and fixed point problem of demicontractive mappings (shortly, SMVIFP) in infinite-dimensional Hilbert spaces, that is formulated to find a point $x^* \in \mathcal{H}_1$ such that

$$0 \in f_1(x^*) + B_1(x^*),\ x^* \in Fix(T) \text{ and } 0 \in f_2(Ax^*) + B_2(Ax^*),$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ are set-valued maximal monotone mappings, $f_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $f_2 : \mathcal{H}_2 \to \mathcal{H}_2$ are single-valued inverse strongly monotone mappings, $T : \mathcal{H}_1 \to \mathcal{H}_1$ is a demicontractive mapping, and $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. Meanwhile, assume that the solution set $\Gamma$ of this problem is nonempty. Under effects of an adaptive step size and the inertial method, two modified iterative algorithms are proposed for approximating the solution of the SMVIFP, and their strong convergence is obtained under mild conditions. Moreover, the effectiveness of our algorithms is illustrated in terms of numerical experiments.

The remainder of this paper is organized as follows. Section 2 provides some useful definitions and lemmas. Two new inertial algorithms for solving the SMVIFP and their convergence theorems are proposed in Sect. 3. In Sect. 4, some corollaries and remarks arise from our results are given. Finally, in Sect. 5, the validity and authenticity of the convergence behavior of the proposed algorithms are demonstrated by some applicable numerical examples.

## 2. Preliminaries

In this section, some required definitions and lemmas are stated for the proofs in Sect. 3. Assume that $\mathcal{H}$ is a real Hilbert space with the inner product $\langle \cdot , \cdot \rangle$ and the induced norm $\| \cdot \|$ induced by the inner product, and $C$ is a nonempty closed convex subset of $\mathcal{H}$. The symbols $x_n \to x$ and $x_n \rightharpoonup x$ represent strong convergence and weak convergence of the sequence $\{x_n\}$ to $x$, respectively. The fixed point set of a mapping $T$ is denoted by $Fix(T)$, i.e., $Fix(T) = \{x \mid x = Tx\}$. The metric projection $P_C$ of $\mathcal{H}$ onto $C$, is defined by $P_C x := \operatorname{argmin}_{y \in C} \|x - y\|,\ \forall x \in \mathcal{H}$. It is known that

$$\langle P_C x - x, P_C x - y\rangle \le 0, \forall y \in C \Leftrightarrow \|y - P_C x\|^2 + \|x - P_C x\|^2 \le \|x - y\|^2. \tag{2.1}$$

Recall that $B : \mathcal{H} \to 2^{\mathcal{H}}$ is a set-valued mapping with domain $D(B) := \{x \in \mathcal{H} \mid B(x) \neq \emptyset\}$ and graph $Graph(B) := \{(x, w) \in \mathcal{H} \times \mathcal{H} \mid x \in D(B), w \in B(x)\}$. A mapping $B : \mathcal{H} \to 2^{\mathcal{H}}$ is monotone if and only if $\langle x - y, w - v\rangle \ge 0$ for any $w \in B(x)$, $v \in B(y)$. Further, a monotone mapping $B : \mathcal{H} \to 2^{\mathcal{H}}$ is maximal, that is, $Graph(B)$ is not properly contained in the graph of any other monotone mapping. In this case, $B$ is a maximal monotone mapping if and only if for any $(x, w) \in Graph(B)$ and $(y, v) \in \mathcal{H} \times \mathcal{H}$, $\langle x - y, w - v\rangle \ge 0$ implies $v \in B(y)$.

**Definition 2.1.** For any $x,\ y \in \mathcal{H}$, a mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be

(1) firmly nonexpansive, if

$$\|Tx - Ty\|^2 \le \langle Tx - Ty, x - y\rangle,$$

or equivalently

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|x - Tx - (y - Ty)\|^2.$$

(2) quasi-nonexpansive if $Fix(T) \neq \emptyset$ and

$$\|Tx - p\| \le \|x - p\|,\ \forall p \in Fix(T).$$

(3) directed if $Fix(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \leq \|x - p\|^2 - \|Tx - x\|^2, \ \forall p \in Fix(T).$$

(4) $L$-Lipschitz continuous with $L > 0$, if

$$\|Tx - Ty\| \leq L\|x - y\|.$$

In particular, if $L = 1$, it is nonexpansive. If $L \in [0, 1)$, it is contraction.

(5) $\varpi$-averaged with $\varpi \in (0, 1)$, if there exists a nonexpansive mapping $S : \mathcal{H} \to \mathcal{H}$ and an identity mapping $I : \mathcal{H} \to \mathcal{H}$ such that

$$T = (1 - \varpi)I + \varpi S.$$

(6) $\eta$-strongly monotone, if there exists $\eta > 0$ such that

$$\eta\|x - y\|^2 \leq \langle Tx - Ty, x - y\rangle.$$

(7) $\vartheta$-inverse strongly monotone, if there exists $\vartheta > 0$ such that

$$\vartheta\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle.$$

The study of the above mappings has appeared in many important literatures, for more detail, see [27, 28]. The following crucial properties deserve our attention.

**Remark 2.2.** (I) The classes of quasi-nonexpansive mappings and directed mappings are contained in the class of $k$-demicontractive mappings.

(II) The class of firmly nonexpansive mappings is included in the class of $k$-strictly pseudo-contractive mappings.

(III) $T$ is firmly nonexpansive mapping if and only if $T$ is $\frac{1}{2}$-averaged, i.e., $T = \frac{1}{2}(I + S)$.

(IV) If $T$ is $\vartheta$-inverse strongly monotone, then $\omega T$ is $\frac{\vartheta}{\omega}$-inverse strongly monotone for $\omega > 0$.

(V) If $T$ is averaged, then $T$ is nonexpansive.

**Definition 2.3.** Let $T$ be a mapping from $\mathcal{H}$ to $\mathcal{H}$ with $Fix(T) \neq \emptyset$. The complement $I - T$ is said to be demiclosed at zero if for an any sequence $\{x_n\}$ in $\mathcal{H}$ satisfying $x_n \rightharpoonup x^*$ and $(I - T)x_n \to 0$, then $x^* \in Fix(T)$.

**Lemma 2.4** ([25])**.** *Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero.*

**Lemma 2.5** ([15, 24])**.** *Let $T : C \to \mathcal{H}$ be a $k$-strictly pseudo-contractive mapping with $Fix(T) \neq \emptyset$. Then $I - T$ is demiclosed at 0 and $Fix(T)$ is closed and convex.*

**Lemma 2.6** ([22])**.** *Let $T : C \to \mathcal{H}$ be a $k$-demicontractive mapping. Then $Fix(T)$ is closed and convex.*

**Lemma 2.7** ([9])**.** *Let $T : \mathcal{H} \to \mathcal{H}$ be a $k$-demicontractive mapping and $T_\mu := (1 - \mu)I + \mu T$. For any $\mu \in (0, 1 - k)$,*

$$\|T_\mu x - x^*\|^2 \leq \|x - x^*\|^2 - \mu(1 - k - \mu)\|(I - T)x\|^2, \ \forall x \in \mathcal{H}, \ x^* \in Fix(T).$$

**Remark 2.8.** From Lemma 2.7, it is obvious that $T_\mu$ is nonexpansive and

$$x^* \in Fix(T) \Leftrightarrow x^* \in Fix(T_\mu).$$

**Lemma 2.9** ([4,17]). (I) *The composite of finitely many averaged mappings is averaged;*

(II) *If the mappings* $\{T_i\}_{i=1}^N$ *are averaged and have a nonempty common fixed point, then*

$$\bigcap_{i=1}^N Fix(T_i) = Fix\left(T_1 \cdots T_N\right);$$

(III) $T$ *is averaged* $\Leftrightarrow$ *its complement* $I - T$ *is* $\vartheta$*-inverse strongly monotone for some* $\vartheta > \frac{1}{2}$.

For any $x, y \in \mathcal{H}$, the following equalities hold.

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle \le \|x\|^2 + 2\langle y, x+y\rangle, \tag{2.2}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.3}$$

**Lemma 2.10** ( [8]). *Let* $T : \mathcal{H} \to \mathcal{H}$ *be a nonexpansive mapping. For any* $x, y \in \mathcal{H}$,

$$\langle (x - Tx) - (y - Ty), Ty - Tx\rangle \le \frac{1}{2}\|(x - Tx) - (y - Ty)\|^2$$

*and consequently if* $y \in Fix(T)$, *then*

$$\langle x - Tx, y - Tx\rangle \le \frac{1}{2}\|x - Tx\|^2.$$

**Definition 2.11** ([16]). Let $(\mathcal{X}, d)$ be a metric space. $f : \mathcal{X} \to \mathcal{X}$ is said to be a Meir-Keeler contraction mapping if for each $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$\varepsilon \le d(x, y) < \varepsilon + \delta \Rightarrow d(f(x), f(y)) < \varepsilon, \ \forall x, y \in \mathcal{X}.$$

**Remark 2.12.** It is clear that the class of Meir-Keeler contraction mappings contains the class of contraction mappings. Besides, more examples and generalizations of Meir-Keeler contraction mappings can be found in [20, 27, 28].

**Lemma 2.13** ( [20]). *Let* $C$ *be a closed convex subset of a Banach space* $\mathcal{B}$. $f : C \to C$ *is a Meir-Keeler contraction mapping if and only if for each* $\varepsilon > 0$, *there exists a number* $\delta \in (0, 1)$ *such that*

$$\|x - y\| \ge \varepsilon \Rightarrow \|f(x) - f(y)\| \le \delta\|x - y\|, \ \forall x, y \in C.$$

**Lemma 2.14** ([17]). *Let* $f : \mathcal{H} \to \mathcal{H}$ *be a mapping and* $B : \mathcal{H} \to 2^{\mathcal{H}}$ *be a maximal monotone mapping. The following properties hold.*

(I) $0 \in f(x^*) + B(x^*)$ *if and only if* $x^* = J_\gamma^B(I - \gamma f)x^*$, *i.e.,* $x^* \in Fix(J_\gamma^B(I - \gamma f))$, *for* $\gamma > 0$;

(II) *If* $f : \mathcal{H} \to \mathcal{H}$ *is* $\vartheta$*-inverse strongly monotone, then* $J_\gamma^B(I - \gamma f)$ *is average for* $\gamma \in (0, 2\vartheta)$.

**Remark 2.15.** From Lemma 2.6 and Lemma 2.14 (I), the solution set of the SMV-IFP is described in the following form

$$\Gamma = \{x^* \in \mathcal{H}_1 \mid x^* \in Fix(J_\gamma^{B_1}(I - \gamma f_1)) \cap Fix(T) \text{ and } Ax^* \in Fix(J_\gamma^{B_2}(I - \gamma f_2))\}$$

and it is closed and convex.

**Lemma 2.16** ( [10]). *Let $\{P_n\}$ and $\{c_n\}$ be two nonnegative real numbers sequences such that*

$$P_{n+1} \leq (1 - a_n)P_n + a_n b_n \quad \text{and} \quad P_{n+1} \leq P_n - c_n + d_n, n \geq 1,$$

*where $\{b_n\}$, $\{d_n\}$ and $\{a_n\}$ are real sequences with $0 < a_n < 1$. If $\sum_{n=1}^\infty a_n = \infty$, $\lim_{n\to\infty} d_n = 0$, and $\lim_{k\to\infty} c_{n_k} = 0$ implies $\limsup_{k\to\infty} b_{n_k} \leq 0$ ($\{n_k\}$ is any subsequence of real numbers of $\{n\}$). The sequence $\{P_n\}$ converges to $0$ as $n \to \infty$.*

## 3. Inertial algorithms and their convergence analysis

In this section, two algorithms are presented for finding the common solution of the split monotone variational inclusion and fixed point problem. The common advantage is that the strong convergence of the proposed algorithms are guaranteed under reasonable constraints. Until then, suppose that the solution set $\Gamma$ is nonempty and the following assumptions are satisfied:

- (A1) $\mathcal{H}_1$, $\mathcal{H}_2$ are two real Hilbert spaces and $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator with the adjoint operator $A^*$;
- (A2) $f_1 : \mathcal{H}_1 \to \mathcal{H}_1$ is an $\vartheta_1$-inverse strongly monotone mapping and $f_2 : \mathcal{H}_2 \to \mathcal{H}_2$ is an $\vartheta_2$-inverse strongly monotone mapping;
- (A3) $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ are two set-valued maximal monotone mappings;
- (A4) $f : \mathcal{H}_1 \to \mathcal{H}_1$ is a Meir-Keeler contraction mapping;
- (A5) $T : \mathcal{H}_1 \to \mathcal{H}_1$ is a $k$-demicontractive mapping and $I - T$ is demiclosed at 0.

For convenience, set $K_1 = J_\gamma^{B_1}(I - \gamma f_1)$ and $K_2 = J_\gamma^{B_2}(I - \gamma f_2)$, then $\Gamma = \{x^* \in \mathcal{H}_1 \mid x^* \in Fix(K_1) \cap Fix(T) \text{ and } Ax^* \in Fix(K_2)\}$ for $\gamma > 0$. Under the premise that Conditions (A1)-(A5) hold, two inertial algorithms employing a Meir-Keeler contraction mapping and an adaptive step size selection are presented. Some important lemmas will be given and can be used to show the convergence of the proposed algorithms.

**Algorithm 1**

**Initialization:** Put $\alpha_n \in [0, \alpha] \subset [0, 1)$, $\sigma_n \in [a, b] \subset (0, 1)$, $\theta_n \in (0, 1)$, $\mu \in (0, 1-k)$ and any initial guesses $x_0, x_1 \in \mathcal{H}_1$.
Compute $x_{n+1}$ by the following way:

$$\begin{cases} u_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = K_1(u_n - \lambda_n A^*(I - K_2)Au_n), \\ x_{n+1} = \theta_n f(y_n) + (1-\theta_n)((1-\mu)I + \mu T)y_n, n \geq 1, \end{cases} \tag{3.1}$$

where

$$\lambda_n = \sigma_n \tau_n \text{ and } \tau_n := \begin{cases} \dfrac{\|(I - K_2)Au_n\|^2}{\|A^*(I - K_2)Au_n\|^2}, & Au_n \notin Fix(K_2), \\ 0, & \text{otherwise}, \end{cases} \tag{3.2}$$

and the following control conditions are met:
**(C1):** $\lim_{n\to\infty} \theta_n = 0$ and $\sum_{n=1}^{\infty} \theta_n = \infty$;
**(C2):** $\lim_{n\to\infty} \dfrac{\alpha_n}{\theta_n}\|x_n - x_{n-1}\| = 0$;
**(C3):** $0 < \gamma < 2\vartheta$ with $\vartheta = \min\{\vartheta_1, \vartheta_2\}$.

**Algorithm 2**

**Initialzation:** Put $\alpha_n \in [0, \alpha] \subset [0, 1)$, $\sigma_n \in [a, b] \subset (0, 1)$, $\theta_n \in (0, 1)$, $\mu \in (0, 1-k)$ and any initial guesses $x_0, x_1 \in \mathcal{H}_1$.
Compute $x_{n+1}$ by the following way:

$$\begin{cases} u_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = u_n - \lambda_n[(I - K_1)u_n + A^*(I - K_2)Au_n], \\ x_{n+1} = \theta_n f(y_n) + (1-\theta_n)((1-\mu)I + \mu T)y_n, n \geq 1, \end{cases} \tag{3.3}$$

where

$$\lambda_n = \sigma_n \tau_n \text{ and } \tau_n := \begin{cases} \min\left\{\dfrac{1}{2}, \dfrac{\|(I - K_2)Au_n\|^2}{2\|A^*(I - K_2)Au_n\|^2}\right\}, & Au_n \notin Fix(K_2), \\ \dfrac{1}{2}, & \text{otherwise}, \end{cases} \tag{3.4}$$

and Conditions (C1)-(C3) are met.

**Lemma 3.1.** *The step size sequence* $\{\lambda_n\}$ *defined by* (3.2) *or* (3.4) *is well-defined.*

*Proof.* From Lemma 2.9 (III) and Lemma 2.14 (II), we have that $K_2$ is average and $I - K_2$ is $\tilde{\vartheta}_2$-inverse strongly monotone for $\tilde{\vartheta}_2 > 1/2$. For any $x^* \in \Gamma$, we have $x^* \in Fix(K_1) \bigcap Fix(T)$ and $Ax^* \in Fix(K_2)$. Further, we get

$$\begin{aligned} \|A^*(I - K_2)Au_n\|\|u_n - x^*\| &\geq \langle A^*(I - K_2)Au_n, u_n - x^*\rangle \\ &= \langle (I - K_2)Au_n, Au_n - Ax^*\rangle \\ &\geq \tilde{\vartheta}_2\|(I - K_2)Au_n\|^2. \end{aligned}$$

If $Au_n \notin Fix(K_2)$, we get $\|A^*(I-K_2)Au_n\| > 0$. This means that $\{\lambda_n\}$ generated by (3.2) and (3.4) is well-defined. □

## 3.1. **The convergence analysis of Algorithm 1.**

**Lemma 3.2.** *For any $x^* \in \Gamma$, the results below hold in Algorithm 1:*

(E1) $\|y_n - x^*\|^2 \leq \|u_n - x^*\|^2 - \lambda_n(1-\sigma_n)\|(I-K_2)Au_n\|^2$ *for some* $Au_n \notin Fix(K_2)$.

(E2) *The sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are bounded.*

*Proof.* **Proof of *(E1)*.** For any $x^* \in \Gamma$, we have $x^* \in Fix(K_1)\bigcap Fix(T)$ and $Ax^* \in Fix(K_2)$. Since $K_1$ and $K_2$ are averaged in Lemma 2.14 (II), then are also nonexpansive. Hence, by Lemma 2.10, we get

$$\begin{aligned} 2\lambda_n\langle (I-K_2)Au_n, Au_n - Ax^*\rangle &= 2\lambda_n\langle (I-K_2)Au_n, K_2Au_n - Ax^*\rangle \\ &\quad + 2\lambda_n\|(I-K_2)Au_n\|^2 \\ &\geq -\lambda_n\|(I-K_2)Au_n\|^2 \\ &\quad + 2\lambda_n\|(I-K_2)Au_n\|^2 \\ &= \lambda_n\|(I-K_2)Au_n\|^2. \end{aligned} \tag{3.5}$$

Further, from (3.5) and the definition of $\lambda_n$ in (3.2), we have

$$\begin{aligned} \|y_n - x^*\|^2 &\leq \|u_n - \lambda_n A^*(I-K_2)Au_n - x^*\|^2 \\ &= \|u_n - x^*\|^2 - 2\lambda_n\langle A^*(I-K_2)Au_n, u_n - x^*\rangle \\ &\quad + \lambda_n^2\|A^*(I-K_2)Au_n\|^2 \\ &= \|u_n - x^*\|^2 - 2\lambda_n\langle (I-K_2)Au_n, Au_n - Ax^*\rangle \\ &\quad + \lambda_n^2\|A^*(I-K_2)Au_n\|^2 \\ &\leq \|u_n - x^*\|^2 - \lambda_n(1-\sigma_n)\|(I-K_2)Au_n\|^2. \end{aligned} \tag{3.6}$$

**Proof of *(E2)*.** If for any $\varepsilon > 0$, $\|x_n - x^*\| \leq \varepsilon$, then $\{x_n\}$ is a bounded sequence. On the contrary, $\|x_n - x^*\| \geq \varepsilon$, there exists a number $\delta \in (0,1)$ by Lemma 2.13 such that $\|f(x_n) - f(x^*)\| \leq \delta\|x_n - x^*\|$. Moreover, from (3.6) and the definition of $\lambda_n$ in (3.2), we get $\lambda_n(1-\sigma_n)\|(I-K_2)Au_n\|^2 \geq 0$, which means that $\|y_n - x^*\| \leq \|u_n - x^*\|$. This inequality also satisfies when $Au_n$ belongs to $Fix(K_2)$. It follows from Remark 2.8 that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \theta_n\|f(y_n) - f(x^*)\| + \theta_n\|f(x^*) - x^*\| \\ &\quad + (1-\theta_n)\|((1-\mu)I + \mu T)y_n - x^*\| \\ &\leq (1-\theta_n(1-\delta))\|y_n - x^*\| + \theta_n\|f(x^*) - x^*\| \\ &\leq (1-\theta_n(1-\delta))\|x_n - x^*\| + \theta_n\|f(x^*) - x^*\| \\ &\quad + (1-\theta_n(1-\delta))\alpha_n\|x_n - x_{n-1}\| \\ &\leq (1-\theta_n(1-\delta))\|x_n - x^*\| \\ &\quad + \theta_n(1-\delta)\left(\frac{\|f(x^*) - x^*\|}{1-\delta} + \frac{\alpha_n\|x_n - x_{n-1}\|}{\theta_n(1-\delta)}\right). \end{aligned}$$

In view of $\delta \in (0,1)$ and Conditions (C1) and (C2), we obtain $\lim_{n\to\infty} \frac{\alpha_n\|x_n-x_{n-1}\|}{\theta_n(1-\delta)} = 0$. Therefore, there exists a non-negative constant $M$ such that

$$M/2 = \max\left\{\frac{\|f(x^*)-x^*\|}{1-\delta}, \frac{\alpha_n\|x_n-x_{n-1}\|}{\theta_n(1-\delta)}\right\}.$$

Hence,

$$\|x_{n+1}-x^*\| \le \max\{\|x_n-x^*\|, M\} \le \cdots \le \max\{\|x_1-x^*\|, M\}.$$

This implies that $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{u_n\}$. □

**Theorem 3.3.** *The sequence $\{x_n\}$ generated by Algorithm 1 converges in norm a point $x^* = P_\Gamma \circ f(x^*)$, i.e.,*

$$\langle f(x^*)-x^*, x-x^*\rangle \le 0, \ \forall x \in \Gamma.$$

*Proof.* It is obvious that the solution set $\Gamma$ is closed and convex, that is, $P_\Gamma$ is well-defined. Since $K_1$ is averaged, there exists a constant $\varpi \in (0,1)$ and a nonexpansive mapping $S$ such that $K_1 = (1-\varpi)I+\varpi S$. First of all, suppose that $Au_n \notin Fix(K_2)$ for any $n \ge 1$ and $\Delta_n = u_n - \lambda_n A^*(I-K_2)Au_n$, it follows from (2.3) and (3.6) that

$$\begin{aligned}\|y_n-x^*\|^2 &= \|(1-\varpi)\Delta_n + \varpi S\Delta_n - x^*\|^2 \\ &\le \|\Delta_n-x^*\|^2 - 2\varpi(1-\varpi)\|(I-S)\Delta_n\|^2 \\ &\le \|u_n-x^*\|^2 - \lambda_n(1-\sigma_n)\|(I-K_2)Au_n\|^2 \\ &\quad - 2\varpi(1-\varpi)\|(I-S)\Delta_n\|^2.\end{aligned} \tag{3.7}$$

Let $H_n = \lambda_n(1-\sigma_n)\|(I-K_2)Au_n\|^2 + 2\varpi(1-\varpi)\|(I-S)\Delta_n\|^2$. Using (2.2), we get

$$\begin{aligned}\|u_n-x^*\|^2 &\le \|x_n-x^*\|^2 + 2\alpha_n\langle u_n-x^*, x_n-x_{n-1}\rangle \\ &\le \|x_n-x^*\|^2 + 2\alpha_n\|u_n-x^*\|\|x_n-x_{n-1}\|.\end{aligned} \tag{3.8}$$

Further, from (3.7), (3.8) and Lemma 2.7, we can obtain

$$\begin{aligned}&\|\theta_n(f(y_n)-f(x^*)) + (1-\theta_n)(((1-\mu)I+\mu T)y_n - x^*)\|^2 \\ &\quad\le \theta_n\|f(y_n)-f(x^*)\|^2 + (1-\theta_n)\|((1-\mu)I+\mu T)y_n - x^*\|^2 \\ &\quad\le (1-\theta_n(1-\delta^2))\|y_n-x^*\|^2 - \mu(1-\theta_n)(1-k-\mu)\|(I-T)y_n\|^2 \\ &\quad\le (1-\theta_n(1-\delta))\|x_n-x^*\|^2 + 2(1-\theta_n(1-\delta))\alpha_n\|u_n-x^*\|\|x_n-x_{n-1}\| \\ &\qquad - (1-\theta_n(1-\delta))H_n - \mu(1-\theta_n)(1-k-\mu)\|(I-T)y_n\|^2.\end{aligned} \tag{3.9}$$

By (2.2) and (3.9), we get

$$\begin{aligned}&\|x_{n+1}-x^*\|^2 \\ &\quad\le \|\theta_n(f(y_n)-f(x^*)) + (1-\theta_n)(((1-\mu)I+\mu T)y_n - x^*)\|^2 \\ &\qquad + 2\theta_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle \\ &\quad\le (1-\theta_n(1-\delta))\|x_n-x^*\|^2 + 2(1-\theta_n(1-\delta))\alpha_n\|u_n-x^*\|\|x_n-x_{n-1}\| \\ &\qquad + 2\theta_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle - (1-\theta_n(1-\delta))H_n \\ &\qquad - \mu(1-\theta_n)(1-k-\mu)\|(I-T)y_n\|^2.\end{aligned}$$

Set $\Delta_n = \|x_n - x^*\|^2$, $a_n = \theta_n(1-\delta)$, $c_n = (1-a_n)H_n + \mu(1-\theta_n)(1-k-\mu)\|(I-T)y_n\|^2$, $d_n = 2(1-a_n)\alpha_n\|u_n - x^*\|\|x_n - x_{n-1}\| + 2\theta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle$ and $b_n = d_n/a_n$. It is easy to check that

$$\Delta_{n+1} \le (1-a_n)\Delta_n + a_n b_n \text{ and } \Delta_{n+1} \le \Delta_n - c_n + d_n,\ n \ge 1.$$

By $\delta \in (0,1)$, Conditions (C1) and (C2) and the boundedness of $\{x_n\}$ and $\{u_n\}$, the following observations are satisfied:

$$\{a_n\} \subset (0,1),\ \sum_{n=1}^{\infty} a_n = \infty \text{ and } \lim_{n\to\infty} d_n = 0. \tag{3.10}$$

In addition, suppose that $\lim_{l\to\infty} c_{n_l} = 0$ for any subsequence $\{n_l\}$ of $\{n\}$, we have

$$\lim_{l\to\infty} H_{n_l} = \lim_{l\to\infty} \|(I-T)y_{n_l}\| = 0.$$

Using the definition of $\lambda_n$ and $H_n$ we get $\lim_{l\to\infty}\|(I-K_1)u_{n_l}\| = \lim_{l\to\infty}\|(I-S)\Delta_{n_l}\| = 0$. Besides,

$$\begin{aligned}\|y_{n_l} - u_{n_k}\| &\le \|y_{n_l} - \Delta_{n_l}\| + \|\Delta_{n_l} - u_{n_l}\|\\ &\le \kappa\|(I-S)\Delta_{n_l}\| + \lambda_{n_l}\|A\|\|(I-K_2)Au_{n_l}\| \to 0, \text{ as } l\to\infty,\end{aligned}$$

and

$$\begin{aligned}\|u_{n_l} - K_1u_{n_l}\| &\le \|u_{n_l} - y_{n_l}\| + \|y_{n_l} - K_1u_{n_l}\|\\ &\le \|u_{n_l} - y_{n_l}\| + \lambda_{n_l}\|A\|\|(I-K_2)Au_{n_l}\| \to 0, \text{ as } l\to\infty.\end{aligned}$$

Furthermore, $\lim_{l\to\infty}\|u_{n_l} - x_{n_l}\| = \lim_{l\to\infty}\alpha_{n_l}\|x_{n_l} - x_{n_l-1}\| = 0$. Since the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_{l_j}}\}$ of $\{x_{n_l}\}$ such that $x_{n_{l_j}} \rightharpoonup \bar{x}$ and $\limsup_{l\to\infty}\langle f(x^*) - x^*, x_{n_l} - x^*\rangle = \lim_{j\to\infty}\langle f(x^*) - x^*, x_{n_{l_j}} - x^*\rangle$. Then, $u_{n_{l_j}} \rightharpoonup \bar{x}$, $y_{n_{l_j}} \rightharpoonup \bar{x}$ and $Au_{n_{l_j}} \rightharpoonup A\bar{x}$ by the linearity of $A$. Since $I-T$ is demiclosed at 0 and $K_1$, $K_2$ are nonexpansive, we get $\bar{x} \in Fix(K_1)\bigcap Fix(T)$ and $A\bar{x} \in Fix(K_2)$ by Definition 2.3 and Lemma 2.4. More precisely, $\bar{x} \in \Gamma$ and $\lim_{j\to\infty}\langle f(x^*) - x^*, x_{n_{l_j}} - x^*\rangle = \langle f(x^*) - x^*, \bar{x} - x^*\rangle \le 0$ by (2.1). On the other hand,

$$\begin{aligned}\|x_{n_l+1} - y_{n_l}\| &\le \theta_{n_l}\|f(y_{n_l}) - y_{n_l}\| + (1-\theta_{n_l})\|((1-\mu)I + \mu T)y_{n_l} - y_{n_l}\|\\ &= \theta_{n_l}\|f(y_{n_l}) - y_{n_l}\| + (1-\theta_{n_l})\mu\|(I-T)y_{n_l}\| \to 0, \text{ as } l\to\infty,\end{aligned}$$

and

$$\|x_{n_l+1} - x_{n_l}\| \le \|x_{n_l+1} - y_{n_l}\| + \|y_{n_l} - u_{n_l}\| + \|u_{n_l} - x_{n_l}\| \to 0, \text{ as } l\to\infty.$$

So, we have

$$\limsup_{l\to\infty}\langle f(x^*) - x^*, x_{n_l+1} - x^*\rangle \le 0 \text{ and } \lim_{n\to\infty}\frac{\alpha_n\|u_n - x^*\|\|x_n - x_{n-1}\|}{\theta_n(1-\delta)} = 0,$$

which shows that $\limsup_{l\to\infty} b_{n_l} \le 0$. Thus, $\|x_n - x^*\| \to 0$ as $n\to\infty$ by Lemma 2.16, that is to say, $\{x_n\}$ converges in norm to $x^* = P_\Gamma \circ f(x^*)$. In the case that $Au_n$ belongs to $Fix(K_2)$, the same result can be obtained by using the above method. This completes the proof. □

## 3.2. The convergence analysis of Algorithm 2.

**Lemma 3.4.** *For any $x^* \in \Gamma$, the following conclusions hold in Algorithm 2:*

(L1) $\|y_n - x^*\|^2 \le \|u_n - x^*\|^2 - W_n$ *for some* $Au_n \notin Fix(K_2)$, *where*

$$W_n = (\lambda_n - 2\lambda_n^2)\|(I-K_1)u_n\|^2 + \lambda_n\|(I-K_2)Au_n\|^2 - 2\lambda_n^2\|A^*(I-K_2)Au_n\|^2.$$

(L2) *The sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *are bounded.*

(L3) *Assume that* $a_n = \theta_n(1-\delta)$, $d_n = 2(1-a_n)\alpha_n\|u_n - x^*\|\|x_n - x_{n-1}\| + 2\theta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle$ *and* $c_n = (1-a_n)W_n + \mu(1-\theta_n)(1-k-\mu)\|(I-T)y_n\|^2$, *we have*

$$\|x_{n+1} - x^*\|^2 \le (1-a_n)\|x_n - x^*\|^2 - c_n + d_n.$$

*Proof. Proof of* (L1). Similarly, for any $x^* \in \Gamma$, using the same way in (3.5), we have

$$2\langle Au_n - Ax^*, (I-K_2)Au_n\rangle \ge \|(I-K_2)Au_n\|^2$$

and $2\langle u_n - x^*, (I-K_1)u_n\rangle \ge \|(I-K_1)u_n\|^2$. So

$$\begin{aligned} &2\langle u_n - x^*, (I-K_1)u_n + A^*(I-K_2)Au_n\rangle \\ &= 2\langle u_n - x^*, (I-K_1)u_n\rangle + 2\langle Au_n - Ax^*, (I-K_2)Au_n\rangle \\ &\ge \|(I-K_1)u_n\|^2 + \|(I-K_2)Au_n\|^2, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \|y_n - x^*\|^2 &= \|u_n - x^*\|^2 + \lambda_n^2\|(I-K_1)u_n + A^*(I-K_2)Au_n\|^2 \\ &\quad - 2\lambda_n\langle u_n - x^*, (I-K_1)u_n + A^*(I-K_2)Au_n\rangle \\ &\le \|u_n - x^*\|^2 + (2\lambda_n^2 - \lambda_n)\|(I-K_1)u_n\|^2 \\ &\quad + 2\lambda_n^2\|A^*(I-K_2)Au_n\|^2 - \lambda_n\|(I-K_2)Au_n\|^2. \end{aligned} \tag{3.12}$$

*Proof of* (L2). From the definition of $\lambda_n$ in (3.4), we get

$(\lambda_n - 2\lambda_n^2)\|(I-K_1)u_n\|^2 \ge 0$ and $\lambda_n\|(I-K_2)Au_n\|^2 - 2\lambda_n^2\|A^*(I-K_2)Au_n\|^2 \ge 0$,

which means that $\|y_n - x^*\| \le \|u_n - x^*\|$. Using the same method as in Lemma 3.2 (E2), we also obtain that $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{u_n\}$.

*Proof of* (L3). Combining (3.8) and (3.12), we get

$$\|y_n - x^*\|^2 \le \|x_n - x^*\|^2 + 2\alpha_n\|u_n - x^*\|\|x_n - x_{n-1}\| - W_n. \tag{3.13}$$

Applying the same method as in Theorem 3.3 to (3.13), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le (1-\theta_n(1-\delta))\|x_n - x^*\|^2 + 2\theta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\quad - (1-\theta_n(1-\delta))W_n - \mu(1-\theta_n)(1-k-\mu)\|(I-T)y_n\|^2 \\ &\quad + 2(1-\theta_n(1-\delta))\alpha_n\|u_n - x^*\|\|x_n - x_{n-1}\| \\ &= (1-a_n)\|x_n - x^*\|^2 - c_n + d_n. \end{aligned}$$

The proof is completed. □

**Theorem 3.5.** *The sequence* $\{x_n\}$ *formed by Algorithm 2 converges in norm a point* $x^* = P_\Gamma \circ f(x^*)$, *i.e.,*

$$\langle f(x^*) - x^*, x - x^*\rangle \le 0, \ \forall x \in \Gamma.$$

*Proof.* Let $\Delta_n = \|x_n - x^*\|^2$ and $b_n = d_n/a_n$. It follows from Lemma 3.4 (L3) that

$$\Delta_{n+1} \le (1-a_n)\Delta_n + a_n b_n \text{ and } \Delta_{n+1} \le \Delta_n - c_n + d_n,\ n \ge 1.$$

Naturally, $\{a_n\}$ and $\{d_n\}$ also satisfy (3.10). In the same way, assume that $\lim_{l\to\infty} c_{n_l} = 0$ for any subsequence $\{n_l\}$ of $\{n\}$, we have

$$\lim_{l\to\infty} W_{n_l} = \lim_{l\to\infty} \|(I-T)y_{n_l}\| = 0.$$

Using the definition of $\lambda_n$ and $W_n$, we get

$$\lim_{l\to\infty} \|(I-K_1)u_{n_l}\| = \lim_{l\to\infty} \|(I-K_2)Au_{n_l}\| = 0.$$

Furthermore,

$$\lim_{l\to\infty} \|y_{n_l} - u_{n_l}\| \le \lim_{l\to\infty} \lambda_{n_l}(\|(I-K_1)u_{n_l}\| + \|A\|\|(I-K_2)Au_{n_l}\|) = 0$$

and $\lim_{l\to\infty} \|u_{n_l} - x_{n_l}\| = \lim_{l\to\infty} \alpha_{n_l}\|x_{n_l} - x_{n_l-1}\| = 0$. Combining the above results with the proof in Theorem 3.3, it can also be obtained that the sequence $\{x_n\}$ converges in norm to a point $x^* = P_\Gamma \circ f(x^*)$. This completes the proof. □

According to Remark 2.12, the following propositions are obtained by Theorems 3.3 and 3.5.

**Proposition 3.6.** *Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ be a contraction mapping in Algorithms 1 and 2. The sequence $\{x_n\}$ converges in norm to $x^* = P_\Gamma \circ f(x^*)$, i.e.,*

$$\langle f(x^*) - x^*, x - x^*\rangle \le 0,\ \forall x \in \Gamma.$$

Further, if the mapping $f$ is a zero mapping, Algorithms 1 and 2 can be reduced to

$$\begin{cases} u_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = K_1(u_n - \lambda_n A^*(I-K_2)Au_n), \\ x_{n+1} = (1-\theta_n)((1-\mu)I + \mu T)y_n, n \ge 1, \end{cases} \tag{3.14}$$

and

$$\begin{cases} u_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = u_n - \lambda_n[(I-K_1)u_n + A^*(I-K_2)Au_n], \\ x_{n+1} = (1-\theta_n)((1-\mu)I + \mu T)y_n, n \ge 1, \end{cases} \tag{3.15}$$

respectively. Here $\{\alpha_n\}$, $\{\sigma_n\}$, $\{\theta_n\}$ and $\mu$ are defined as in Algorithm 1, Conditions (C1)-(C3) are satisfied, and the adaptive step size $\lambda_n$ in (3.14) and (3.15) is selected as in (3.2) and (3.4), respectively.

**Proposition 3.7.** *Assume that (A1)-(A3) and (A5) hold. The sequence $\{x_n\}$ generated by the algorithms* (3.14) *and* (3.15) *converges in norm to a point $x^* = P_\Gamma(0)$, i.e., the minimum-norm element of $\Gamma$.*

## 4. Some important corollaries and remarks

In what follows, some important corollaries arise from Theorems 3.3 and 3.5 are given and these results are easily verified using the previous proof procedure and related lemmas. First of all, the SMVIP is a special case that the mapping $T$ is a identity mapping in the SMVIFP, it is easy to obtain the following corollaries.

**Corollary 4.1.** *Assume that (A1)-(A4) hold. For any $x_0,\ x_1 \in \mathcal{H}_1$, the sequence $\{x_n\}$ is defined by the following algorithm*

$$\begin{cases} u_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = K_1(u_n - \lambda_n A^*(I - K_2)Au_n), \\ x_{n+1} = \theta_n f(y_n) + (1-\theta_n)y_n,\ n \geq 1, \end{cases}$$

*where $\lambda_n$ is defined as* (3.2) *and Conditions (C1)-(C3) are satisfied. Suppose that the solution set of the split monotone variational inclusion problem is nonempty and is represented by $\Psi$. Then the sequence $\{x_n\}$ converges in norm to $x^* = P_\Psi \circ f(x^*)$, i.e., $\langle f(x^*) - x^*, x - x^*\rangle \leq 0,\ \forall x \in \Psi$.*

**Corollary 4.2.** *Assume that (A1)-(A4) hold. For any $x_0,\ x_1 \in \mathcal{H}_1$, the sequence $\{x_n\}$ is defined by the following algorithm*

$$\begin{cases} u_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = u_n - \lambda_n[(I - K_1)u_n + A^*(I - K_2)Au_n], \\ x_{n+1} = \theta_n f(y_n) + (1-\theta_n)y_n,\ n \geq 1, \end{cases}$$

*where $\lambda_n$ is defined as* (3.4) *and Conditions (C1)-(C3) are satisfied. Suppose that the solution set $\Psi$ of the SMVIP is nonempty. Then the sequence $\{x_n\}$ converges in norm to $x^* = P_\Psi \circ f(x^*)$, i.e., $\langle f(x^*) - x^*, x - x^*\rangle \leq 0,\ \forall x \in \Psi$.*

For further discussion, the SMVIP also generalizes the split variational inclusion problem, the split variational inequality problem and the split feasibility problem. In the same way, the main results in Theorems 3.3 and 3.5 can be applied to these problems. These results also promote some existing work, such as Moudafi [17], Byrne et al. [5], Censor et al. [7], Long et al. [21] and Anh et al. [2]. Moreover, all applications of these problems are also covered in the split monotone variational inclusion problem. In addition, there are some special cases about the demicontractive mapping that should be noted:

**Case 1:** The $k$-strictly pseudo-contractive mapping with nonempty fixed point sets is the $k$-demicontractive mapping and the complement of the $k$-strictly pseudo-contractive mapping is demiclosed at 0 since Lemma 2.5.

**Case 2:** The directed mapping is the $-1$-demicontractive mapping and includes the firmly nonexpansive mapping with nonempty fixed point sets.

**Case 3:** The quasi-nonexpansive mapping is the 0-demicontractive mapping and includes the nonexpansive mapping with nonempty fixed point sets.

Hence, when $T$ is one of the above cases in Algorithms 1 and 2, the strong convergence of iterative sequence $\{x_n\}$ is still true. That is to say, Theorems 3.3 and 3.5 generalize the known results in Shehu and Ogbuisi [19] and Kazmi and Rizvi [12].

The following remarks are some interesting explanations and considerations of the inertial coefficient $\alpha_n$ and adaptive step size $\lambda_n$ in our algorithms.

**Remark 4.3.** (I) Since the norm of $(x_n - x_{n-1})$ is known in each iteration, the above conditional constraints on the inertial extrapolation term are easy to achieve in practical applications. Further, the coefficient $\alpha_n$ can be considered in the following form:

$$\alpha_n := \begin{cases} \min\left\{\alpha, \dfrac{\beta_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise}, \end{cases} \tag{4.1}$$

where $\alpha \in [0,1)$ and $\lim_{n\to\infty} \frac{\beta_n}{\theta_n} = 0$. For more detail, see [27].
(II) In particular, when $\alpha_n = 0$, the proposed algorithms become the case without inertial extrapolation term, that is, $u_n = x_n$ in (3.1) and (3.3). But, our results in Theorems 3.3 and 3.5 are still correct.

**Remark 4.4.** By virtue of (3.11), we have modified (3.12) as follows:

$$\begin{aligned} \|y_n - x^*\|^2 &= \|u_n - x^*\|^2 + \lambda_n^2 \|(I-K_1)u_n + A^*(I-K_2)Au_n\|^2 \\ &\quad - 2\lambda_n \langle u_n - x^*, (I-K_1)u_n + A^*(I-K_2)Au_n \rangle \\ &\le \|u_n - x^*\|^2 + \lambda_n^2 \|(I-K_1)u_n + A^*(I-K_2)Au_n\|^2 \\ &\quad - \lambda_n \|(I-K_1)u_n\|^2 - \lambda_n \|(I-K_2)Au_n\|^2. \end{aligned}$$

In the same way, set adaptive step size

$$\lambda_n = \sigma_n \tau_n \text{ and } \tau_n := \frac{\|(I-K_1)u_n\|^2 + \|(I-K_2)Au_n\|^2}{\|(I-K_1)u_n + A^*(I-K_2)Au_n\|^2}. \tag{4.2}$$

When (4.2) replaces (3.4) in Algorithms 2, the strong convergence of $\{x_n\}$ is still satisfied.

## 5. Numerical examples

In this section, we provide some numerical examples to demonstrate the effectiveness and realization of Algorithms 1 and 2. All the programs were implemented in Matlab 2018a on a Intel(R) Core(TM) i5-8250U CPU @1.60 GHz computer with RAM 8.00 GB. Our results compare the existing conclusions below. Firstly, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real Hilbert spaces and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with the adjoint operator $A^*$. Let $f_1 : \mathcal{H}_1 \to \mathcal{H}_1$ be a $\vartheta_1$-inverse strongly monotone mapping and $f_2 : \mathcal{H}_2 \to \mathcal{H}_2$ be a $\vartheta_2$-inverse strongly monotone mapping. Let $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be set-valued maximal monotone mappings.

**Theorem 5.1** ( [19]). *Let $T : \mathcal{H}_1 \to \mathcal{H}_1$ be a $k$-strictly pseudo-contractive mapping and $\Gamma \neq \emptyset$. For any $x_1 \in \mathcal{H}_1$, $\{x_n\}$ is generated by the following process*

$$\begin{cases} w_n = (1-\theta_n)\, x_n, \\ y_n = J_\gamma^{B_1}(I - \gamma f_1)\left(w_n + \lambda A^*\left(J_\gamma^{B_2}(I-\gamma f_2) - I\right) A w_n\right), \\ x_{n+1} = (1-\mu_n)\, y_n + \mu_n T y_n, \end{cases} \tag{SO}$$

*where $0 < \gamma < 2\vartheta_1$, $2\vartheta_2$ and $\lambda \in (0, 1/L)$, $L$ is the spectral radius of $AA^*$. Suppose that $\{\theta_n\}$ and $\{\mu_n\}$ are real sequences in $(0,1)$ satisfying $\lim_{n\to\infty} \theta_n = 0, \sum_{n=1}^{\infty} \theta_n = \infty$ and $0 < \liminf \mu_n \le \limsup \mu_n < 1-k$. Then $\{x_n\}$ converges in norm to $x^* \in \Gamma$.*

**Example 5.2.** Assume that $A, A_1, A_2 : \mathbb{R}^m \to \mathbb{R}^m$ are created from a normal distribution with mean zero and unit variance. Let $B_1 : \mathbb{R}^m \to \mathbb{R}^m$ and $B_2 : \mathbb{R}^m \to \mathbb{R}^m$ be defined by $B_1(x) = A_1^* A_1 x$ and $B_2(y) = A_2^* A_2 y$, respectively. Consider the problem of finding a point $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)^\mathsf{T} \in \mathbb{R}^m$ such that $B_1(\bar{x}) = (0, \dots, 0)^\mathsf{T}$ and $B_2(A\bar{x}) = (0, \dots, 0)^\mathsf{T}$. Let mapping $T$ be defined by $Tx = 0.5x$. Obviously, $x^* = (0, \dots, 0)^\mathsf{T}$is a solution of this problem. In this example, we compare the presented algorithms with Algorithm (SO) introduced by Shehu and Ogbuisi [19]. In the proposed Algorithms 1 and 2, take $f_1 = f_2 = 0$, $\gamma = 1$, $\theta_n = 1/(n+1)$, $\sigma_n = 0.5$, $\mu = 0.9$, $f(x) = 0.1x$, and $\alpha_n$ is defined as (4.1) with $\alpha = 0.5$ and $\beta_n = 1/(n+1)^2$. Choose $f_1 = f_2 = 0$, $\gamma = 1$, $\theta_n = 1/(n+1)$, $\lambda = 0.5/\|A^*A\|$ and $\mu_n = 0.9$ for Algorithm (SO). The start points with the initial values $x_0 = x_1 = 5rand(m, 1)$. The stopping condition is $D_n = \|x_n - x^*\| < 10^{-5}$. Table 1 shows that the numerical results of all algorithms with four different dimensions.

TABLE 1. Numerical results for Example 5.2

| Algorithms | $m = 100$ | | $m = 200$ | | $m = 500$ | | $m = 1000$ | |
|---|---|---|---|---|---|---|---|---|
| | Iter. | Time ($s$) | Iter. | Time ($s$) | Iter. | Time ($s$) | Iter. | Time ($s$) |
| Our Alg. 1 | 15 | 0.0199 | 16 | 0.0428 | 16 | 0.2158 | 16 | 1.3006 |
| Our Alg. 2 | 19 | 0.0212 | 20 | 0.0540 | 20 | 0.2633 | 20 | 1.5697 |
| Alg. (SO) | 23 | 0.0408 | 25 | 0.1343 | 24 | 0.8310 | 25 | 6.9339 |

**Example 5.3.** Assume that $\mathcal{H}_1 = \mathcal{H}_2 = L^2([0,1])$ with the inner product $\langle x, y \rangle := \int_0^1 x(t)y(t)\,\mathrm{d}t$ and the induced norm $\|x\| := \big(\int_0^1 |x(t)|^2\,\mathrm{d}t\big)^{1/2}$, for any $x, y \in L^2([0,1])$. Consider the following nonempty closed and convex subsets $C_1$ and $Q_1$ in $L^2([0,1])$:

$$C_1 := \Big\{ x \in L_2([0,1]) \mid \int_0^1 x(t)\,\mathrm{d}t \le 1 \Big\}$$

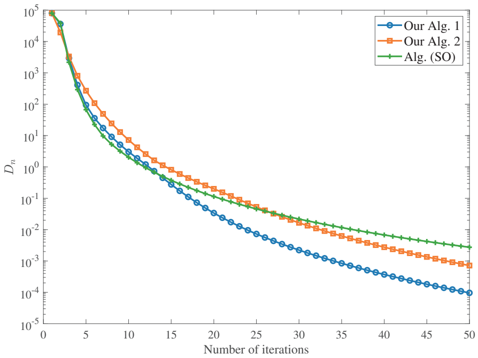
and

$$Q_1 := \Big\{ y \in L_2([0,1]) \mid \int_0^1 |y(t) - \sin(t)|^2\,\mathrm{d}t \le 16 \Big\}.$$

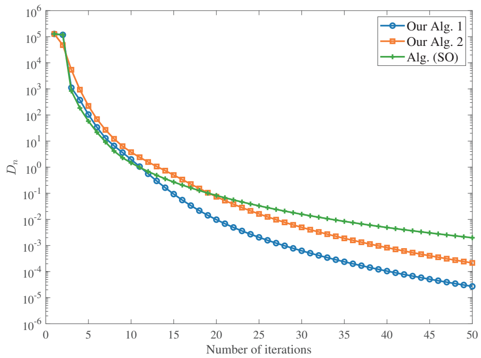
Suppose that $A : L^2([0,1]) \to L^2([0,1])$ is the Volterra integration operator, and is defined by $(Ax)(t) = \int_0^t x(s)\,\mathrm{d}s$, $\forall t \in [0,1]$, $x \in \mathcal{H}_1$. So, $A$ is a bounded linear operator with the norm $\|A\| = 2/\pi$. Moreover, the adjoint operator $A^*$ of $A$ is defined by $(A^*x)(t) = \int_t^1 x(s)\,\mathrm{d}s$. Meanwhile, its projections on sets $C_1$ and $Q_1$ have explicit forms, i.e.,

$$P_{C_1}(x) = \begin{cases} 1 - a + x\,, & a > 1\,; \\ x\,, & a \le 1\,, \end{cases} \text{ and } P_{Q_1}(y) = \begin{cases} \sin(\cdot) + \frac{4(y - \sin(\cdot))}{\sqrt{b}}\,, & b > 16\,; \\ y\,, & b \le 16\,, \end{cases}$$

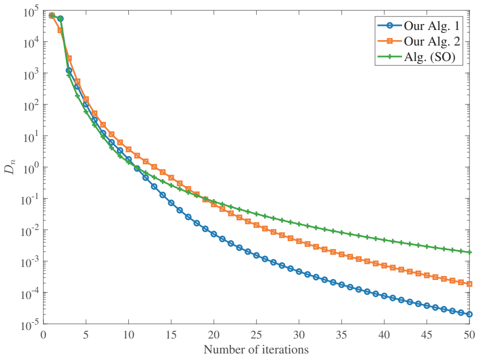
where $a := \int_0^1 x(t)\,\mathrm{d}t$ and $b := \int_0^1 |y(t) - \sin(t)|^2\,\mathrm{d}t$. Let the operator $T$ be defined by $Tx = x$. The parameters of all algorithms are set the same as in Example 5.2. The function $D_n = \|x_{n+1} - x_n\|^2$ is used to measure the error of the $n$-th iteration step. The maximum number of iterations 50 is a common stopping criterion for all algorithms. Figure 1 shows the numerical behavior of all algorithms with four different initial values.



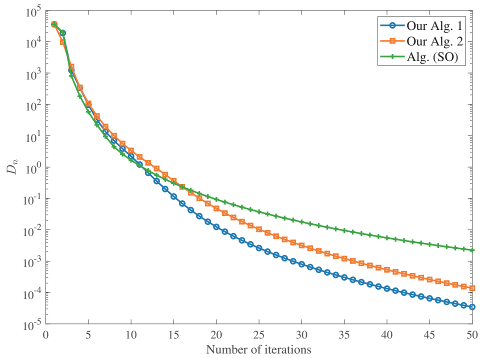

(a) $x_0 = x_1 = 200\log(t)$




(b) $x_0 = x_1 = 200\mathrm{e}^t$




(c) $x_0 = x_1 = 500\sin(t)$




(d) $x_0 = x_1 = 500t^3$

FIGURE 1. Numerical behavior of all algorithms for Example 5.3

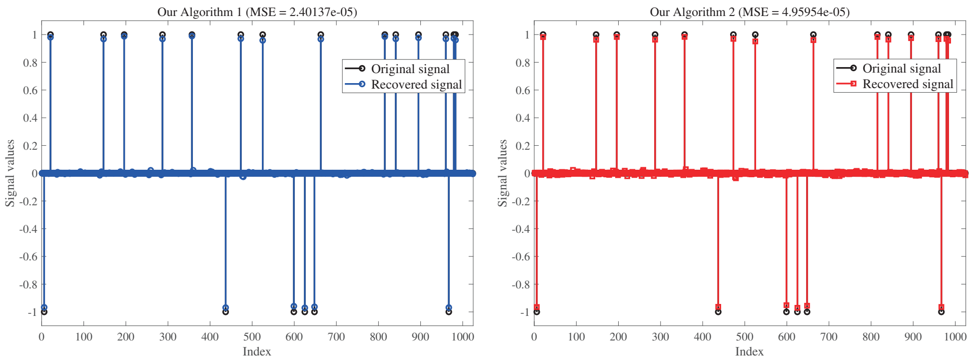
**Example 5.4.** In signal recovery problems, the following underdetermined system problem is often considered and resolved:

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \varepsilon\,, \tag{5.1}$$

where $\mathbf{y} \in \mathbb{R}^M$ is the observed noise data, $\mathbf{A} \in \mathbb{R}^{M\times N}$ is a bounded linear observation operator, $\mathbf{x} \in \mathbb{R}^N$ with $k$ ($k \ll N$) non-zero elements is the original and clean data that needs to be restored, and $\varepsilon$ is the noise observation encountered during data transmission. The difficulty is that the signal $\mathbf{x}$ is sparse, that is, the number of non-zero elements in the signal $\mathbf{x}$ is much smaller than the dimension of the signal $\mathbf{x}$. Thus, the problem (5.1) can be expressed as the following convex constraint minimization problem:

$$\min_{\mathbf{x}\in\mathbb{R}^N} \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2 \;\text{ subject to }\; \|\mathbf{x}\|_1 \le t\,, \tag{5.2}$$

where $t$ is a positive constant and $\|\cdot\|_1$ is $\ell_1$ norm. More precisely, the problem (5.2) is equivalent to the split feasibility problem when $C_1 = \{\mathbf{x} \in \mathbb{R}^N : \|\mathbf{x}\|_1 \le t\}$ and $Q_1 = \{\mathbf{y}\}$. In this numerical experiment, the matrix $\mathbf{A} \in \mathbb{R}^{M\times N}$ is created from a standard normal distribution with zero mean and unit variance and then orthonormalizing the rows. The clean signal $\mathbf{x} \in \mathbb{R}^N$ contains $k$ ($k \ll N$) randomly generated $\pm 1$ spikes. The observation $\mathbf{y}$ is formed by $\mathbf{y} = \mathbf{Ax} + \varepsilon$ with white Gaussian noise $\varepsilon$ of variance $10^{-4}$. The recovery process starts with the initial signals $\mathbf{x_0} = \mathbf{x_1} = \mathbf{0}$ and ends after 1000 iterations. We use the mean squared error $\text{MSE} = (1/N)\left\|\mathbf{x}^* - \mathbf{x}\right\|^2$ ($\mathbf{x}^*$ is an estimated signal of $\mathbf{x}$) to measure the restoration accuracy of all algorithms. Set $M = 512$, $N = 1024$, $k = 20$ and $T$ is an identity mapping in suggested Algorithms 1 and 2. The other parameters of the proposed algorithms are the same as those in Example 5.2. The recovery results of the suggested algorithms are shown in Fig. 2.




(a) Recovered by our Algorithm 1 (b) Recovered by our Algorithm 2

FIGURE 2. The original signal and the signal recovered by our algorithms

Through the above experiment results on Examples 5.2–5.4, the following conclusions can be easily obtained.

(1) For different dimensions and initial values, Table 1 and Fig. 1 show that our algorithms are stable, efficient and easy to implement without using operator norms.
(2) From Table 1, under the same error accuracy, the number of iterations and CPU time of our algorithms is less than that of Algorithm (SO).
(3) From Fig. 1, under the same number of iterations, the error of our algorithms is much lower than Algorithm (SO).
(4) Figure 2 shows the original signal and the signal recovered by our algorithms, and verifies the usability of our algorithms in signal recovery problems.

## 6. Conclusions

In this paper, two strongly convergent algorithms with the Meir-Keeler contraction mapping and the inertial method are proposed for finding the common solution of the split monotone variational inclusion and the fixed point problem of demicontractive mappings. More importantly, an adaptive step size independent of the operator norm is considered in our algorithms, which solves the problem that the operator norm cannot be estimated in practical applications. Our results also improve the existing results in many mathematical problems, such as the split monotone variational inclusion problem, the split variational inclusion problem, the split variational inequality problem and the split feasibility problem. Moreover, the convergence performance of our algorithms is demonstrated by some numerical experiments including signal recovery problems.

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